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Results in Mathematics

Direct and Converse Voronovskaya Estimates for the Bernstein Operator

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Abstract. As is known, if $f \in C^2[0,1]$, then, for the Bernstein operator B_n, there holds

$$\lim_{n\to\infty} n(B_n f(x) - f(x)) = \frac{x(1-x)}{2} f''(x)$$

uniformly on $[0,1]$. We characterize the rate of this convergence in terms of K-functionals and moduli of smoothness.

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1. Main Results

The Bernstein polynomial of degree $n \in \mathbb{N}$ of $f \in C[0,1]$ is defined for $x \in [0,1]$ by

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}.$$

A quite well known result due to Voronovskaya [23] (or see e.g. [8, p. 307], or [20, p. 22]) states that if $f \in C^2[0,1]$, then

$$\lim_{n\to\infty} n(B_n f(x) - f(x)) = \frac{x(1-x)}{2} f''(x)$$

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uniformly on $[0,1]$. Our goal is to estimate the rate of this convergence. To this purpose, we introduce the linear operator

$$D_n f(x) := n(B_n f(x) - f(x))$$

and the Sobolev-type function spaces

$$W^m_\infty(\varphi)[0,1] := \{f \in C[0,1] : f \in AC^{m-1}_{loc}(0,1), \varphi^m f^{(m)} \in L_\infty[0,1]\},$$

where $\varphi(x) = \sqrt{x(1-x)}$. For $f \in W^2_\infty(\varphi)[0,1]$ we set $\mathcal{D}f(x) := \frac{\varphi^2(x)}{2} f''(x)$.

We denote the L_∞-norm on $[0,1]$ by $\| \circ \|$, and $\| \circ \|_J$ is L_∞-norm on the interval J. By c we denote absolute positive constants, not necessarily the same at each occurrence.

It is known that (see [9, Lemma 8.3])

$$\left\| B_n f - f - \frac{1}{2n}\,\varphi^2 f'' \right\| \le \frac{c}{n^{3/2}} \|\varphi^3 f^{(3)}\|, \tag{1.1}$$

which can be written in the form

$$\|D_n f - \mathcal{D}f\| \le \frac{c}{n^{1/2}} \|\varphi^3 f^{(3)}\|.$$

We will show, assuming a higher degree of smoothness, that

$$\left\| B_n f - f - \frac{1}{2n}\,\varphi^2 f'' \right\| \le \frac{c}{n^2} \left(\|\varphi^2 f^{(3)}\| + \|\varphi^4 f^{(4)}\| \right), \quad f \in W^4_\infty(\varphi)[0,1],$$

that is,

$$\|D_n f - \mathcal{D}f\| \le \frac{c}{n} \left(\|\varphi^2 f^{(3)}\| + \|\varphi^4 f^{(4)}\| \right).$$

Let us note that if $f \in W^4_\infty(\varphi)[0,1]$, then $\varphi^2 f'', \varphi^2 f^{(3)} \in L_\infty[0,1]$ (see Lemma 2.2(a) below). That slightly improves the estimate

$$\left\| B_n f - f - \frac{1}{2n}\,\varphi^2 f'' \right\| \le \frac{c}{n^2} \left(\|f^{(3)}\| + \|f^{(4)}\| \right), \quad f \in C^4[0,1],$$

established in [16] (see also [15]).

To state our main results we will use the K-functionals

$$K_{2,\varphi}(F,t)_{w,\infty} := \inf_{G \in AC^1_{loc}(0,1)} \{\|w(F-G)\| + t\|w\varphi^2 G''\|\}$$

where w is a weight on $(0,1)$, and

$$\widetilde{K}(F,t) := \inf_{g \in W^4_\infty(\varphi)[0,1]} \left\{ \|F - \mathcal{D}g\| + t \left(\|\varphi^2 g^{(3)}\| + \|\varphi^4 g^{(4)}\| \right) \right\}.$$

We establish the following characterization of the rate of approximation of $\mathcal{D}f$ by means of $D_n f$.

Theorem 1.1. *For all $f \in W^2_\infty(\varphi)[0,1]$ and all $n \in \mathbb{N}$ there holds*

$$\|D_n f - \mathcal{D}f\| \le c\,\widetilde{K}(\mathcal{D}f, n^{-1}) \le c\left(K_{2,\varphi}(f'', n^{-1})_{\varphi^2,\infty} + \frac{1}{n}\,\|\varphi^2 f''\|\right). \quad (1.2)$$

Conversely, for all $f \in W^2_\infty(\varphi)[0,1]$ and all $k, n \in \mathbb{N}$ there holds

$$K_{2,\varphi}(f'', n^{-1})_{\varphi^2,\infty} \le 2\,\|D_k f - \mathcal{D}f\| + c\frac{k}{n}\,K_{2,\varphi}(f'', k^{-1})_{\varphi^2,\infty} + \frac{c}{n}\,\|\varphi^2 f''\|. \quad (1.3)$$

The above two estimates can also be written in the form

$$\left\|B_n f - f - \frac{1}{2n}\,\varphi^2 f''\right\| \le \frac{c}{n}\,\widetilde{K}(\mathcal{D}f, n^{-1}) \le \frac{c}{n}\,K_{2,\varphi}(f'', n^{-1})_{\varphi^2,\infty} + \frac{c}{n^2}\,\|\varphi^2 f''\| \quad (1.4)$$

and

$$\frac{c}{k}\,K_{2,\varphi}(f'', n^{-1})_{\varphi^2,\infty} \le 2\left\|B_k f - f - \frac{1}{2k}\,\varphi^2 f''\right\| + \frac{c}{n}\,K_{2,\varphi}(f'', k^{-1})_{\varphi^2,\infty} + \frac{c}{nk}\,\|\varphi^2 f''\|. \quad (1.5)$$

We will refer to (1.2) and (1.4) as *direct Voronovskaya inequalities*, and to (1.3) and (1.5) as *weak converse Voronovskaya inequalities*.[1]

Similar direct point-wise estimates were established in [14, Theorem 3.2] and [22, Theorem 2] ([14] contains an overview of other related results). The assumptions on the functions made there are more restrictive. However, the first of these results is very general and both give explicit values to the absolute constant.

Remark 1.2. Since the quantities $D_n f - \mathcal{D}f$ and $K_{2,\varphi}(f'', t)_{\varphi^2,\infty}$ are invariant to translations of f by a quadratic polynomial, the relations above directly imply the following slight improvement:

$$\|D_n f - \mathcal{D}f\| \le c\left(K_{2,\varphi}(f'', n^{-1})_{\varphi^2,\infty} + \frac{1}{n}\,E_0(f'')_{\varphi^2,\infty}\right)$$

and

$$K_{2,\varphi}(f'', n^{-1})_{\varphi^2,\infty} \le 2\,\|D_k f - \mathcal{D}f\| + c\left(\frac{k}{n}\,K_{2,\varphi}(f'', k^{-1})_{\varphi^2,\infty} + \frac{1}{n}\,E_0(f'')_{\varphi^2,\infty}\right),$$

where $E_0(F)_{\varphi^2,\infty} = \inf_{\alpha\in\mathbb{R}} \|\varphi^2(F - \alpha)\|$.

From Theorem 1.1 we shall derive the following equivalence relation.

[1] The term "inverse Voronovskaya theorem" is also used for a different type of results (see [1,4]).

Corollary 1.3. *Let $f \in W^2_\infty(\varphi)[0,1]$ and $0 < \alpha < 1$. Then*

$$\|D_n f - \mathcal{D} f\| = O(n^{-\alpha}) \quad \Longleftrightarrow \quad K_{2,\varphi}(f'', t)_{\varphi^2,\infty} = O(t^\alpha).$$

Bernstein [6] proved that if $f \in C^{2r}[0,1]$, then

$$\lim_{n\to\infty} n^r \left(B_n f(x) - f(x) - \sum_{i=1}^{2r} B_n\left((\circ - x)^i\right)(x) \, \frac{f^{(i)}(x)}{i!} \right) = 0$$

uniformly on $[0,1]$ (see also [21]). A quantitative estimate of this convergence for positive linear operators on $C[0,1]$ was established by Gonska [14] (see also [2,3,13]).

Setting $r = 2$ above we have for $f \in C^4[0,1]$ (see (2.3) below)

$$\lim_{n\to\infty} n(D_n f(x) - \mathcal{D} f(x)) = D' f(x) \tag{1.6}$$

uniformly on $[0,1]$, where

$$D' f(x) := \frac{(1-2x)\varphi^2(x)}{3!} \, f^{(3)}(x) + \frac{3\,\varphi^4(x)}{4!} \, f^{(4)}(x).$$

This shows that the operator D_n (restricted to $C^4[0,1]$) is saturated, as its saturation order is n^{-1} and its trivial class is the set of the algebraic polynomials of degree at most 2.

We will establish the following quantitative estimate of the convergence in (1.6).

Theorem 1.4. *For all $f \in W^4_\infty(\varphi)[0,1]$ and all $n \in \mathbb{N}$ there holds*

$$\left\| D_n f - \mathcal{D} f - \frac{1}{n} D' f \right\| \le \frac{c}{n} K_{2,\varphi^2}(f^{(4)}, n^{-1})_{\varphi^4,\infty} + \frac{c}{n^2} \|\varphi^4 f^{(4)}\|.$$

Instead of $K_{2,\varphi}(F,t)_{\varphi^r,\infty}$ one can use the weighted Ditzian–Totik modulus of smoothness $\omega^2_\varphi(F,t)_{\varphi^r,\infty}$, defined in [10, Chapter 6], or its modification $\omega^\varphi_{2,r}(F,t)_\infty$, given in [12, Chapter 3, Section 10] (see also [17–19]). Both moduli are equivalent to $K_{2,\varphi}(F,t^2)_{\varphi^2,\infty}$. More precisely, [10, Theorem 6.1.1] states

$$c^{-1}\omega^2_\varphi(F,t)_{\varphi^r,\infty} \le K_{2,\varphi}(F,t^2)_{\varphi^r,\infty} \le c\,\omega^2_\varphi(F,t)_{\varphi^r,\infty}, \quad 0 < t \le t_0,$$

with some $t_0 > 0$; and [17, Theorem 2.7] states that

$$c^{-1}\omega^\varphi_{2,r}(F,t)_\infty \le K_{2,\varphi}(F,t^2)_{\varphi^r,\infty} \le c\,\omega^\varphi_{2,r}(F,t)_\infty, \quad 0 < t \le 1. \tag{1.7}$$

In this regard, the weighted Ditzian-Totik main-part modulus of smoothness allows us to restate the characterization in Corollary 1.3 in a simpler form. Corollary 1.3 and [10, (6.2.6) and (6.2.10)] yield

Corollary 1.5. *Let $f \in W^2_\infty(\varphi)[0,1]$ and $0 < \alpha < 1$. Then*

$$\|D_n f - \mathcal{D} f\| = O(n^{-\alpha}) \quad \Longleftrightarrow \quad \|\varphi^2 \Delta^2_{h\varphi} f''\|_{[2h^2, 1-2h^2]} = O(h^{2\alpha}),$$

where $\Delta^2_{h\varphi(x)} F(x) := F(x + h\varphi(x)) - 2F(x) + F(x - h\varphi(x))$, $x \in [2h^2, 1-2h^2]$.

Let us also note that (1.2), [17, Theorem 7.1] and (1.7) imply (1.1).

We refer the reader to the new monograph by Bustamante [7], which includes, besides the classical, also the most recent results on the Bernstein polynomials.

The contents of the paper are organized as follows. In Sect. 2 we recall several pertinent properties of the Bernstein operator and establish Jackson, Bernstein, and Voronovskaya-type inequalities concerning D_n. Then, in the next and last section, we present proofs of Theorems 1.1 and 1.4 and of Corollary 1.3.

2. Basic Properties of D_n

First, we recall several basic properties of the Bernstein operators. Direct computation yields the following formula for the derivatives of the polynomials $p_{n,k}$, $k = 0, \dots, n$ (see e.g. [8, Chapter 10, (2.1)]):

$$p'_{n,k}(x) = \varphi^{-2}(x)(k - nx)p_{n,k}(x). \tag{2.1}$$

We will use the quantities

$$T_{n,\ell}(x) := \sum_{k=0}^{n} (k - nx)^{\ell} p_{n,k}(x).$$

It is known (see [8, Chapter 10, Theorem 1.1]) that

$$T_{n,\ell}(x) = \sum_{1\le\rho\le\ell/2} t_{\ell,\rho}(x) \left(n\varphi^2(x)\right)^{\rho}, \quad \ell \in \mathbb{N}, \tag{2.2}$$

where $t_{\ell,\rho}(x)$ are polynomials, whose coefficients are independent of n.

In particular (see e.g. [8, p. 304] and [20, p. 14]),

$$\begin{aligned} &T_{n,0}(x) = 1, \quad T_{n,1}(x) = 0, \quad T_{n,2}(x) = n\varphi^2(x),\\ &T_{n,3}(x) = (1 - 2x)n\varphi^2(x), \quad T_{n,4}(x) = 3n^2\varphi^4(x) + n\varphi^2(x)(1 - 6\varphi^2(x)). \end{aligned} \tag{2.3}$$

The identity (2.2) implies

$$0 \le T_{n,2m}(x) \le c \begin{cases} n\varphi^2(x), & n\varphi^2(x) \le 1,\\ \left(n\varphi^2(x)\right)^m, & n\varphi^2(x) \ge 1. \end{cases} \tag{2.4}$$

Then by Cauchy's inequality and the identity $\sum_{k=0}^{n} p_{n,k}(x) \equiv 1$ we have

$$\sum_{k=0}^{n} |k - nx|^m p_{n,k}(x) \le c \begin{cases} 1, & n\varphi^2(x) \le 1,\\ \left(n\varphi^2(x)\right)^{m/2}, & n\varphi^2(x) \ge 1. \end{cases} \tag{2.5}$$

After these preliminaries, we proceed to the basic properties of D_n.

First, we note that the operator D_n is bounded in the following sense.

Proposition 2.1. *For all $f \in W^2_\infty(\varphi)[0,1]$ and all $n \in \mathbb{N}$ there holds*

$$\|D_n f\| \le 2\,\|\mathcal{D} f\|.$$

Proof. As is known (see e.g. [9, p. 87]),

$$\|B_n f - f\| \le \frac{1}{n}\,\|\varphi^2 f''\|.$$

Hence the assertion immediately follows. □

Next, we will establish a Jackson-type estimate. Before that we make a technical observation.

Lemma 2.2. (a) *If $g \in W^4_\infty(\varphi)[0,1]$, then $\varphi^2 g'', \varphi^2 g^{(3)} \in L_\infty[0,1]$, as moreover*

$$\|\varphi^2 g^{(3)}\| \le c\left(\|\varphi^2 g''\| + \|\varphi^4 g^{(4)}\|\right). \tag{2.6}$$

(b) *If $g \in W^6_\infty(\varphi)[0,1]$, then $\varphi^2 g^{(4)}, \varphi^4 g^{(5)} \in L_\infty(\varphi)[0,1]$, as moreover*

$$\|\varphi^2 g^{(4)}\| \le c\left(\|\varphi^4 g^{(4)}\| + \|\varphi^6 g^{(6)}\|\right) \tag{2.7}$$

and

$$\|\varphi^4 g^{(5)}\| \le c\left(\|\varphi^4 g^{(4)}\| + \|\varphi^6 g^{(6)}\|\right) \tag{2.8}$$

Proof. To prove (a), we first apply [11, Proposition 2.1] with $p = \infty$, $w_1 = \varphi^2$, $w_2 = \varphi^4$, $j = 2$ and $m = 4$ and thus get $\varphi^2 g'' \in L_\infty[0,1]$. Again in virtue of [11, Proposition 2.1] with $p = \infty$, $w_1 = \varphi^2$, $w_2 = \varphi^4$, $j = 1$, $m = 2$, and g'' in place of g, we have $\varphi^2 g^{(3)} \in L_\infty[0,1]$ and (2.6).

Similarly, (b) follows from [11, Proposition 2.1] with $p = \infty$, $w_1 = \varphi^{2(j+1)}$, $w_2 = \varphi^6$, $j = 0, 1$, $m = 2$, and $g^{(4)}$ in place of g. □

Proposition 2.3. *For all $g \in W^4_\infty(\varphi)[0,1]$ and all $n \in \mathbb{N}$ there holds*

$$\|D_n g - \mathcal{D} g\| \le \frac{c}{n}\left(\|\varphi^2 g^{(3)}\| + \|\varphi^4 g^{(4)}\|\right).$$

Proof. First, we note that in virtue of Lemma 2.2(a) we have $\varphi^2 g'', \varphi^2 g^{(3)} \in L_\infty[0,1]$ too.

Applying Taylor's formula, we have for $x \in (0,1)$

$$\begin{aligned} g\left(\frac{k}{n}\right) &= g(x) + \left(\frac{k}{n} - x\right) g'(x) + \frac{1}{2}\left(\frac{k}{n} - x\right)^2 g''(x) \\ &\quad + \frac{1}{6}\left(\frac{k}{n} - x\right)^3 g^{(3)}(x) + \frac{1}{6}\int_x^{k/n} \left(\frac{k}{n} - v\right)^3 g^{(4)}(v)\,dv. \end{aligned}$$

Multiplying both sides by $p_{n,k}(x)$, summing with respect to k and using the identities (2.3) we obtain

$$\begin{aligned}
&|D_n g(x) - \mathcal{D}g(x)| \\
&\quad = \left| n(B_n g(x) - g(x)) - \frac{1}{2}\,\varphi^2(x) g''(x) \right| \\
&\quad = \left| \frac{(1-2x)\varphi^2(x)}{6n} g^{(3)}(x) + \frac{n}{6} \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - v\right)^3 g^{(4)}(v)\, dv \right| \\
&\quad \le \frac{1}{6n} \|\varphi^2 g^{(3)}\| + \frac{n}{6} \|\varphi^4 g^{(4)}\| \left| \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - v\right)^3 \varphi^{-4}(v)\, dv \right|
\end{aligned}$$

We will show that

$$R_n(x) := \left| \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - v\right)^3 \varphi^{-4}(v)\, dv \right| \le \frac{c}{n^2}.$$

Obviously, it is enough to prove it for $0 < x \le 1/2$. We consider two cases.

Case 1. $1/n \le x \le 1/2$.

Then $\varphi^2(x) \ge 1/2n$ and by using (for v between x and k/n) the inequality [10, p. 141]

$$\frac{\left|\frac{k}{n} - v\right|}{\varphi^2(v)} \le \frac{\left|\frac{k}{n} - x\right|}{\varphi^2(x)}$$

and (2.3), we obtain

$$\begin{aligned}
R_n(x) &\le \sum_{k=0}^{n} p_{n,k}(x) \frac{\left(\frac{k}{n} - x\right)^2}{\varphi^4(x)} \left| \int_x^{k/n} \left(\frac{k}{n} - v\right) dv \right| \\
&= \frac{\varphi^{-4}(x)}{2} \sum_{k=0}^{n} p_{n,k}(x) \left(\frac{k}{n} - x\right)^4 \\
&= \frac{\varphi^{-4}(x)}{2} \left[\frac{3\varphi^4(x)}{n^2} + \frac{(1-6\varphi^2(x))\varphi^2(x)}{n^3} \right] \le \frac{c}{n^2}.
\end{aligned}$$

Case 2. $0 < x \le 1/n$.

Analogously to [9, Lemma 8.3], we will estimate the terms in the sum of $R_n(x)$ separately for $k = 0, 1$ and $k \ge 2$. We have for $k = 0$

$$\begin{aligned}
p_{n,0}(x) \int_0^x v^3 \varphi^{-4}(v)\, dv &= (1-x)^n \int_0^x \frac{v^3\, dv}{\big(v(1-v)\big)^2} \\
&\le (1-x)^{n-2} \int_0^x v\, dv = \frac{x^2(1-x)^{n-2}}{2} \le \frac{c}{n^2}.
\end{aligned}$$

For $k = 1$ and $n \geq 2$ we have

$$\begin{aligned} p_{n,1}(x) &\int_x^{1/n} \left(\frac{1}{n} - v\right)^3 \varphi^{-4}(v)\, dv \\ &= nx(1-x)^{n-1} \int_x^{1/n} \frac{\left(\frac{1}{n} - v\right)^3 dv}{\big(v(1-v)\big)^2} \\ &\leq nx(1-x)^{n-1} \left(1 - \frac{1}{n}\right)^{-2} \int_x^{1/n} \frac{\left(\frac{1}{n}\right)^3 dv}{v^2} \leq \frac{c}{n^2}. \end{aligned}$$

Trivially, for $n = k = 1$ we have

$$\begin{aligned} p_{1,1}(x) \int_x^1 (1-v)^3 \varphi^{-4}(v)\, dv &= x \int_x^1 \frac{(1-v)^3 dv}{\big(v(1-v)\big)^2} \\ &\leq x \int_x^1 \frac{dv}{v^2} \leq 1. \end{aligned}$$

For $k \geq 2$ and $n \geq 3$ we have

$$\begin{aligned} &\left| \sum_{k=2}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - v\right)^3 \varphi^{-4}(v)\, dv \right| \\ &\leq c x^{-2} \sum_{k=2}^{n} p_{n,k}(x) \left(\frac{k}{n} - x\right)^4 \leq c x^{-2} \sum_{k=2}^{n} p_{n,k}(x) \left(\frac{k}{n}\right)^4 \\ &= c x^{-2} \sum_{k=0}^{n-2} \frac{n!}{(k+2)!(n-k-2)!}\, x^{k+2}(1-x)^{n-k-2} \left(\frac{k+2}{n}\right)^4 \\ &\leq c \sum_{k=0}^{n-2} p_{n-2,k}(x) \left(\frac{k}{n-2}\right)^2 = c\left(x^2 + \frac{\varphi^2(x)}{n-2}\right) \leq \frac{c}{n^2}, \end{aligned}$$

where at the last but one estimate we have taken into account (2.3). The case $n = k = 2$ is again trivial. The proof is complete. □

To prove the weak converse inequality in Theorem 1.1 we will use the operator A_n defined for $f \in W_\infty^2(\varphi)[0,1]$ by

$$A_n f(x) := 2n \int_{1/2}^{x} (x-t)\varphi^{-2}(t)[B_n f(t) - f(t)]\, dt.$$

It is easy to see that A_n is well-defined. We show that in the lemma below.

Lemma 2.4. *If $f \in W_\infty^2(\varphi)[0,1]$, then $\varphi^{-2}.(B_n f - f) \in L_1[0,1]$.*

Proof. Clearly, $\varphi^{-2}(x)[B_n f(x) - f(x)]$ is continuous on $(0,1)$. To complete the proof of the lemma, we will show that it is dominated by a summable function on $[0,1]$. To this end, we expand $f(t)$ at $x \in (0,1)$ by Taylor's formula to get

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u) f''(u)\, du, \quad t \in [0,1],$$

and apply B_n with respect to t to both sides of this identity to arrive at

$$B_n f(x) = f(x) + \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du.$$

Here we have used that $T_{n,0}(x) = 1$ and $T_{n,1}(x) = 0$ (see (2.3)).

Consequently,

$$B_n f(x) - f(x) = \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du. \tag{2.9}$$

For $k = 1, \ldots, n-1$, $n \geq 2$, we have $\varphi^{-2} p_{n,k} \in C[0,1]$ and

$$\begin{aligned}\left| \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du \right| &\leq \left| \int_x^{k/n} \frac{du}{u(1-u)} \right| \|\varphi^2 f''\| \\ &\leq [\ln n - \ln x - \ln(1-x)]\, \|\varphi^2 f''\|.\end{aligned}$$

For $k = 0$ and $k = n$, we have, respectively,

$$\left| \frac{p_{n,0}(x)}{\varphi^2(x)} \int_0^x u f''(u)\, du \right| \leq -\frac{\ln(1-x)}{x} \|\varphi^2 f''\|$$

and

$$\left| \frac{p_{n,n}(x)}{\varphi^2(x)} \int_x^1 (1-u) f''(u)\, du \right| \leq -\frac{\ln x}{1-x} \|\varphi^2 f''\|.$$

Hence the assertion of the lemma follows. □

To verify the converse inequality we need two inequalities concerning the derivatives of $A_n f$. The first one is a Bernstein-type inequality.

Proposition 2.5. *For all $f \in W^2_\infty(\varphi)[0,1]$ and $n \in \mathbb{N}$ we have $A_n f \in AC^3_{loc}(0,1)$ and*

$$\|\varphi^4 (A_n f)^{(4)}\| \leq cn\, \|\varphi^2 f''\|.$$

Proof. Clearly, if $f \in W^2_\infty(\varphi)[0,1]$, then $A_n f \in AC^3_{loc}(0,1)$. To establish the inequality, we first evaluate the fourth derivative of $A_n f(x)$ for $x \in (0,1)$. Using the representation (2.9), we get

$$(A_n f)^{(3)}(x) = 2n \sum_{k=0}^{n} \left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)' \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du - \frac{2f''(x)}{\varphi^2(x)} T_{n,1}(x)$$

a.e. in $(0,1)$. Taking into account that $T_{n,1}(x) = 0$, $f''(x)$ is finite almost everywhere, and that the functions $(A_n f)^{(3)}(x)$ and the sum on the right above are continuous on $(0,1)$, we deduce that

$$(A_n f)^{(3)}(x) = 2n \sum_{k=0}^{n} \left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)' \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du, \quad x \in (0,1). \tag{2.10}$$

We differentiate once more and arrive at

$$(A_nf)^{(4)}(x) = 2n\sum_{k=0}^{n}\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)''\int_x^{k/n}\left(\frac{k}{n}-u\right)f''(u)\,du$$
$$- 2nf''(x)\sum_{k=0}^{n}\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)'\left(\frac{k}{n}-x\right) \text{ a.e. in } (0,1). \quad (2.11)$$

Let us consider the second sum on the right above. We calculate the derivative

$$\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)' = \frac{2x-1}{\varphi^4(x)}\,p_{n,k}(x) + \frac{p'_{n,k}(x)}{\varphi^2(x)},$$

and apply the identities (2.1), $T_{n,1}(x) = 0$ and $T_{n,2}(x) = n\varphi^2(x)$ (see (2.3)), to arrive at

$$\sum_{k=0}^{n}\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)'\left(\frac{k}{n}-x\right) = \frac{1}{\varphi^2(x)}.$$

Consequently, by (2.11) we get

$$\varphi^4(x)(A_nf)^{(4)}(x) = 2n\varphi^4(x)\sum_{k=0}^{n}\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)''\int_x^{k/n}\left(\frac{k}{n}-u\right)f''(u)\,du$$
$$- 2n\varphi^2(x)f''(x).$$

Thus to complete the proof of the proposition, it remains to show that

$$\left|\varphi^4(x)\sum_{k=0}^{n}\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)''\int_x^{k/n}\left(\frac{k}{n}-u\right)f''(u)\,du\right| \le c\,\|\varphi^2 f''\| \quad (2.12)$$

for $x\in(0,1)$. We use that (see e.g. [11, Lemma 4.2])

$$\left|\int_x^{k/n}\left(\frac{k}{n}-u\right)f''(u)\,du\right| \le \frac{c}{\varphi^2(x)}\left(\frac{k}{n}-x\right)^2\|\varphi^2 f''\|.$$

So, to verify (2.12), it suffices to establish the estimate

$$\varphi^2(x)\sum_{k=0}^{n}\left|\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)''\right|\left(\frac{k}{n}-x\right)^2 \le c, \quad x\in(0,1). \quad (2.13)$$

First, let $n\varphi^2(x)\ge 1$. By means of (2.1) we represent the second derivative of $\varphi^{-2}(x)p_{n,k}(x)$ in the form

$$\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)'' = \left(\frac{2-n}{\varphi^4(x)} + \frac{2(1-2x)^2}{\varphi^6(x)}\right)p_{n,k}(x)$$
$$- 3\frac{1-2x}{\varphi^6(x)}(k-nx)p_{n,k}(x) + \frac{1}{\varphi^6(x)}(k-nx)^2 p_{n,k}(x).$$

Consequently,

$$\begin{aligned}\varphi^2(x)\sum_{k=0}^{n}\left|\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)''\right|\left(\frac{k}{n}-x\right)^2 &\\ \le \frac{1}{n^2}\left(\frac{n+2}{\varphi^2(x)}+\frac{2}{\varphi^4(x)}\right)T_{n,2}(x) &\\ +\frac{3}{n^2\varphi^4(x)}\sum_{k=0}^{n}|k-nx|^3p_{n,k}(x)+\frac{1}{n^2\varphi^4(x)}T_{n,4}(x)&\end{aligned}$$

and (2.13) for $n\varphi^2(x)\ge 1$ follows from (2.5).

Now, let $n\varphi^2(x)\le 1$. Since

$$\begin{aligned}[x^{k-1}(1-x)^{n-k-1}]'' = {}& (k-1)(k-2)x^{k-3}(1-x)^{n-k-1}\\ &-2(k-1)(n-k-1)x^{k-2}(1-x)^{n-k-2}\\ &+(n-k-1)(n-k-2)x^{k-1}(1-x)^{n-k-3},\end{aligned}$$

in order to verify (2.13), it is enough to show for $x\in(0,1)$ that

$$\sum_{k=0}^{n}\binom{n}{k}(k-1)(k-2)x^{k-3}(1-x)^{n-k-1}(k-nx)^2\le\frac{cn^2}{\varphi^2(x)},$$
$$\sum_{k=0}^{n}\binom{n}{k}(k-1)(n-k-1)x^{k-2}(1-x)^{n-k-2}(k-nx)^2\le\frac{cn^2}{\varphi^2(x)},$$
$$\sum_{k=0}^{n}\binom{n}{k}(n-k-1)(n-k-2)x^{k-1}(1-x)^{n-k-3}(k-nx)^2\le\frac{cn^2}{\varphi^2(x)}.$$

We will prove the first of these inequalities. The proof of the other two is quite similar. We directly see that the terms for $k=0$ and $k=n$ are estimated above by $cn^2\varphi^{-2}(x)$. Hence it remains to show

$$\sum_{k=3}^{n-1}\binom{n}{k}(k-1)(k-2)x^{k-3}(1-x)^{n-k-1}(k-nx)^2\le cn^3, \tag{2.14}$$

where $n\ge 4$.

We change the summation index and use that $n/[\ell(n-\ell)]\le c$ for $\ell=1,\ldots,n-1$, to deduce

$$\begin{aligned}&\sum_{k=3}^{n-1}\binom{n}{k}(k-1)(k-2)x^{k-3}(1-x)^{n-k-1}(k-nx)^2\\ &\quad=\sum_{k=0}^{n-4}\binom{n}{k+3}(k+1)(k+2)x^{k}(1-x)^{n-4-k}(k+3-nx)^2\\ &\quad\le cn^3\sum_{k=0}^{n-4}p_{n-4,k}(x)(k+3-nx)^2\end{aligned}$$

$$= cn^3 \sum_{k=0}^{n-4} p_{n-4,k}(x)\big[(k-(n-4)x)+(3-4x)\big]^2$$
$$= cn^3 \big[T_{n-4,2}(x) + 2(3-4x)T_{n-4,1}(x) + (3-4x)^2 T_{n-4,0}(x)\big] \le cn^3,$$

where at the last step we have taken into account (2.3). Thus (2.14) is verified.

This completes the proof of (2.13) for $n\varphi^2(x) \le 1$, and the proof of the proposition. □

Next, we will estimate $\|\varphi^4(A_n g)^{(4)}\|$ using higher order derivatives. In preparation, we establish the following auxiliary result.

Lemma 2.6. *If $f \in C^3[0,1]$, then the second derivative of $\varphi^{-2}(x)[B_n f(x) - f(x)]$ is continuous and bounded on $(0,1)$.*

Proof. Clearly, $\varphi^{-2}(x)[B_n f(x) - f(x)]$ is twice continuously differentiable on $(0,1)$. By (2.10) we have

$$\left(\frac{B_n f(x) - f(x)}{\varphi^2(x)}\right)' = \sum_{k=0}^{n} \left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)' \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du, \quad x \in (0,1).$$

The summands on the right above for $k = 1, \dots, n-1$, $n \ge 2$, are in $C^1[0,1]$. Also, it is clear that the first derivatives of the terms with $k = 0$ and $k = n$ are continuous on $(0,1)$. It remains to show that they are bounded on $(0,1)$. We will demonstrate this only for $k = 0$; the case of $k = n$ is treated in a similar way.

For $k = 0$ we have

$$\left(\frac{p_{n,0}(x)}{\varphi^2(x)}\right)' \int_0^x u f''(u)\, du = -(n-1)\frac{(1-x)^{n-2}}{x} \int_0^x u f''(u)\, du$$
$$- \frac{(1-x)^{n-1}}{x^2} \int_0^x u f''(u)\, du$$

Set

$$F_1(x) := \frac{1}{x} \int_0^x u f''(u)\, du, \quad F_2(x) := \frac{1}{x^2} \int_0^x u f''(u)\, du.$$

For the derivative of $F_1(x)$ we have

$$F_1'(x) = f''(x) - \frac{1}{x^2} \int_0^x u f''(u)\, du$$

and since

$$\left| \frac{1}{x^2} \int_0^x u f''(u)\, du \right| \le \frac{1}{2} \|f''\|, \quad x \in (0,1),$$

then $F_1'(x)$ is bounded on $(0,1)$.

To show this for $F_2(x)$, we integrate by parts and get

$$F_2(x) = \frac{1}{2x^2} \int_0^x f''(u) d(u^2) = \frac{1}{2} f''(x) - \frac{1}{2x^2} \int_0^x u^2 f^{(3)}(u)\, du.$$

So, it remains to show that the derivative of the function

$$F_3(x) := \frac{1}{x^2}\int_0^x u^2 f^{(3)}(u)\,du$$

is bounded on $(0,1)$. This is verified again straightforwardly since we have

$$F_3'(x) = f^{(3)}(x) - \frac{2}{x^3}\int_0^x u^2 f^{(3)}(u)\,du$$

and

$$\left|\frac{1}{x^3}\int_0^x u^2 f^{(3)}(u)\,du\right| \le \frac{1}{3}\,\|f^{(3)}\|, \quad x \in (0,1).$$

□

Proposition 2.7. *For all $g \in C^4[0,1]$ and all $n \in \mathbb{N}$ we have*

$$\|\varphi^4(A_n g)^{(4)}\| \le c\left(\|\varphi^2 g''\| + \|\varphi^4 g^{(4)}\|\right).$$

Proof. In virtue of Lemma 2.6 we have $A_n g \in AC^3[0,1]$. To establish the inequality, we apply [11, Proposition 2.4, (2.13)] with $p=\infty$, $r=1$, $s=2$, $w=\varphi^2$, and $A_n g$ in place of g, and get (note that $D = 2\mathcal{D}$)

$$\|\varphi^4(A_n g)^{(4)}\| \le c\,\|\mathcal{D}^2(A_n g)\| = c\,\|\varphi^2(D_n g)''\|.$$

Finally, we get by means of [11, Proposition 4.3, (4.17)] with $p=\infty$, $s=2$ and $w=\varphi^2$, the inequality

$$\|\varphi^2(D_n g)''\| = n\,\|\varphi^2(B_n g - g)''\| \le c\left(\|\varphi^2 g''\| + \|\varphi^4 g^{(4)}\|\right).$$

□

We proceed to a Voronovskaya-type estimate for the operator D_n.

Proposition 2.8. *For all $g \in W^6_\infty(\varphi)[0,1]$ and all $n \in \mathbb{N}$ there holds*

$$\left\|D_n g - \mathcal{D}g - \frac{1}{n}\,D'g\right\| \le \frac{c}{n^2}\left(\|\varphi^4 g^{(4)}\| + \|\varphi^6 g^{(6)}\|\right).$$

Proof. First, we note that in virtue of Lemma 2.2 we have $\varphi^2 g'', \varphi^2 g^{(3)}, \varphi^2 g^{(4)}$, $\varphi^4 g^{(5)} \in L_\infty[0,1]$ too.

Again we consider two cases.

Let $n\varphi^2(x) \ge 1$. We expand $g(t)$ at $x \in (0,1)$ by Taylor's formula to get for $t \in [0,1]$

$$g(t) = \sum_{i=0}^{5} g^{(i)}(x)\frac{(t-x)^i}{i!} + \frac{1}{5!}\int_x^t (t-u)^5 g^{(6)}(u)\,du.$$

We apply B_n to both sides of the above identity, take into account (2.3), multiply by n, and rearrange the terms to get

$$\begin{aligned} D_n g(x) - \mathcal{D}g(x) - \frac{1}{n}\, D' g(x) = {} & \frac{1-6\varphi^2(x)}{4! n^2}\, \varphi^2(x) g^{(4)}(x) \\ & + \frac{1}{5! n^4}\, T_{n,5}(x) g^{(5)}(x) \\ & + \frac{n}{5!} \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^5 g^{(6)}(u)\, du. \end{aligned} \tag{2.15}$$

By (2.2) with $\ell = 5$ and (2.8) we have

$$\left| T_{n,5}(x) g^{(5)}(x) \right| \le c(n\varphi^2(x))^2 \left| g^{(5)}(x) \right| \le cn^2 \left(\|\varphi^4 g^{(4)}\| + \|\varphi^6 g^{(6)}\| \right). \tag{2.16}$$

Further, we use that (see e.g. [11, Lemma 4.2])

$$\left| \int_x^{k/n} \left(\frac{k}{n} - u\right)^5 g^{(6)}(u)\, du \right| \le \frac{c}{\varphi^6(x)} \left(\frac{k}{n} - x\right)^6 \|\varphi^6 g^{(6)}\|.$$

Therefore, taking into account (2.4) with $m = 3$, we have

$$\begin{aligned} & \left| \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^5 g^{(6)}(u)\, du \right| \\ & \quad \le \frac{c}{n^6 \varphi^6(x)} T_{n,6}(x) \, \|\varphi^6 g^{(6)}\| \le \frac{c}{n^3}\, \|\varphi^6 g^{(6)}\|. \end{aligned} \tag{2.17}$$

Combining (2.15), (2.7), (2.16) and (2.17), we arrive at the inequality

$$\left| D_n g(x) - \mathcal{D}g(x) - \frac{1}{n}\, D' g(x) \right| \le \frac{c}{n^2} \left(\|\varphi^4 g^{(4)}\| + \|\varphi^6 g^{(6)}\| \right).$$

for $n\varphi^2(x) \ge 1$.

Now, let $n\varphi^2(x) \le 1$. Using symmetry, we can also assume that $0 < x \le 1/2$. Then $x \le 2/n$. In this case we start with the expansion

$$g(t) = \sum_{i=0}^{4} g^{(i)}(x) \frac{(t-x)^i}{i!} + \frac{1}{4!} \int_x^t (t-u)^4 g^{(5)}(u)\, du, \quad t \in [0,1].$$

We apply B_n to both sides of the above identity, take into account (2.3), multiply by n, and rearrange the terms to get

$$\begin{aligned} D_n g(x) - \mathcal{D}g(x) - \frac{1}{n}\, D' g(x) = {} & \frac{1-6\varphi^2(x)}{4! n^2}\, \varphi^2(x) g^{(4)}(x) \\ & + \frac{n}{4!} \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^4 g^{(5)}(u)\, du. \end{aligned} \tag{2.18}$$

In order to estimate the sum on the right above, we consider separately the terms with $k = 0$ and $k = 1$.

For $k = 0$ we have, bearing in mind that $0 < x \leq 1/2$ and $x \leq 2/n$,

$$\left|(1-x)^n \int_0^x u^4 g^{(5)}(u)\,du\right| \leq \int_0^x \frac{u^2}{(1-u)^2}\,du\,\|\varphi^4 g^{(5)}\| \leq \frac{c}{n^3}\,\|\varphi^4 g^{(5)}\|.$$

Similarly, for $k = 1$ and $n \geq 2$ we have

$$\begin{aligned}&\left|nx(1-x)^{n-1} \int_x^{1/n} \left(\frac{1}{n}-u\right)^4 g^{(5)}(u)\,du\right|\\ &\quad\leq \frac{cx}{n^3}\left|\int_x^{1/n} \frac{du}{u^2(1-u)^2}\right|\|\varphi^4 g^{(5)}\|\\ &\quad\leq \frac{cx}{n^3}\left|\int_x^{1/n} \frac{du}{u^2}\right|\|\varphi^4 g^{(5)}\| \leq \frac{c}{n^3}\,\|\varphi^4 g^{(5)}\|.\end{aligned}$$

Straightforward calculations yield for $n = k = 1$ the estimate

$$\begin{aligned}\left|x\int_x^1 (1-u)^4 g^{(5)}(u)\,du\right| &\leq x\int_x^1 \frac{(1-u)^2}{u^2}\,du\,\|\varphi^4 g^{(5)}\|\\ &\leq x\int_x^1 \frac{du}{u^2}\,\|\varphi^4 g^{(5)}\| \leq \|\varphi^4 g^{(5)}\|.\end{aligned}$$

Thus we have established that

$$\left|p_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n}-u\right)^4 g^{(5)}(u)\,du\right| \leq \frac{c}{n^3}\,\|\varphi^4 g^{(5)}\|, \quad k = 0,1,\ n \in \mathbb{N}. \tag{2.19}$$

In order to estimate the remaining part of the sum on the right of (2.18), we take into account that (see e.g. [11, Lemma 4.2])

$$\left|\int_x^{k/n}\left(\frac{k}{n}-u\right)^4 g^{(5)}(u)\,du\right| \leq \frac{c}{\varphi^4(x)}\left|\frac{k}{n}-x\right|^5 \|\varphi^4 g^{(5)}\|.$$

Hence, for $n \geq 2$, we have

$$\begin{aligned}&\left|\sum_{k=2}^n p_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n}-u\right)^4 g^{(5)}(u)\,du\right|\\ &\quad\leq \frac{c}{n^5\varphi^4(x)}\sum_{k=2}^n p_{n,k}(x)|k-nx|^5\|\varphi^4 g^{(5)}\|.\end{aligned}$$

We estimate the sum on the right. We have

$$\begin{aligned}
&\sum_{k=2}^{n} p_{n,k}(x)|k-nx|^5\\
&\qquad = \sum_{k=0}^{n-2} \frac{n!}{(k+2)!(n-k-2)!}\, x^{k+2}(1-x)^{n-k-2}|k+2-nx|^5\\
&\qquad \le n^2x^2 \sum_{k=0}^{n-2} p_{n-2,k}(x)\,|(k-(n-2)x)+2(1-x)|^5\\
&\qquad \le c\,n^2x^2 \sum_{k=0}^{n-2} p_{n-2,k}(x)\,\left[|k-(n-2)x|^5+1\right]\\
&\qquad \le c\,n^2x^2,
\end{aligned}$$

as at the last step we have taken into account (2.5).

Thus we have established that

$$\left|\sum_{k=2}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n}-u\right)^4 g^{(5)}(u)\,du\right| \le \frac{c}{n^3}\,\|\varphi^4 g^{(5)}\|. \tag{2.20}$$

Combining (2.18), (2.19), (2.20), (2.7) and (2.8), we arrive at

$$\left|D_n g(x) - \mathcal{D}g(x) - \frac{1}{n}\,D'g(x)\right| \le \frac{c}{n^2}\left(\|\varphi^4 g^{(4)}\| + \|\varphi^6 g^{(6)}\|\right).$$

for $n\varphi^2(x) \le 1$.

The proof of the proposition is completed. □

3. Proofs of the Main Theorems

Proof of Theorem 1.1. The direct inequality

$$\|D_n f - \mathcal{D}f\| \le c\,\widetilde{K}(\mathcal{D}f, n^{-1}) \tag{3.1}$$

follows from Propositions 2.1 and 2.3 and Lemma 2.2(a) by means of a standard technique. For any $g \in W^4_\infty(\varphi)[0,1]$ we have

$$\begin{aligned}
\|D_n f - \mathcal{D}f\| &\le \|D_n(f-g)\| + \|D_n g - \mathcal{D}g\| + \|\mathcal{D}(f-g)\|\\
&\le c\left[\|\mathcal{D}f - \mathcal{D}g\| + \frac{1}{n}\left(\|\varphi^2 g^{(3)}\| + \|\varphi^4 g^{(4)}\|\right)\right].
\end{aligned}$$

We take the infimum on g and arrive at (3.1).

Next, we observe that (2.6) directly implies for $g \in C^4[0,1]$ and $0 < t \le 1$

$$\begin{aligned}
\widetilde{K}(\mathcal{D}f, t) &\le c\left[\|\varphi^2(f''-g'')\| + t\left(\|\varphi^2 g''\| + \|\varphi^4 g^{(4)}\|\right)\right]\\
&\le c\left(\|\varphi^2(f''-g'')\| + t\,\|\varphi^4 g^{(4)}\|\right) + ct\,\|\varphi^2 f''\|.
\end{aligned}$$

Taking the infimum on g, we get

$$\begin{aligned}\widetilde{K}(\mathcal{D}f,t) &\le c \inf_{g\in C^4[0,1]} \left\{\|\varphi^2(f''-g'')\| + t\,\|\varphi^4 g^{(4)}\|\right\} + ct\,\|\varphi^2 f''\| \\ &\le c\left(K_{2,\varphi}(f'',t)_{\varphi^2} + t\,\|\varphi^2 f''\|\right),\end{aligned}$$

as, to get the last estimate, we have applied [11, Lemma 5.1] with $p=\infty$, $r=1$, $s=2$ and $w=\varphi^2$.

That completes the proof of (1.2). We proceed to the weak converse one given in (1.3).

For any $g\in C^4[0,1]$ we have

$$\begin{aligned}K_{2,\varphi}(f'',n^{-1})_{\varphi^2} &\le \|\varphi^2[f''-(A_k f)'']\| + \frac{1}{n}\,\|\varphi^4 (A_k f)^{(4)}\| \\ &\le 2\,\|D_k f - \mathcal{D}f\| \\ &\quad + \frac{1}{n}\left(\|\varphi^4(A_k(f-g))^{(4)}\| + \|\varphi^4(A_k g)^{(4)}\|\right). \qquad (3.2)\end{aligned}$$

We estimate the second term by Proposition 2.5 to get

$$\|\varphi^4(A_k(f-g))^{(4)}\| \le ck\,\|\varphi^2(f''-g'')\|. \qquad (3.3)$$

For the third term, by means of Proposition 2.7, we derive the estimate

$$\begin{aligned}\|\varphi^4(A_k g)^{(4)}\| &\le c\left(\|\varphi^2 g''\| + \|\varphi^4 g^{(4)}\|\right) \\ &\le c\left(\|\varphi^2(f''-g'')\| + \|\varphi^4 g^{(4)}\| + \|\varphi^2 f''\|\right).\end{aligned} \qquad (3.4)$$

Now, combining (3.2)–(3.4), we arrive at

$$\begin{aligned}K_{2,\varphi}(f'',n^{-1})_{\varphi^2} &\le 2\,\|D_k f - \mathcal{D}f\| \\ &\quad + c\frac{k}{n}\left(\|\varphi^2(f''-g'')\| + \frac{1}{k}\,\|\varphi^4 g^{(4)}\|\right) + \frac{c}{n}\,\|\varphi^2 f''\|.\end{aligned}$$

Consequently,

$$\begin{aligned}K_{2,\varphi}(f'',n^{-1})_{\varphi^2} &\le 2\,\|D_k f - \mathcal{D}f\| \\ &+ c\left(\frac{k}{n}\inf_{g\in C^4[0,1]}\left\{\|\varphi^2(f''-g'')\| + \frac{1}{k}\,\|\varphi^4 g^{(4)}\|\right\} + \frac{1}{n}\,\|\varphi^2 f''\|\right).\end{aligned}$$

Now, (1.3) follows from [11, Lemma 5.1] with $p=\infty$, $r=1$, $s=2$ and $w=\varphi^2$. □

Proof of Corollary 1.3. If $K_{2,\varphi}(f'',t)_{\varphi^2,\infty} = O(t^\alpha)$ for some $\alpha\in(0,1]$, then (1.2) implies $\|D_n f - \mathcal{D}f\| = O(n^{-\alpha})$.

To establish the converse implication, we use a standard method based on the Berens–Lorentz lemma (see [5], or e.g. [8, Chapter 10, Lemma 5.2]).

Let $\|D_n f - \mathcal{D}f\| = O(n^{-\alpha})$ for some $\alpha\in(0,1)$. Then (1.3) implies

$$K_{2,\varphi}(f'',n^{-1})_{\varphi^2,\infty} \le C_f k^{-\alpha} + c\frac{k}{n}\,K_{2,\varphi}(f'',k^{-1})_{\varphi^2,\infty} + \frac{c}{n}\,\|\varphi^2 f''\|, \qquad (3.5)$$

where C_f is a positive constant that generally may depend on f, but not on k or n.

Let $0 < s \le t \le 1$. Set $n :=]1/s[$ and $k :=]1/t[$, where $]\gamma[$ denotes the smallest integer not less than the positive real γ. Then

$$1 \le sn \le 2, \quad 1 \le tk \le 2. \tag{3.6}$$

Using (3.5), (3.6) and the subadditivity of the K-functional on its second argument, that is,

$$K_{2,\varphi}(F,\delta_1)_{\varphi^2,\infty} \le \max\left\{1, \frac{\delta_1}{\delta_2}\right\} K_{2,\varphi}(F,\delta_2)_{\varphi^2,\infty},$$

we arrive at the estimate

$$\begin{aligned} K_{2,\varphi}(f'',s)_{\varphi^2,\infty} &\le 2K_{2,\varphi}(f'',n^{-1})_{\varphi^2,\infty} \\ &\le C_f t^\alpha + c\frac{s}{t}\, K_{2,\varphi}(f'',t)_{\varphi^2,\infty} + cs\,\|\varphi^2 f''\|, \end{aligned}$$

where, to recall, the constant c is independent of f, s and t, and the constant C_f may depend on f, but is independent of s and t.

Consequently,

$$\begin{aligned} &K_{2,\varphi}(f'',s)_{\varphi^2,\infty} + s\,\|\varphi^2 f''\| \\ &\quad \le C_f t^\alpha + c\frac{s}{t}\left(K_{2,\varphi}(f'',t)_{\varphi^2,\infty} + t\,\|\varphi^2 f''\|\right). \end{aligned} \tag{3.7}$$

We set $\phi(y) := K_{2,\varphi}(f'',y^2)_{\varphi^2,\infty} + y^2\|\varphi^2 f''\|$. Let $0 < x \le y \le 1$. We put $s := x^2$ and $t := y^2$ in (3.7) and get

$$\phi(x) \le C_f\left(y^{2\alpha} + \frac{x^2}{y^2}\,\phi(y)\right), \quad 0 < x \le y \le 1, \tag{3.8}$$

with some constant C_f, which may depend on f, but is independent of x and y.

Now, the Berens-Lorentz lemma yields

$$\phi(y) \le C_f y^{2\alpha}, \quad 0 \le y \le 1,$$

with some constant C_f, which may depend on f, but is independent of y.

Consequently,

$$K_{2,\varphi}(f'',t)_{\varphi^2,\infty} = O(t^\alpha).$$

□

Proof of Theorem 1.4. We proceed similarly to the proof of the direct inequality in Theorem 1.1.

Let $g \in C^6[0,1]$ and $h(x) := \alpha x^3/6$ with $\alpha := (f-g)^{(3)}(1/2)$. We have, in virtue of Propositions 2.3 and 2.8,

$$\begin{aligned}\left\| D_n f - \mathcal{D} f - \frac{1}{n} D' f \right\| &\le \| D_n(f-g-h) - \mathcal{D}(f-g-h)\| \\ &\quad + \frac{1}{n} \| D'(f-g-h)\| \\ &\quad + \left\| D_n(g+h) - \mathcal{D}(g+h) - \frac{1}{n} D'(g+h) \right\| \\ &\le \frac{c}{n} \left(\|\varphi^2[(f-g)^{(3)} - \alpha]\| + \|\varphi^4 (f-g)^{(4)}\| \right) \\ &\quad + \frac{c}{n^2} \left(\|\varphi^4 g^{(4)}\| + \|\varphi^6 g^{(6)}\| \right). \end{aligned} \tag{3.9}$$

Trivially, we have

$$\|\varphi^4 g^{(4)}\| \le \|\varphi^4 (f-g)^{(4)}\| + \|\varphi^4 f^{(4)}\|. \tag{3.10}$$

Also, for $F \in AC_{loc}(0,1)$ such that $\varphi^4 F' \in L_\infty[0,1]$, and $x \in (0,1)$ there holds

$$\begin{aligned}|\varphi^2(x)(F(x) - F(1/2))| &= \left| \varphi^2(x) \int_{1/2}^{x} F'(u)\, du \right| \\ &\le \left| \varphi^2(x) \int_{1/2}^{x} \frac{du}{\varphi^4(u)} \right| \|\varphi^4 F'\| \le c\, \|\varphi^4 F'\|. \end{aligned} \tag{3.11}$$

The estimates (3.9), (3.10), and (3.11) with $F = (f-g)^{(3)}$ imply

$$\left\| D_n f - \mathcal{D} f - \frac{1}{n} D' f \right\| \le \frac{c}{n} \left(\|\varphi^4 (f^{(4)} - g^{(4)})\| + \frac{1}{n} \|\varphi^6 g^{(6)}\| \right) + \frac{c}{n^2} \|\varphi^4 f^{(4)}\|.$$

We complete the proof as we take the infimum on $g \in C^6[0,1]$ in the relation above and apply [11, Lemma 5.1] with $p = \infty$, $r = 1$, $s = 4$ and $w = \varphi^4$. $\square$

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