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Results in Mathematics

Generalized Statistically Almost Convergence Based on the Difference Operator which Includes the (p,q)-Gamma Function and Related Approximation Theorems

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Abstract. This paper is devoted to extend the notion of almost convergence and its statistical forms with respect to the difference operator involving (p,q)-gamma function and an increasing sequence (λ_n) of positive numbers. We firstly introduce some new concepts of almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistical convergence, statistical almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergence and strong almost $[\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)]_r$-convergence. Moreover, we present some inclusion relations between these newly proposed methods and give some counterexamples to show that these are non-trivial generalizations of existing literature on this topic. We then prove a Korovkin type approximation theorem for functions of two variables through statistically almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergence and also present an illustrative example via bivariate non-tensor type Meyer–König and Zeller generalization of Bernstein power series. Furthermore, we estimate the rate of almost convergence of approximating linear operators by means of the modulus of continuity and derive some Voronovskaja type results by using the generalized Meyer–König and Zeller operators. Finally, some computational and geometrical interpretations for the convergence of operators to a function are presented.

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1. Introduction and Preliminaries

The theory of sequence spaces has wide application areas to several branches of functional analysis such as summability theory, matrix transformations, the theory of functions, and the theory of locally convex spaces. By a sequence space, we understand a linear subspace of the space ω of all complex sequences. Let ℓ_∞, c and c_0 denote the linear spaces of bounded, convergent and null sequences with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$ where $k \in \mathbb{N}$, the set of positive integers. With this norm, it is proved that these are all Banach spaces.

It is well known that Hahn-Banach Extension Theorem has many useful applications in the theory of sequence spaces. One of the most important of these applications is Banach limit [1] which gives rise to the concept of almost convergence. Based on the notion of Banach limit, as a non-negative linear functional, Lorentz [2] introduced a new type of convergence which is called almost convergence.

Now, we will give some well known definitions related to the above concepts.

A linear functional $\mathcal{L}$ on ℓ_∞ is said to be a Banach limit if it has the properties:

(i) $\mathcal{L}(x) \geq 0$ if $x \geq 0$ (i.e $x_n \geq 0$ for all n);
(ii) $\mathcal{L}(e) = 1$, where $e = (1, 1, \dots)$;
(iii) $\mathcal{L}(Px) = \mathcal{L}(x)$, where P is the shift operator defined by $(Px)_n = x_{n+1}$ for each $n \in \mathbb{N}$.

A sequence $x = (x_k) \in \ell_\infty$ is said to be *almost convergent* to the generalized limit L if all Banach limits of x are L, and we denote it by $L = f\text{-}\lim x_k$. Assume that P^i is the composition of P and define $t_{mn}(x)$ by

$$t_{mn}(x) = \frac{1}{m+1} \sum_{i=0}^{m} (P^i x)_n = \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i}$$

for all $n, m \in \mathbb{N}$. It is shown that $f\text{-}\lim x_k = L$ if and only if $\lim_m t_{mn}(x) = L$, uniformly in n (see [2]). That is, we say that a sequence (x_k) is almost convergent to L if and only if $t_{mn}(x) \to L$ as $m \to \infty$, uniformly in n.

There is another approach for convergent sequence known as the statistical convergence which was introduced by Fast [3] and Steinhaus [4] independently in the same year 1951. Many years later it has been discussed satisfactorily in the theory of functional analysis, number theory and ergodic theory. There are useful applications of statistical convergence even in remote fields such as probability theory. Later on, in recent years, the idea of statistical convergence was investigated as some generalizations on summability methods and approximation by linear positive operators, we refer to [5–15].

Let us first recall the definition of statistical convergence and its some new generalizations.

Let K be a subset of the set $\mathbb{N}$ of natural numbers. Then, the asymptotic density $\delta(K)$ of K is defined as

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n \ : \ k \in K\}\right|$$

where the vertical bars denote the cardinality of the enclosed set. A number sequence $x = (x_k)$ said to be statistically convergent to the number L if for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \leq n : |x_k - L| > \varepsilon\}$ has asymptotic density zero, that is, $\delta(K(\varepsilon)) = 0$.

Very recently, Aktuğlu [16] has introduced the notion of $\alpha\beta$-statistical convergence for single sequences by using two sequences $\alpha(n)$ and $\beta(n)$ of positive numbers which satisfy the following conditions:

(i) α and β are both non-decreasing sequences;
(ii) $\beta(n) \geqq \alpha(n)$;
(iii) $\beta(n) - \alpha(n) \longrightarrow \infty$ as $n \longrightarrow \infty$.

Let Λ denote the set of pairs (α, β) satisfying (i)-(iii). For $(\alpha, \beta) \in \Lambda$, we say that a sequence $x = (x_k)$ is $\alpha\beta$-statistically convergent to L, if for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\beta(n) - \alpha(n) + 1} \left|\left\{k \in P_n^{\alpha,\beta} : |x_k - L| \geq \epsilon\right\}\right| = 0, \ P_n^{\alpha,\beta} = [\alpha(n), \beta(n)].$$

In recent years, new extensions of several well-known positive linear operators to quantum-calculus have been introduced by various researchers. In particular, the notion of post-quantum calculus, namely the (p, q)-calculus, has recently been used in approximation theory as a fruitful application of quantum variant (see [17]). Recent studies reveal that (p, q)-calculus has provided useful applications for approximation by positive linear operators. For this reason, many researchers have focused on this direction (see [18–23]). For basic definitions and notations of (p, q)-calculus mentioned below the reader is referred to [20] and references therein.

We begin by recalling some certain definitions and notations of (p, q)-calculus:
For any $n \in \mathbb{N}$ and $0 < q < p \leq 1$, the (p, q)-numbers are defined by

$$\begin{aligned} [n]_{p,q} &:= p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} \\ &= \begin{cases} \frac{p^n - q^n}{p-q}, & \text{if } p \neq q \neq 1; \\ n, & \text{if } p = q = 1. \end{cases} \end{aligned}$$

The (p, q)-factorial is defined by

$$[0]_{p,q}! := 1 \text{ and } [n]!_{p,q} = [1]_{p,q}[2]_{p,q} \cdots [n]_{p,q} \text{ if } n \geq 1.$$

The (p, q)-binomial coefficient satisfies

$$\begin{bmatrix} n \\ r \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[r]_{p,q}! \ [n-r]_{p,q}!}, \quad 0 \leq r \leq n.$$

For a non-negative integer n, the (p,q)-Gamma function (see [24]) is defined as

$$\Gamma_{p,q}(x) = (p-q)^{1-x} \prod_{n=0}^{\infty} \frac{p^{n+1}-q^{n+1}}{p^{n+x}-q^{n+x}} = (p-q)^{1-x} \frac{(p;q)_{p,q}^{\infty}}{(p^x;q^x)_{p,q}^{\infty}} \tag{1.1}$$

where

$$(a;b)_{p,q}^{\infty} = \prod_{n=0}^{\infty} (ap^n - bq^n), \ (a,b \in (0,1]).$$

Note that for $p=1$, (p,q)-gamma function turns out to be q-gamma function. Also if n is a nonnegative integer, it is clear that $\Gamma_{p,q}(n+1) = [n]!_{p,q}$ where $[\cdot]!_{p,q}$ is (p,q)-factorial function.

The concept of difference sequence space was firstly introduced by Kızmaz [25]. This idea was further extended by Et and Çolak [26] to the difference sequence of order m for real sequences by defining $\nu(\Delta^m) = \{(x_k) : \Delta^m(x) \in \nu\}$ for $\nu \in \{\ell_\infty, c, c_0\}$, where $m \in \mathbb{N}$, $\Delta^0 x = (x_k), \Delta^m x = (\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1})$ and $\Delta^m x_k = \sum_{i=0}^{m}(-1)^i \binom{m}{i} x_{k+i}$. Recently, Baliarsingh [27] has introduced certain difference sequence spaces based on fractional difference operator involving Euler Gamma function. Also, this operator has been modified by Baliarsingh and Nayak [28]. Very recently, Kadak has introduced weighted statistical convergence using (p,q)-numbers [29]. In the year 2017, based on a generalized difference operator (p,q)-gamma function, the concept of weighted statistical convergence has been extended by Kadak [30]. For more details on difference sequence spaces, see [31–36].

Definition 1 [30]. Let $x = (x_k)$ be any sequence in ω and $0 < q < p \leq 1$. For non-negative integers a, b, c and any constant $h > 0$, the generalized difference sequence is defined by

$$(\Delta_{h,p,q}^{a,b,c} x)_k = \sum_{i=0}^{\infty} (-1)^i \frac{[a]_{p,q}^{\underline{i}} \ [b]_{p,q}^{\underline{i}}}{[i]_{p,q}! \ [c]_{p,q}^{\underline{i}} \ h^{a+b-c}} \ x_{k-i}, \ ([c]_{p,q}^{\underline{i}} \neq 0, \forall i \in \mathbb{N}) \tag{1.2}$$

where $[s]_{p,q}^{\underline{i}}$ is (p,q)-shifted factorial which is being defined (p,q)-gamma function as

$$[s]_{p,q}^{\underline{i}} := \begin{cases} 1, & (s=0 \text{ or } i=0), \\ \frac{\Gamma_{p,q}(s+1)}{\Gamma_{p,q}(s-i+1)} = [s]_{p,q}[s-1]_{p,q}[s-2]_{p,q} \ldots [s-i+1]_{p,q}, & (s \in \mathbb{N}). \end{cases}$$

It should be noted that the condition $[c]_{p,q}^{\underline{i}} \neq 0$ can not be removed from (1.2). Throughout the text, we assume that the series defined in (1.2) converges for all $a+b > c$ and $[c]_{p,q}^{\underline{i}} \neq 0$ for all $i \in \mathbb{N}$.

Example 1 [30]. Define the sequence (x_k) by $x_k = k$ for all $k \in \mathbb{N}$. Although the sequence (x_k) is not convergent, ordinary difference sequence $\{\Delta^3(x_k)\}$ is convergent i.e.

$$\Delta^3(x_k) = x_k - 3x_{k-1} + 3x_{k-2} - x_{k-3} \to 0 \text{ as } k \to \infty.$$

For $a = 3$, $b = c$ and $h = 1$, we find that

$$\begin{aligned}(\Delta^{3,b,b}_{1,p,q}x)_k &= \sum_{i=0}^{\infty} (-1)^i \frac{[3]^{\underline{i}}_{p,q}}{i!}\ x_{k-i} \\ &= x_k - [3]^{\underline{1}}_{p,q}x_{k-1} + \frac{[3]^{\underline{2}}_{p,q}}{[2]_{p,q}!}x_{k-2} + \frac{[3]^{\underline{3}}_{p,q}}{[3]_{p,q}!}x_{k-3} + \cdots \\ &= 3 - (p^2 + pq + q^2).\end{aligned}$$

Obviously, the difference sequence $\{\Delta^{[3]}_{p,q}(x_k)\}$ has different limits based on the values p and q $(0 < q < p \leq 1)$. The reason belongs to the definition of (p, q)-numbers. For this reason, in order to obtain a convergence result for the different operators defined in (1.2), we replace $p = (p_k)$ and $q = (q_k)$ such that

$$\lim_{k\to\infty} p_k = P \quad \text{and} \quad \lim_{k\to\infty} q_k = Q \quad \text{where} \quad P, Q \in (0, 1]. \tag{1.3}$$

For example, if we take $q_k = \left(\frac{k}{k+n}\right) < \left(\frac{k}{k+m}\right) = p_k$ such that $0 < q_k < p_k \leq 1$ for $n > m > 0$. It is clear that $P = Q = 1$ and hence

$$\lim_{k\to\infty} (\Delta^{3,b,b}_{1,p,q}x)_k = \lim_{k\to\infty} \left\{3 - (p_k^2 + p_kq_k + q_k^2)\right\} = 0.$$

Remark 1. We remark that if we take $a = 1$, $b = c$, $h = 1$ and $P = Q = 1$, the difference operator $\Delta^{1,b,b}_{1,p,q}$ is reduced to the difference operator $\Delta^{(1)}$ (see [25]). For the case $a = m \in \mathbb{N}$, $b = c$, $h = 1$, then $\Delta^{a,b,c}_{h,p,q}$ can be represented by Δ^m (see [26]) for $P = Q = 1$. Furthermore, taking $a = m \in \mathbb{N}$, the difference operator $\Delta^{a.b,c}_{1,p,q}$ is reduced to the difference operator $\Delta^{[m]}_{p,q}$ involving (p, q)-integers introduced in [29].

Our main focus of the present study is to extend the concepts of almost (statistical) convergence and its strong versions by using the difference operator involving (p, q)-gamma function and an increasing sequence (λ_n) of positive numbers. In fact, newly proposed methods (or new kinds of summability methods) which will actually provide many new interesting and useful results based on the choice of the values a, b, c, h as well as the sequences $(p_k), (q_k), (\lambda_k), \alpha(k)$ and $\beta(k)$. These methods are not only generalize the earlier works but also present some new perspectives regarding the development of almost convergence and its statistical versions. We also establish some important approximation results associating with statistically almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergence and investigate Korovkin and Voronovskaja type approximation results for functions of two variables by the help of bivariate non-tensor type Meyer–König and Zeller operator. Moreover, we estimate the rate of statistical almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergence by means of the modulus of continuity. We also present geometrical interpretations to illustrate some of our approximation results in this paper.

This study is organized as follows.

In Sect. 2, some new concepts of almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistical convergence, statistical almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergence and strong almost $[\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)]_r$-convergence are introduced and some important inclusion relations between newly proposed methods are presented with corresponding counterexamples. In addition, to show the effectiveness of these methods some special cases are presented in this section. Section 3 is devoted to the Korovkin type approximation theorem for functions of two variables through statistical almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergence. As an application to this theorem, an illustrative example is given using bivariate non-tensor type Meyer–König and Zeller generalization of Bernstein power series. In Sect. 4, the rate of statistical almost convergence of approximating linear operators by means of modulus of continuity is estimated. In Sect. 5, a Voronovskaja type approximation theorem is derived by using statistical almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergence. In the final section of this article, we present some computational and geometrical approaches to illustrate some of our approximation results.

2. Some New Definitions and Inclusion Relations

In this section, we first define the summation $t^{\alpha,\beta}_{mn}(\Delta^{a,b,c}_{h,p,q}(\lambda x))$ involving the difference operator $\Delta^{a,b,c}_{h,p,q}$ where (λ_n) is an increasing sequence of positive numbers. Secondly, we give the definitions of almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistical convergence, statistical almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergence and strong almost $[\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)]_r$-convergence $(0 < r < \infty)$ and also present counterexamples related to these methods. Additionally, we state and prove some theorems involving some inclusion relations.

Let h be any positive constant, $(\alpha,\beta) \in \Lambda$ and $a,b,c \in \mathbb{N}$. Let $p=(p_k)$ and $q=(q_k)$ be two real valued sequences such that $0<q_k<p_k\le 1$ for all $k\in\mathbb{N}$ and $\lim_k q_k = Q$ and $\lim_k p_k = P$ where $P,Q\in(0,1]$. Assume further that $(\lambda_k)_{k=0}^{\infty}$ is a given strictly increasing sequence of positive numbers such that

$$|(\Delta^{a,b,c}_{h,p,q}\lambda)_k)| \ge 0 \quad ([c]^{\underline{i}}_{p,q} \neq 0, \forall i \in \mathbb{N}) \quad \text{and} \quad \lim_{k\to\infty}\lambda_k = \infty.$$

For a sequence $x=(x_k)$ of real or complex number, we define the following sum involving the difference operator $\Delta^{a,b,c}_{h,p,q}(\lambda x)$ as follows:

$$t^{\alpha,\beta}_{mn}(\Delta^{a,b,c}_{h,p,q}(\lambda x)) = \frac{1}{P_m^{(\alpha,\beta)}} \sum_{k=\alpha(m)+n}^{\beta(m)+n} \Delta^{a,b,c}_{h,p,q}(\lambda_k x_k),\ t^{\alpha,\beta}_{-1,n}=0$$

where $P_m^{(\alpha,\beta)} = \beta(m)-\alpha(m)+1$. Here

$$\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k) = \sum_{i=0}^{\infty} \frac{(-1)^i}{[i]_{p,q}!} \frac{[a]^{\underline{i}}_{p,q}\ [b]^{\underline{i}}_{p,q}}{[c]^{\underline{i}}_{p,q}\ h^{a+b-c}}\ \lambda_{k-i}x_{k-i} \tag{2.1}$$

where $\lambda_{-k}=0$ and $[c]_{p,q}^{\underline{i}}\neq 0$ for all $k,i\in\mathbb{N}$. Equivalently, we write

$$
\begin{aligned}
&t_{mn}^{\alpha,\beta}\left(\Delta_{h,p,q}^{a,b,c}(\lambda x)\right)\\
&=\frac{1}{P_m^{(\alpha,\beta)}h^{a+b-c}}\sum_{k=\alpha(m)}^{\beta(m)}\Bigg\{\lambda_{n+k}x_{n+k}-\frac{[a]_{p,q}[b]_{p,q}}{[c]_{p,q}}\lambda_{n+k-1}x_{n+k-1}\\
&\quad+\frac{[a]_{p,q}[a-1]_{p,q}[b]_{p,q}[b-1]_{p,q}}{[c]_{p,q}[c-1]_{p,q}[2]_{p,q}!}\lambda_{n+k-2}x_{n+k-2}\\
&\quad-\frac{[a]_{p,q}[a-1]_{p,q}[a-2]_{p,q}[b]_{p,q}[b-1]_{p,q}[b-2]_{p,q}}{[c]_{p,q}[c-1]_{p,q}[c-2]_{p,q}[3]_{p,q}!}\lambda_{n+k-3}x_{n+k-3}\\
&\quad+\cdots+(-1)^r\,\frac{[a]_{p,q}\cdots[a-r+1]_{p,q}[b]_{p,q}\cdots[b-r+1]_{p,q}}{[c]_{p,q}\cdots[c-r+1]_{p,q}[r]_{p,q}!}\lambda_{n+k-r}x_{n+k-r}+\cdots\Bigg\}\\
&=\frac{1}{P_m^{(\alpha,\beta)}h^{a+b-c}}\sum_{k=\alpha(m)}^{\beta(m)}\Bigg\{\lambda_{n+k}x_{n+k}-\frac{(p^a-q^a)(p^b-q^b)}{(p^c-q^c)\ (p-q)}\lambda_{n+k-1}x_{n+k-1}\\
&\quad+\frac{(p^a-q^a)(p^{a-1}-q^{a-1})(p^b-q^b)(p^{b-1}-q^{b-1})}{(p^c-q^c)\ (p^{c-1}-q^{c-1})(p-q)^2(p+q)}\lambda_{n+k-2}x_{n+k-2}-\cdots\\
&\quad+(-1)^r\frac{(p^a-q^a)\cdots(p^{a-r+1}-q^{a-r+1})(p^b-q^b)\cdots(p^{b-r+1}-q^{b-r+1})}{(p^c-q^c)\cdots(p^{c-r+1}-q^{c-r+1})(p-q)^r[r]_{p,q}!}(\lambda x)_{n+k-r}+\cdots\Bigg\}.
\end{aligned}
$$

Throughout the text, we assume that the series defined in (2.1) converges for all $c<a+b$ and $[c]_{p,q}^{\underline{i}}\neq 0$ for all $i\in\mathbb{N}$.

Definition 2. Let $(\alpha,\beta)\in\Lambda$, $h>0$ and $a,b,c\in\mathbb{N}$. Also let $p=(p_k)$ and $q=(q_k)$ be two real valued sequences satisfying (1.3).

(i) A sequence $x=(x_k)\in\ell_\infty$ is said to be almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-statistically convergent to a number L, if for every $\epsilon>0$,

$$\lim_{m\to\infty}\frac{1}{P_m^{(\alpha,\beta)}}\left|\left\{\alpha(m)\le k\le\beta(m):|\Delta_{h,p,q}^{a,b,c}(\lambda_{n+k}x_{n+k})-L|\ge\epsilon\right\}\right|=0,\ \text{uniformly in } n.$$

In this case we write $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)(stat)\text{-}\lim x=L$.

(ii) A sequence $x=(x_k)\in\ell_\infty$ is said to be statistically almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-convergent to a number L, if for every $\epsilon>0$,

$$\lim_{m\to\infty}\frac{1}{m}\left|\left\{k\le m:|t_{kn}^{\alpha,\beta}\left(\Delta_{h,p,q}^{a,b,c}(\lambda x)\right)-L|\ge\epsilon\right\}\right|=0,\ \text{uniformly in } n.$$

In this case we write $F_{\Delta,\lambda}^{\alpha,\beta}(stat)\text{-}\lim x=L$.

(iii) A sequence $x=(x_k)\in\ell_\infty$ is said to be strongly almost $[\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)]_r$-convergent $(0<r<\infty)$ to the limit L, if

$$\lim_{m\to\infty}\frac{1}{P_m^{(\alpha,\beta)}}\sum_{k=\alpha(m)}^{\beta(m)}|\Delta_{h,p,q}^{a,b,c}(\lambda_{n+k}x_{n+k})-L|^r=0,\ \text{uniformly in } n,$$

and we write this as $[\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)]_r\text{-}\lim x=L$. In this case L is called the strongly almost $[\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)]_r$- limit of x.

Remark 2. Let $(\alpha, \beta) \in \Lambda$, $h > 0$ and $a, b, c \in \mathbb{N}$. Let $p = (p_k)$ and $q = (q_k)$ be two real valued sequences satisfying (1.3).

(i) If x is almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistically convergent then x is $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistically convergent.

(ii) $\Delta^{a,b,c}_{\alpha,\beta,h}$-statistical convergence implies statistical almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergence, and hence by (i) we get that almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistical convergence $\Rightarrow$ $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistical convergence $\Rightarrow$ statistical almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergence.

(iii) Almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergence implies statistical almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergence but not almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistical convergence.

The following examples give the relations between almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistical convergence, statistical almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergence and $\Delta^{a,b,c}_{\alpha,\beta,h}$-statistical convergence for sequences.

Example 2. Consider the case $a = 1$, $b = c$, $h = 1$, $\alpha(m) = 1$, $\beta(m) = m$ and $\lambda_k = k^2$ for all $k, m \in \mathbb{N}$. Define the sequence (x_k) by

$$x_k = \begin{cases} 1 & (k = F_i) \\ \frac{1}{k^2} & (\text{otherwise}) \end{cases}$$

where $(F_i)_{i \geq 0}$ be the sequence of Fibonacci numbers given by $F_0 = 1, F_1 = 1$ and $F_{i+2} = F_{i+1} + F_i$ for all $i \geq 0$. In this case, it is verified that (x_k) is bounded and divergent. Obviously, (x_k) is almost $\Delta^{[1,b,b]}_{1,1,m}(\lambda)$-statistically convergent to 0. Therefore from Remark (ii) it is $\Delta^{[1,b,b]}_{1,1,m}(\lambda)$-statistically convergent and statistically almost $\Delta^{[1,b,b]}_{1,1,m}(\lambda)$-convergent.

Example 3. Let us take the sequence $x = (x_k)$ which is defined by

$$x_k = \begin{cases} \frac{1}{m^4} & , k = m^3 - m^2, m^3 - m^2 + 1, \ldots, m^3 - 1, \ m = 2, 3, 4, \ldots; \\ -\frac{1}{m^4} & , k = m^3, \ m = 2, 3, 4, \ldots; \\ 0 & , \text{otherwise}. \end{cases}$$

In addition, we are also assuming that $a = 1$, $b = c$, $h = 1$, $\alpha(m) = 1$, $\beta(m) = m$ and $\lambda_k = k$ for all $k, m \in \mathbb{N}$. Under these conditions,we get

$$t^{\alpha,\beta}_{mn}(\Delta^{1,b,b}_{h,p,q}(\lambda x)) = \frac{1}{m} \sum_{k=n+1}^{n+m} \Delta^{1,b,b}_{1,p,q}(\lambda_k \ x_k) = \frac{1}{m} \big[(n+m) \ x_{n+m} - n \ x_n\big].$$

Hence $x = (x_k)$ is statistically almost $\Delta^{[1,b,b]}_{1,1,m}(\lambda)$-convergent to 0 but it is neither $\Delta^{[1,b,b]}_{1,1,m}(\lambda)$-statistically convergent nor almost $\Delta^{[1,b,b]}_{1,1,m}(\lambda)$-statistically convergent.

Example 4. Let $H = \cup_{k=1}^{\infty}\{5^{k^2}+1, 5^{k^2}+2, \ldots, 5^{k^2}+k\}$ and $\widetilde{H} = \cup_{k=1}^{\infty}\{5^{k^2}\}$. Let us consider the case when $h = 1$, $\alpha(m) = 1$ and $\beta(m) = m$ for all $m \in \mathbb{N}$. Let $x = (x_k)$ be a sequence and consider $(y_k) = \Delta_{1,1,m}^{a,b,c}(\lambda_k x_k)$ defined by

$$y_k = \begin{cases} 1 & , (k \in H) \\ -1 & , (k \in \widetilde{H}) \\ 0 & , (otherwise) \end{cases} , \ k = 1, 2, 3, \ldots .$$

Then, it is easy to verify that (y_k) is divergent and $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-statistically convergent. In fact, its $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-statistical limit is 0. On the other hand it is not almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-statistically convergent.

Now, we shall give the following special cases to show the effectiveness of proposed method.

- Let us take $a = 0$, $b = c$, $h = 1$, $\alpha(m) = 1$, $\beta(m) = m$ and $\lambda_k = 1$ for all $k, m \in \mathbb{N}$, i.e. $P_m^{(\alpha,\beta)} = m$ and $\Delta_{1,p,q}^{0,b,b}(\lambda_k x_k) = x_k$, the almost $\Delta_{1,1,m}^{[0,b,b]}(\lambda)$-statistical convergence given in Definition 2 (i) is reduced to ordinary almost statistical convergence [2]. Based upon the above choices, we obtain the notion of almost (statistical) convergence for sequences (see [13]). If we take $a, b, c \in \mathbb{N}$, $P, Q \in (0, 1]$, $h = 1$, then these convergences reduce almost (statistical) convergence of difference sequences (cf.[36,37]).
- Let (ν_n) be a strictly increasing sequence of positive numbers tending to infinity as $n \to \infty$ such that $\nu_{n+1} \leq \nu_n + 1$ for $\nu_1 = 1$. Also let us take $a = 0$, $b = c$, $h = 1$, $\alpha(m) = m - \nu_m + 1, \beta(m) = m$ and $\lambda_k = 1$ for all $k, m \in \mathbb{N}$, almost $\Delta_{1,\alpha,\beta}^{[0,b,b]}(\lambda)$-statistical convergence is reduced to almost ν-statistical convergence in [14]. Similarly, in view of Definition 2 (ii), we have the notion of statistical almost ν-convergence.
- Let $\theta = (k_n)$ be the sequence of positive integers such that $k_0 = 0$, $0 < k_n < k_{n+1}$ and $h_n = (k_n - k_{n-1}) \to \infty$ as $n \to \infty$. Then θ is called a lacunary sequence. Taking $a = 0$, $b = c$, $h = 1$, $\alpha(m) = k_{m-1}+1, \beta(m) = k_m$ and $\lambda_k = 1$ for all $k, m \in \mathbb{N}$, almost $\Delta_{1,\alpha,\beta}^{[0,b,b]}(\lambda)$-statistical convergence is reduced to almost lacunary statistical convergence, (cf. [15]).

We prove here some relations between almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-statistical convergence, statistical almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-convergence and strong almost $[\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)]_r$-convergence.

In our first theorem we establish the relation between our two newly defined concepts of almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-statistical convergence and statistical almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-convergence.

Theorem 3. *Let $(\alpha,\beta)\in\Lambda$, $h>0$ and $a,b,c\in\mathbb{N}$. Also let $p=(p_k)$ and $q=(q_k)$ be two real valued sequences such that $0<q_k<p_k\leq 1$ with the property* (1.3). *Let*

$$|\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k)-L|\leq M \ (k\in\mathbb{N}).$$

If a sequence $x=(x_k)$ is almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistically convergent to L then it is statistically almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergent to L but not conversely.

Proof. Let $|\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k)-L|\leq M$ for all $k\in\mathbb{N}$. Let us set

$$K(\epsilon;n):=\left\{\alpha(m)+n\leq k\leq\beta(m)+n : |\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k)-L|\geq\epsilon\right\}$$

and

$$K^C(\epsilon;n):=\left\{\alpha(m)+n\leq k\leq\beta(m)+n : |\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k)-L|<\epsilon\right\}.$$

By the hypothesis we know that

$$\lim_{m\to\infty}\frac{1}{P_m^{(\alpha,\beta)}}|K(\epsilon;n)|=0,\ \text{uniformly in } n.$$

We thus find that

$$\begin{aligned}
&|t^{\alpha,\beta}_{mn}(\Delta^{a,b,c}_{h,p,q}(\lambda x))-L|\\
&=\left|\frac{1}{P_m^{(\alpha,\beta)}}\sum_{k=\alpha(m)+n}^{\beta(m)+n}\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k)-L\right|\\
&\leq\left|\frac{1}{P_m^{(\alpha,\beta)}}\sum_{k=\alpha(m)+n}^{\beta(m)+n}(\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k)-L)\right|+\left|\frac{1}{P_m^{(\alpha,\beta)}}\sum_{k=\alpha(m)+n}^{\beta(m)+n}L-L\right|\\
&\leq\frac{1}{P_m^{(\alpha,\beta)}}\sum_{\substack{k=\alpha(m)+n\\(k\in K(\epsilon;n))}}^{\beta(m)+n}|\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k)-L|+\frac{1}{P_m^{(\alpha,\beta)}}\sum_{\substack{k=\alpha(m)+n\\(k\in K^C(\epsilon;n))}}^{\beta(m)+n}|\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k)-L|\\
&\leq\frac{M}{P_m^{(\alpha,\beta)}}|K(\epsilon;n)|+\frac{\epsilon}{P_m^{(\alpha,\beta)}}\left|K^C(\epsilon;n)\right|\to 0+\epsilon\cdot 1=\epsilon,\ \text{uniformly in } n\,(m\to\infty).
\end{aligned}$$

Hence the sequence $x=(x_k)$ is almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergent to L, which implies $F^{\alpha,\beta}_{\Delta,\lambda}(stat)\text{-}\lim x=L$. That is, the sequence $x=(x_k)$ is statistically almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergent to L.

For the converse, consider the case $a=1$, $b=c$, $h=1$, $\alpha(m)=1$, $\beta(m)=m$ and $\lambda_k=k$ for all $k,m\in\mathbb{N}$ and the sequence $x=(x_k)$ is defined as in Example 3. Of course this sequence is not almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistically convergent. On the other hand x is almost $\Delta^{[1,b,b]}_{1,1,m}(\lambda)$-convergent to 0 and hence statistically almost $\Delta^{[1,b,b]}_{1,1,m}(\lambda)$-convergent to 0.

This completes the proof of the theorem. □

The following theorem gives the relation between almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistical convergence and strongly almost $[\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)]_r$-convergence $(0 < r < \infty)$.

Theorem 4. *Let $(\alpha,\beta) \in \Lambda$, $h > 0$ and $a, b, c \in \mathbb{N}$. Also let $p = (p_k)$ and $q = (q_k)$ be two real valued sequences such that $0 < q_k < p_k \leq 1$ with the property (1.3).*
(a) Let us suppose that the sequence $x = (x_k)$ is strongly almost $[\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)]_r$-convergent to the limit L. If

$$r \in (0,1) \text{ and } 0 \leqq |\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k) - L| < 1 \text{ or}$$
$$r \in [1,\infty) \text{ and } 1 \leqq |\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k) - L| < \infty$$

for all $k \in \mathbb{N}$, then $x = (x_k)$ is almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistically convergent to L.
(b) Assume that $x = (x_k)$ is almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistically convergent to L and let

$$|\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k) - L| \leq M$$

for all $k \in \mathbb{N}$. If

$$r \in (0,1] \text{ and } M \in [1,\infty) \text{ or } r \in [1,\infty) \text{ and } M \in [0,1),$$

then the sequence $x = (x_k)$ is strongly almost $[\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)]_r$-convergent to L.

Proof. (a) Suppose that $x = (x_k)$ is strongly almost $[\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)]_r$-convergent $(0 < r < \infty)$ to L. Under the above conditions, as $m \to \infty$, we have

$$\begin{aligned} 0 \leftarrow\ & \frac{1}{P^{(\alpha,\beta)}_m} \sum_{k=\alpha(m)+n}^{\beta(m)+n} \left|\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k) - L\right|^r \\ \geq\ & \frac{1}{P^{(\alpha,\beta)}_m} \sum_{\substack{k=\alpha(m)+n \\ (k \in K(\epsilon;n))}}^{\beta(m)+n} \left|\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k) - L\right|^r \\ \geq\ & \frac{1}{P^{(\alpha,\beta)}_m} \sum_{\substack{k=\alpha(m)+n \\ (k \in K(\epsilon;n))}}^{\beta(m)+n} \left|\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k) - L\right| \\ \geq\ & \frac{1}{P^{(\alpha,\beta)}_m} \sum_{\substack{k=\alpha(m)+n \\ (k \in K(\epsilon;n))}}^{\beta(m)+n} \epsilon = \frac{\epsilon}{P^{(\alpha,\beta)}_m} |K(\epsilon;n)| \quad \text{(uniformly in } n\text{)}. \end{aligned}$$

Hence, the sequence $x = (x_k)$ is almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistically convergent to L.

(b) Let us suppose that $x = (x_k)$ is almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-statistically convergent to L. Also we get, for all $k \in \mathbb{N}$ satisfying $|\Delta^{a,b,c}_{h,p,q}(\lambda_k x_k) - L| \leq M$, that

$$\begin{aligned}\frac{1}{P_m^{(\alpha,\beta)}}\sum_{k=\alpha(m)+n}^{\beta(m)+n}\left|\Delta_{h,p,q}^{a,b,c}(\lambda_k x_k)-L\right|^r &= \frac{1}{P_m^{(\alpha,\beta)}}\sum_{\substack{k=\alpha(m)+n\\(k\in K(\epsilon;n))}}^{\beta(m)+n}\left|\Delta_{h,p,q}^{a,b,c}(\lambda_k x_k)-L\right|^r\\ &\quad+\frac{1}{P_m^{(\alpha,\beta)}}\sum_{\substack{k=\alpha(m)+n\\(k\in K^C(\epsilon;n))}}^{\beta(m)+n}\left|\Delta_{h,p,q}^{a,b,c}(\lambda_k x_k)-L\right|^r\\ &= S_1(m,n)+S_2(m,n)\end{aligned}$$

where

$$S_1(m,n)=\frac{1}{P_m^{(\alpha,\beta)}}\sum_{\substack{k=\alpha(m)+n\\(k\in K(\epsilon;n))}}^{\beta(m)+n}\left|\Delta_{h,p,q}^{a,b,c}(\lambda_k x_k)-L\right|^r$$

and

$$S_2(m,n)=\frac{1}{P_m^{(\alpha,\beta)}}\sum_{\substack{k=\alpha(m)+n\\(k\in K^C(\epsilon;n))}}^{\beta(m)+n}\left|\Delta_{h,p,q}^{a,b,c}(\lambda_k x_k)-L\right|^r.$$

Now if $k\in K^C(\epsilon;n)$, then

$$\begin{aligned}S_2(m,n)&=\frac{1}{P_m^{(\alpha,\beta)}}\sum_{\substack{k=\alpha(m)+n\\(k\in K^C(\epsilon;n))}}^{\beta(m)+n}\left|\Delta_{h,p,q}^{a,b,c}(\lambda_k x_k)-L\right|^r\\ &\le\frac{1}{P_m^{(\alpha,\beta)}}\sum_{\substack{k=\alpha(m)+n\\(k\in K^C(\epsilon;n))}}^{\beta(m)+n}\left|\Delta_{h,p,q}^{a,b,c}(\lambda_k x_k)-L\right|=\frac{\epsilon}{P_m^{(\alpha,\beta)}}|K^C(\epsilon;n)|.\end{aligned}$$

Also, for $k\in K(\epsilon;n)$, we get

$$S_1(m,n)=\frac{1}{P_m^{(\alpha,\beta)}}\sum_{\substack{k=\alpha(m)+n\\(k\in K(\epsilon;n))}}^{\beta(m)+n}\left|\Delta_{h,p,q}^{a,b,c}(\lambda_k x_k)-L\right|^r\le M\frac{1}{P_m^{(\alpha,\beta)}}|K(\epsilon;n)|.$$

Obviously, $\{S_1(m,n)+S_2(m,n)\}\to 0$ uniformly in n, as $m\to\infty$. Therefore, the sequence $x=(x_k)$ is strongly almost $[\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)]_r$-convergent to L for $0<r<\infty$.

This step completes the proof. □

In the next result we characterize statistically almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-convergent sequences by means of the almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-convergence of subsequences.

Theorem 5. *Let $h > 0$ be any constant, $(\alpha, \beta) \in \Lambda$ and $a, b, c \in \mathbb{N}$. Also let $p = (p_k)$ and $q = (q_k)$ be two real valued sequences such that $0 < q_k < p_k \leq 1$ with the property (1.3). A sequence $x = (x_k)$ is statistically almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergent to the number L if and only if there exists a set $K = \{k_1 < k_2 < \cdots < k_m < \cdots\} \subseteq \mathbb{N}$ such that*

$$\delta(K) = 1 \text{ and } \lim_{m\to\infty} t^{\alpha,\beta}_{k_m,n}(\Delta^{a,b,c}_{h,p,q}(\lambda x)) = L, \text{ uniformly in } n,$$

where $\delta(K)$ denotes the natural density of the set $K \subset \mathbb{N}$.

Proof. Let us suppose that there exists a set $K = \{k_1 < k_2 < \cdots < k_m < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and (x_{k_m}) is almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergent to L with respect to the subsequence (λ_{k_m}) of (λ_k). Then there is a positive integer N such that, for every $m > N(\epsilon)$, we get

$$\frac{1}{\beta(k_m) - \alpha(k_m) + 1} \sum_{i=\alpha(k_m)+n}^{\beta(k_m)+n} \left|\Delta^{a,b,c}_{h,p,q}(\lambda_i x_i) - L\right| < \epsilon \text{ for each } n \in \mathbb{N}.$$

Let us set

$E_\epsilon = \{m : \ m \in \mathbb{N} \text{ and } \left|t^{\alpha,\beta}_{k_m,n}(\Delta^{a,b,c}_{h,p,q}(\lambda x)) - L\right| \geq \epsilon\}$ and $E'_\epsilon = \{k_{N+1}, k_{N+2}, \ldots\}$.

Then, $\delta(E'_\epsilon) = 1$ and $E_\epsilon \subset \mathbb{N} \setminus E'_\epsilon$ which yields that $\delta(E_\epsilon) = 0$, uniformly in n. Hence, the sequence $x = (x_k)$ is statistically almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergent to L.

Conversely, let $x = (x_k)$ be statistically almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergent to the number L. For $\nu = \{1, 2, 3, \ldots\}$, put

$$K_\nu = \{j \in \mathbb{N} : \left|t^{\alpha,\beta}_{k_j,n}(\Delta^{a,b,c}_{h,p,q}(\lambda x)) - L\right| \geq 1/\nu\}$$

and

$$M_\nu = \{j \in \mathbb{N} : \left|t^{\alpha,\beta}_{k_j,n}(\Delta^{a,b,c}_{h,p,q}(\lambda x)) - L\right| < 1/\nu\}.$$

Then

$$\delta(K_\nu) = 0, \ M_1 \supseteq M_2 \supseteq \cdots \supseteq M_i \supseteq M_{i+1} \supseteq \cdots \text{ and } \delta(M_\nu) = 1 \ (\nu \in \mathbb{N}). \tag{2.2}$$

We will now show that (x_{k_j}) is almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergent to L for $j \in M_\nu$. Suppose that the subsequence (x_{k_j}) is not almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergent to L. Therefore, there exists $\epsilon > 0$ such that

$$\left|t^{\alpha,\beta}_{k_j,n}(\Delta^{a,b,c}_{h,p,q}(\lambda x)) - L\right| \geq \epsilon, \ (n \in \mathbb{N})$$

for infinitely many terms. Let

$$M(\epsilon; n) = \{j \in \mathbb{N} : \left|t^{\alpha,\beta}_{k_j,n}(\Delta^{a,b,c}_{h,p,q}(\lambda x)) - L\right| < \epsilon\} \text{ and } \epsilon > 1/\nu \ (\nu \in \mathbb{N}).$$

Then

$$\delta(M(\epsilon; n)) = 0, \text{ uniformly in } n$$

and by (2.2), $M_\nu \subset M(\epsilon; n)$. Hence $\delta(M_\nu) = 0$, which contradicts $\delta(M_\nu) = 1$ and therefore (x_{k_j}) is almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergent to L. □

3. A Korovkin-Type Approximation Theorem

At the beginning of the 1950s, the study of some particular approximation by means of the positive linear operators was extended to general approximation sequences of such operators. The foundation of the theory of approximation by positive linear operators or functionals was constructed by Bohman [38] and Korovkin [39]. It is known that the theory of approximation by positive linear operators is an important research field in mathematics i.e. functional analysis, measure theory, real analysis, harmonic analysis, probability theory, summability theory, and partial differential equations. Even today, the development of this type approximation is far from complete. First, Korovkin established the necessary and sufficient conditions for the uniform convergence of a sequence of positive linear operators acting on $C[a, b]$ based on some test functions. For example the test functions $\{1, e^{-x}, e^{-2x}\}$ are used in logarithmic version and $\{1, \sin x, \cos x\}$ are also used trigonometric form of Korovkin type theorem. For statistical and other versions of Korovkin type approximation theorems, we refer the reader to [40–43].

In this section, we prove a Korovkin type approximation theorem for functions of two variables through statistically almost $\Delta^{[a,b,c]}_{h,\alpha,\beta}(\lambda)$-convergence with the test functions given by

$$f_0(s,t) = 1, \quad f_1(s,t) = \frac{s}{1-s-t}, \quad f_2(s,t) = \frac{t}{1-s-t}$$
$$\text{and} \qquad f_3(s,t) = \left(\frac{s}{1-s-t}\right)^2 + \left(\frac{t}{1-s-t}\right)^2. \tag{3.1}$$

Note that, in Remark 7, we explain the effects of above test functions and the motivations of using this Korovkin set.

By $C(S_A)$, we denote the space of all continuous real valued functions on a fixed compact subset S_A of $\mathbb{R}^2$ defined by

$$S_A = \left\{(x,y) \in \mathbb{R}^2 : x \in [0, A], y \in [0, A-x], 0 < A \le 1/2\right\}$$

and equipped with the following norm:

$$\|f\|_{C(S_A)} = \sup_{(x,y)\in S_A} |f(x,y)| \ (f \in C(S_A)).$$

Then as usual, we say that T is positive linear operator provided that $f \ge 0$ implies $Tf \ge 0$. Also, we use the notation $T(f(s,t); x, y)$ for the value of Tf at a point $(x, y) \in S_A$.

Theorem 6. *Let h be any positive constant, $a, b, c \in \mathbb{N}$ and $(\alpha, \beta) \in \Lambda$, and let $p = (p_k)$, $q = (q_k)$ be two real valued sequences satisfying (1.3). Also let*

$\{L_k\}_{k\geq 1}$ *be a sequence of positive linear operators from* $C(S_A)$ *into itself. Then,*

$$F^{\alpha,\beta}_{\Delta,\lambda}(stat)\text{-}\lim_{m\to\infty} \|L_k(f;x,y) - f\|_{C(S_A)} = 0, \text{ for all } f \in C(S_A) \tag{3.2}$$

if and only if

$$F^{\alpha,\beta}_{\Delta,\lambda}(stat)\text{-}\lim_{m\to\infty} \|L_k(f_i;x,y) - f_i\|_{C(S_A)} = 0, \quad \text{for each } i = 0,1,2,3. \tag{3.3}$$

Proof. The implication (3.2) $\Rightarrow$ (3.3) is obvious by taking into account that each f_i belongs to $C(S_A)$ where $i = 0,1,2,3$. Let us take $f \in C(S_A)$ and $(x,y) \in S_A$ be fixed. Then, there exists a constant $M > 0$ such that $|f(x,y)| \leq M$ for all $(x,y) \in S_A$. Hence, we get

$$|f(s,t) - f(x,y)| \leq 2M \quad ((s,t),(x,y) \in S_A). \tag{3.4}$$

Since f is a continuous function on S_A, for a given $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that

$$|f(s,t) - f(x,y)| < \epsilon \tag{3.5}$$

whenever

$$\left|\frac{s}{1-s-t} - \frac{x}{1-x-y}\right| < \delta \text{ and } \left|\frac{t}{1-s-t} - \frac{y}{1-x-y}\right| < \delta.$$

Also we obtain for all $(s,t),(x,y) \in S_A$ satisfying

$$\sqrt{\left(\frac{s}{1-s-t} - \frac{x}{1-x-y}\right)^2 + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y}\right)^2} \geq \delta$$

that

$$|f(s,t) - f(x,y)| \leq \frac{2M}{\delta^2}\left(\psi^2(s;x) + \psi^2(t;y)\right) \tag{3.6}$$

where

$$\psi(s;x) = \frac{s}{1-s-t} - \frac{x}{1-x-y} \text{ and } \psi(t;y) = \frac{t}{1-s-t} - \frac{y}{1-x-y}.$$

Now, from the relations (3.4) to (3.6), we get for all $(x,y),(s,t) \in S_A$ and $f \in C(S_A)$ that

$$|f(s,t) - f(x,y)| < \epsilon + \frac{2M}{\delta^2}\left(\psi^2(s;x) + \psi^2(t;y)\right). \tag{3.7}$$

Using the linearity and positivity of L_k, we obtain, from (3.7), that

$$\begin{aligned}
&\left|L_k(f(s,t);x,y) - f(x,y)\right| \\
&\quad = \left|L_k(f(s,t) - f(x,y);x,y) + f(x,y)[L_k(f_0;x,y) - f_0(x,y)]\right| \\
&\quad \leqq L_k\big(|f(s,t) - f(x,y)|;x,y\big) + M\ |L_k(f_0;x,y) - f_0(x,y)| \\
&\quad \leqq \left|L_k\left(\varepsilon + \frac{2M}{\delta^2}(\psi^2(s;x) + \psi^2(t;y);x,y\right)\right| + M\ |L_k(f_0;x,y) - f_0(x,y)| \\
&\quad \leq \epsilon + (\epsilon + M)\ \left|L_k(f_0;x,y) - f_0(x,y)\right|
\end{aligned}$$

$$
\begin{aligned}
&- \frac{4M}{\delta^2}\left(\frac{x}{1-x-y}\right)\left|L_k(f_1;x,y)-f_1(x,y)\right| \\
&- \frac{4M}{\delta^2}\left(\frac{y}{1-x-y}\right)\left|L_k(f_2;x,y)-f_2(x,y)\right| + \frac{2M}{\delta^2}\left|L_k(f_3;x,y)-f_3(x,y)\right| \\
&+ \frac{2M}{\delta^2}\left(\left(\frac{x}{1-x-y}\right)^2+\left(\frac{y}{1-x-y}\right)^2\right)\left|L_k(f_0;x,y)-f_0(x,y)\right| \\
\leqq{}& \varepsilon + \left(\varepsilon + M + \frac{4M}{\delta^2}\right)\left|L_k(f_0;x,y)-f_0\right| + \frac{4M}{\delta^2}\left|L_k(f_1;x,y)-f_1(x,y)\right| \\
&+ \frac{4M}{\delta^2}\left|L_k(f_2;x,y)-f_2(x,y)\right| + \frac{2M}{\delta^2}\left|L_k(f_3;x,y)-f_3(x,y)\right|.
\end{aligned}
$$

Taking the supremum over $(x, y) \in S_A$ in the last inequality, we find that

$$
\|L_k(f;x,y)-f\|_{C(S_A)} \leq \varepsilon + N\sum_{i=0}^{3}\|L_k(f_i;x,y)-f_i\|_{C(S_A)} \tag{3.8}
$$

where $N := \left\{\varepsilon + M + \frac{4M}{\delta^2}\right\}$. We now replace $L_k(\cdot;x,y)$ by

$$
t_{kn}^{\alpha,\beta}(\cdot;x,y) = \frac{1}{P_k^{(\alpha,\beta)}}\sum_{i=\alpha(k)+n}^{\beta(k)+n}\left|\Delta_{h,p,q}^{a,b,c}(\lambda_i L_i(\cdot;x,y))\right| \quad (P_k^{(\alpha,\beta)} = \beta(k)-\alpha(k)+1)
$$

on both sides of the above inequality (3.8). For a given $\varepsilon' > 0$, we choose a number $\varepsilon > 0$ such that $\varepsilon < \varepsilon'$. By defining the following sets:

$$
\begin{aligned}
\mathcal{J} &:= \left\{k \leq m : \| t_{kn}^{\alpha,\beta}(f;x,y) - f \|_{C(S_A)} \geqq \varepsilon'\right\} \\
\mathcal{J}_i &:= \left\{k \leq m : \|t_{kn}^{\alpha,\beta}(f_i;x,y) - f_i\|_{C(S_A)} \geqq \frac{\varepsilon'-\varepsilon}{4N}\right\} \qquad (i = 0,1,2,3).
\end{aligned}
$$

Then the inclusion $\mathcal{J} \subset \mathcal{J}_0 \cup \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$ holds true and their densities (uniformly in n) are satisfy the following relation:

$$
\delta(\mathcal{J}) \leq \delta(\mathcal{J}_0) + \delta(\mathcal{J}_1) + \delta(\mathcal{J}_2) + \delta(\mathcal{J}_3).
$$

Letting $m \to \infty$ and using (3.3), we obtain

$$
F_{\Delta,\lambda}^{\alpha,\beta}(stat)\text{-}\lim_{m\to\infty}\|L_k(f;x,y)-f\|_{C(S_A)} = 0, \text{ for all } f \in C(S_A)
$$

This completes the proof. □

We now present an illustrative example using bivariate non-tensor type Meyer–König and Zeller generalization of the Bernstein power series (see [44, 45]) satisfying the conditions of Theorem 6. Before giving this example, we present a short introduction related with bivariate non-tensor type Meyer–König and Zeller operators defined on S_A (see [46]).

Let us take the following sequence of bivariate non-tensor positive operators:

$$M_n(f(s,t);x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{p_{i,j}^n(s,t)x^iy^j}{\Gamma_n(s,t;x,y)} f\left(\frac{a_{i,j,n}}{a_{i,j,n}+c_{i,j,n}+b_n}, \frac{c_{i,j,n}}{a_{i,j,n}+c_{i,j,n}+b_n}\right) \tag{3.9}$$

where $p_{i,j}^n(s,t) \geq 0$ for all $(s,t) \in S_A \subset \mathbb{R}^2$ and $f \in C(S_A)$. Based on the double indexed function sequence $\{p_{i,j}^n(s,t)\}$, the generating function $\Gamma_n(s,t;x,y)$ is given by

$$\Gamma_n(s,t;x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_{i,j}^n(s,t)x^iy^j,\ ((x,y) \in S_A).$$

We also suppose that the following conditions hold true:

(i) $\Gamma_n(s,t;x,y) = (1-x-y)\ \Gamma_{n+1}(s,t;x,y)$;

(ii) $a_{i+1,j,n}p_{i+1,j}^n(s,t) = b_{n+1}p_{i,j}^{n+1}(s,t)$ and $c_{i,j+1,n}p_{i,j+1}^n(s,t) = b_{n+1}p_{i,j}^{n+1}(s,t)$;

(iii) $b_n \to \infty$, $\frac{b_{n+1}}{b_n} \to 1$ and $b_n \neq 0$ for all $n \in \mathbb{N}$;

(it) $a_{i+1,j,n} - a_{i,j,n+1} = \xi_n$ and $c_{i,j+1,n} - c_{i,j,n+1} = \xi_n'$, $|\xi_n| \leq u < \infty$, $|\xi_n'| \leq u' < \infty$ and $a_{0,j,n} = 0, c_{i,0,n} = 0$ for all $n \in \mathbb{N}$.

The power series that defines M_n defined by (3.9) may not converge so we will consider the subspace

$$\mathcal{E}_M = \{f \in C(S_A) : \forall (x,y) \in S_A,\ \forall n \in \mathbb{N},\ |M_n(f;x,y)| < +\infty\}.$$

Thus, we say that $M_n : \mathcal{E}_M \to C(S_A)$ is a positive linear operator on $\mathcal{E}_M$. We also observe that

$$M_n(f_0;x,y) = 1,$$

$$M_n(f_1;x,y) = \frac{b_{n+1}}{b_n}\frac{x}{1-x-y}$$

$$M_n(f_2;x,y) = \frac{b_{n+1}}{b_n}\frac{y}{1-x-y}$$

$$M_n(f_3;x,y) = \frac{b_{n+1}b_{n+2}}{b_n^2}\frac{x^2+y^2}{(1-x-y)^2} + \frac{b_{n+1}\xi_n}{b_n^2}\frac{x}{1-x-y} + \frac{b_{n+1}\xi_n'}{b_n^2}\frac{y}{1-x-y}.$$

Remark 7. In this remark, we explain the effects of the test functions f_i ($i = 0,1,2,3$) and the motivations of using this Korovkin set. Clearly, for the nodes are given by

$$s = \frac{a_{i,j,n}}{a_{i,j,n}+c_{i,j,n}+b_n} \qquad \text{and} \qquad t = \frac{c_{i,j,n}}{a_{i,j,n}+c_{i,j,n}+b_n},$$

the denominators of

$$f_1(s,t) = \frac{a_{i,j,n}}{b_n},\ f_2(s,t) = \frac{c_{i,j,n}}{b_n}$$

and

$$f_3(s,t) = \left(\frac{a_{i,j,n}}{b_n}\right)^2 + \left(\frac{c_{i,j,n}}{b_n}\right)^2$$

are independent of i and j, respectively. It should be noted that, if one uses the test functions $e_{ij}(x,y) = x^i y^j$, $(i,j) \in \{0,1,2,\dots\} \times \{0,1,2,\dots\}$ with $i+j \leq 2$ in investigating the approximation properties of $M_n(f;x,y)$, it is not easy to calculate the second moment. For this reason, we prefer to use this Korovkin set. For the estimation of the higher order moments of Meyer–König and Zeller operators, we refer to [47].

Example 5. Let $\alpha(m), \beta(m), (\lambda_k), (p_k), (q_k)$ be sequences and a,b,c,h be the same as in Example 3. Let $\{T_k\}$ be a sequence of positive linear operators acting from $\mathcal{E}_M$ to $C(S_A)$ defined by

$$T_k(f(s,t);x,y) = (1+x_k)\ M_k(f(s,t);x,y) \tag{3.10}$$

where $x=(x_k)$ is defined as in Example 3. Since $\{x_k\}$ is statistically almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-convergent to zero, we immediately get that

$$F_{\Delta,\lambda}^{\alpha,\beta}(stat) - \lim_{m\to\infty} \|T_k(1;x,y)-1\|_{C(S_A)} = 0,$$

$$F_{\Delta,\lambda}^{\alpha,\beta}(stat) - \lim_{m\to\infty} \left\| T_k\left(\frac{s}{1-s-t};x,y\right) - \frac{x}{1-x-y}\right\|_{C(S_A)} = 0,$$

$$F_{\Delta,\lambda}^{\alpha,\beta}(stat) - \lim_{m\to\infty} \left\| T_k\left(\frac{t}{1-s-t};x,y\right) - \frac{y}{1-x-y}\right\|_{C(S_A)} = 0.$$

Also, since

$$\begin{aligned} t_{mn}^{\alpha,\beta}(f_3(s,t);x,y) &= \frac{1}{P_m^{(\alpha,\beta)}} \sum_{k=\alpha(m)+n}^{\beta(m)+n} \Delta_{1,p,q}^{1,b,b}(\lambda_k T_k(f_3;x,y)) \\ &= \frac{1}{m}\left[(n+m)T_{n+m}(f_3;x,y) - nT_n(f_3;x,y)\right] \\ &= \frac{1}{m}\left[(n+m)(1+x_{n+m})M_{n+m}(f_3;x,y) - n(1+x_n)M_n(f_3;x,y)\right] \end{aligned}$$

then

$$\begin{aligned} &\left\|t_{mn}^{\alpha,\beta}(f_3;x,y) - f_3\right\|_{C(S_A)} \\ &\quad = \left\| t_{mn}^{\alpha,\beta}\left(\frac{s^2+t^2}{(1-s-t)^2};x,y\right) - \frac{x^2+y^2}{(1-x-y)^2}\right\|_{C(S_A)} \\ &\quad = \left\| \frac{1}{m}\Big((n+m)(1+x_{n+m})M_{n+m}(f_3;x,y) - n(1+x_n)M_n(f_3;x,y)\Big) - f_3\right\|_{C(S_A)}. \end{aligned}$$

By taking the conditions (iii) and (iv) into account, and hence by letting $m \to \infty$, we are led to the fact that

$$F_{\Delta,\lambda}^{\alpha,\beta}(stat) - \lim_{m\to\infty} \left\| T_k\left(\frac{s^2+t^2}{(1-s-t)^2};x,y\right) - \frac{x^2+y^2}{(1-x-y)^2}\right\|_{C(S_A)} = 0,$$

as asserted by Theorem 6.

Thus, clearly, Theorem 6 does not work for each type of classical, statistical and almost statistical convergence in Korovkin type theorems. Hence, it is not difficult to see that our proposed approximation methods are stronger than the existing literature on Korovkin type approximation.

4. Rates of Statistical Almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-Convergence

In this section, we compute the rate of Statistical almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-convergence of sequences of positive linear operators defined from $C(S_A)$ into itself by the help of the modulus of continuity.

We first present the following definition.

Definition 3. Let h be any positive constant, $a, b, c \in \mathbb{N}$ and $(\alpha, \beta) \in \Lambda$. Also let $p = (p_k)$ and $q = (q_k)$ be two real valued sequences such that $0 < q_k < p_k \leq 1$ for all $k \in \mathbb{N}$ and $q_k \to Q$ and $p_k \to P$ as $k \to \infty$ where $P, Q \in (0, 1]$. Assume that $(\lambda_n)_{n=0}^{\infty}$ is a given strictly increasing sequence of positive numbers such that

$$|\Delta_{h,p,q}^{a,b,c}(\lambda_n)| \geq 0 \quad ([c]_{p,q}^{i} \neq 0, \forall i \in \mathbb{N}) \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = \infty.$$

Assume further that (θ_m) be a positive non-increasing sequence. We say that a sequence $x = (x_k) \in \ell_\infty$ is Statistical almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-convergence to a number L with the rate $o(\theta_m)$, if, for every $\epsilon > 0$

$$\lim_{m\to\infty} \frac{1}{m\ \theta_m} \left| \left\{ k \leq m : |t_{kn}^{\alpha,\beta}(\Delta_{h,p,q}^{a,b,c}(\lambda x)) - L| \geq \epsilon \right\} \right| = 0, \text{ uniformly in } n.$$

In this case, we denote it by $x_k - L = F_{\Delta,\lambda}^{\alpha,\beta}(stat)\ o(\theta_m)$.

Lemma 1. *Let (θ_m) and (θ'_m) be two nonincreasing sequences of positive real numbers. Also, let $x = (x_k)$ and $y = (y_k)$ be two sequences such that*

$$x_k - L_1 = F_{\Delta,\lambda}^{\alpha,\beta}(stat)\ o(\theta_m) \text{ and } y_k - L_2 = F_{\Delta,\lambda}^{\alpha,\beta}(stat)\ o(\theta'_m).$$

Then

$$(x_k - L_1) \pm (y_k - L_2) = F_{\Delta,\lambda}^{\alpha,\beta}(stat)\ o(\widetilde{\theta}_m),$$
$$(x_k - L_1)(y_k - L_2) = F_{\Delta,\lambda}^{\alpha,\beta}(stat)\ o(\theta_m \theta'_m),$$

and

$$\mu(x_k - L_1) = F_{\Delta,\lambda}^{\alpha,\beta}(stat)\ o(\theta_m)\ (\mu \in \mathbb{R})$$

where $\widetilde{\theta}_m = \max\{\theta_m, \theta'_m\}$.

Proof. Assume that $x_k - L_1 = F_{\Delta,\lambda}^{\alpha,\beta}(stat)\ o(\theta_m)$ and $y_k - L_2 = F_{\Delta,\lambda}^{\alpha,\beta}(stat)\ o(\theta'_m)$. Also for $\epsilon > 0$, let us set

$$\mathcal{D} := \left\{ k \leq m : \left| [t_{kn}^{\alpha,\beta}(\Delta_{h,p,q}^{a,b,c}(\lambda x)) + t_{kn}^{\alpha,\beta}(\Delta_{h,p,q}^{a,b,c}(\lambda y))] - (L_1 + L_2) \right| \geq \epsilon \right\},$$

$$\mathcal{D}_0 := \left\{k \le m : \left|t_{kn}^{\alpha,\beta}(\Delta_{h,p,q}^{a,b,c}(\lambda x)) - L_1\right| \ge \frac{\epsilon}{2}\right\}$$

and

$$\mathcal{D}_1 := \left\{k \le m : \left|t_{kn}^{\alpha,\beta}(\Delta_{h,p,q}^{a,b,c}(\lambda y)) - L_2\right| \ge \frac{\epsilon}{2}\right\}.$$

We then observe that $\mathcal{D} \subset \mathcal{D}_0 \cup \mathcal{D}_1$, which yields, for $m \in \mathbb{N}$, that

$$\frac{|\mathcal{D}|}{m\widetilde{\theta}_m} \le \frac{|\mathcal{D}_0|}{m\theta_m} + \frac{|\mathcal{D}_1|}{m\theta'_m}. \tag{4.1}$$

Since $\widetilde{\theta}_m = \max\{\theta_m, \theta'_m\}$, by (4.1), we have

$$\lim_{m\to\infty} \frac{1}{m\widetilde{\theta}_m} \left|\left\{k \le m : \left|[t_{kn}^{\alpha,\beta}(\Delta_{h,p,q}^{a,b,c}(\lambda x)) + t_{kn}^{\alpha,\beta}(\Delta_{h,p,q}^{a,b,c}(\lambda y))] - (L_1 + L_2)\right| \ge \epsilon\right\}\right| = 0,$$

uniformly in n, which leads us to the desired result. Since the rest part of the proof is similar, we omitted. □

We now recall the modulus of continuity and auxiliary facts to get the rates of statistically almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-convergence by means of this modulus.

Let $H_\omega(S_A)$ denote the space of all real-valued functions f on S_A such that

$$|f(s,t)-f(x,y)| \leqq \omega(f,\delta)\left[\frac{\sqrt{\left(\frac{s}{1-s-t} - \frac{x}{1-x-y}\right)^2 + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y}\right)^2}}{\delta} + 1\right] \tag{4.2}$$

where $\omega(f;\delta)$ is the modulus of continuity defined by

$$\omega(f;\delta) = \sup_{(s,t),(x,y)\in S_A} \left\{|f(s,t) - f(x,y)| : \sqrt{(s-x)^2 + (t-y)^2} \leqq \delta\right\}, \ (\delta > 0).$$

We then observe that any function in $H_\omega(S_A)$ is continuous and bounded on S_A.

Theorem 8. *Let h be any positive constant, $a, b, c \in \mathbb{N}$ and $(\alpha, \beta) \in \Lambda$. Also let $p = (p_k)$ and $q = (q_k)$ be two real valued sequences with the property (1.3). Suppose that $\{L_k\}$ be a sequence of positive linear operators from $H_\omega(S_A)$ into $C(S_A)$. Assume that the following conditions hold true:*

$$\|L_k(f_0; x, y) - f_0\|_{C(S_A)} = F_{\Delta,\lambda}^{\alpha,\beta}(stat)\ o(\theta_m)$$

and

$$\omega(f, \delta_k) = F_{\Delta,\lambda}^{\alpha,\beta}(stat)\ o(\theta'_m) \text{ on } S_A$$

where

$$\delta_k := \|L_k(\eta; x, y)\|_{C(S_A)}^{1/2}$$

with

$$\eta(s,t) = \left(\frac{s}{1-s-t} - \frac{x}{1-x-y}\right)^2 + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y}\right)^2.$$

Then, we have

$$\|L_k(f;x,y) - f\|_{C(S_A)} = F^{\alpha,\beta}_{\Delta,\lambda}(stat)\ o(\widetilde{\theta}_m),\ (f \in H_\omega(S_A))$$

where $\widetilde{\theta}_m = \max\{\theta_m, \theta'_m\}$

Proof. Let $f \in H_\omega(S_A)$ and $(x,y) \in S_A$ be fixed. By using (4.2) and the linearity of L_k, for any $\delta > 0$, we get

$$\begin{aligned}|L_k(f;x,y) - f(x,y)| = \ & L_k(|f(s,t) - f(x,y)|;x,y) \\ & + |f(x,y)|\ |L_k(f_0;x,y) - f_0(x,y)| \\ & \leq \omega(f,\delta) L_k\left(\frac{\eta(s,t)}{\delta^2} + 1; x,y\right) \\ & + |f(x,y)|\ |L_k(f_0;x,y) - f_0(x,y)| \\ & \leq \omega(f,\delta)\left\{\frac{1}{\delta^2} L_k(\eta;x,y) + L_k(f_0;x,y)\right\} \\ & + |f(x,y)|\ |L_k(f_0;x,y) - f_0(x,y)|\end{aligned}$$

Taking supremum over $(x,y) \in S_A$ on both sides, we have

$$\begin{aligned}&\|L_k(f;x,y) - f\|_{C(S_A)} \\ &\quad \leq \omega(f,\delta)\left\{\frac{1}{\delta^2}\|L_k(\eta;x,y)\|_{C(S_A)} + \|L_k(f_0;x,y) - f_0\|_{C(S_A)} + 1\right\} \\ &\quad\quad + N\|L_k(f_0;x,y) - f_0\|_{C(S_A)}\end{aligned}$$

where $N = \|f\|_{C(S_A)}$. Put $\delta = \delta_k := \|L_k(\eta;x,y)\|^{1/2}_{C(S_A)}$ in the last relation, so we get

$$\begin{aligned}\|L_k(f;x,y) - f\|_{C(S_A)} \leq \ & \omega(f,\delta_k)\big\{\|L_k(f_0;x,y) - f_0\|_{C(S_A)} + 2\big\} \\ & + N\|L_k(f_0;x,y) - f_0\|_{C(S_A)}.\end{aligned}$$

Replacing $L_k(\cdot;x,y)$ by

$$t^{\alpha,\beta}_{mn}(\cdot;x,y) = \frac{1}{P^{(\alpha,\beta)}_m}\sum_{k=\alpha(m)+n}^{\beta(k)+n} |\Delta^{a,b,c}_{h,p,q}(\lambda_k L_k(\cdot;x,y))|$$

on both sides of the last inequality, we find that

$$\begin{aligned}&\|t^{\alpha,\beta}_{mn}(f;x,y) - f\|_{C(S_A)} \\ &\quad \leq \omega(f,\delta_k)\|t^{\alpha,\beta}_{mn}(f_0;x,y) - f_0\|_{C(S_A)} + 2\omega(f,\delta_k) \\ &\quad\quad + N\ \|t^{\alpha,\beta}_{mn}(f_0;x,y) - f_0\|_{C(S_A)}.\end{aligned}$$

For a given $\epsilon > 0$, we consider the following sets:

$$\mathcal{J} := \left\{m : \|t_{mn}^{\alpha,\beta}(f;x,y) - f\|_{C(S_A)} \geq \epsilon\right\},$$
$$\mathcal{J}_0 := \left\{m : \omega(f,\delta_k)\|t_{mn}^{\alpha,\beta}(f_0;x,y) - f_0\|_{C(S_A)} \geqq \frac{\epsilon}{3N}\right\},$$
$$\mathcal{J}_1 := \left\{m : \omega(f,\delta_k) \geqq \frac{\epsilon}{6N}\right\},$$
$$\mathcal{J}_2 := \left\{k \leq m : \ \|t_{mn}^{\alpha,\beta}(f_0;x,y) - f_0\|_{C(S_A)} \geqq \frac{\epsilon}{3N}\right\}.$$

Then the inclusion $\mathcal{J} \subset \cup_{i=0}^{3}\mathcal{J}_i$ holds true and their densities (uniformly in n) are satisfy the relation $\delta(\mathcal{J}) \leq \sum_{i=0}^{3}\delta(\mathcal{J}_i)$. Since $\widetilde{\theta}_m = \max\{\theta_m, \theta'_m\}$, we obtain, for every $m \in \mathbb{N}$, that

$$\frac{|\mathcal{J}|}{m\ \widetilde{\theta}_m} \leq \frac{|\mathcal{J}_0|}{m\ \theta_m\ \theta'_m} + \frac{|\mathcal{J}_1|}{m\theta'_m} + \frac{|\mathcal{J}_2|}{m\theta_m}.$$

Hence, by letting $m \to \infty$ in the last inequality, we are led to the fact that

$$\|L_k(f;x,y) - f\|_{C(S_A)} = F_{\Delta,\lambda}^{\alpha,\beta}(stat)\ o(\widetilde{\theta}_m),\ (f \in H_\omega(S_A)),$$

as asserted by Theorem 8. □

5. A Voronovskaja Type Theorem

For the pointwise convergence of corresponding sequence of linear positive operators, Voronovskaja theorem (see [48]) concerning the asymptotic behavior of Bernstein polynomials has a crucial role (see also [49–52]). Very recently, quantitative Voronovskaja theorem has been studied as well so that the error of approximation in this pointwise convergence are also obtained via appropriate modulus of continuity. In this section, by using the notion of statistically almost $\Delta_{h,\alpha,\beta}^{[a,b,c]}(\lambda)$-convergence we obtain a Voronovskaja-type theorem with the help of J_k family of linear operators based on the generalized form of Meyer–König and Zeller operator (see [53]).

Let us take the following sequence of generalized linear positive operators:

$$L_n(f(s);x) = \frac{1}{h_n(x,s)}\sum_{k=0}^{\infty} f\left(\frac{a_{k,n}}{a_{k,n}+b_n}\right)\mathit{\Gamma}_{k,n}(s)x^k$$

where $0 < \frac{a_{k,n}}{a_{k,n}+b_n} < B$ and $B \in (0,1)$. For the sequence of functions $\{\mathit{\Gamma}_{k,n}(s)\}_{k\in\mathbb{N}}$, the generating function $h_n(x,s)$ is defined by $h_n(x,s) = \sum_{k=0}^{\infty}\mathit{\Gamma}_{k,n}(s)x^k$. It is not hard to show that

$$L_n(1;x) = 1,\ L_n\left(\frac{s}{1-s};x\right) = \frac{x}{1-x},\ L_n\left(\left(\frac{s}{1-s}\right)^2;x\right)$$
$$= \frac{x^2}{(1-x)^2}\frac{b_{n+1}}{b_n} + \frac{1}{b_n}\frac{x}{1-x}.$$

Let $C_B(S)$ be the space of continuous function on the closed interval $S=[0,A]$ where $A\leq \frac{1}{2}$ and bounded on entire line, that is $|f(x)|\leq K_f$, $-\infty<x<\infty$, where K_f is a constant depending on f. Also it is known that $B(S)$ is equipped with the supremum norm

$$\|f\|_{B(S)}=\sup_{x\in S}|f(x)|,\ \ (f\in B(S)).$$

Let $\{J_k\}$ be a sequence of positive linear operators acting from $C_B(S)$ into $B(S)$ defined by

$$J_k(f(s);x)=(1+x_k)L_k(f(s);x) \tag{5.1}$$

where $x=\{x_k\}$ is defined as in Example 3. Obviously,

$$F^{\alpha,\beta}_{\Delta,\lambda}(stat)-\lim\|J_k(1;x)-1\|_{B(S)}=0,$$

$$F^{\alpha,\beta}_{\Delta,\lambda}(stat)-\lim\left\|J_k\left(\frac{s}{1-s};x\right)-\left(\frac{x}{1-x}\right)\right\|_{B(S)}=0,$$

$$F^{\alpha,\beta}_{\Delta,\lambda}(stat)-\lim\left\|J_k\left(\left(\frac{s}{1-s}\right)^2;x\right)-\left(\frac{x}{1-x}\right)^2\right\|_{B(S)}=0.$$

Lemma 2. *Let h be any positive constant, $a,b,c\in\mathbb{N}$ and $(\alpha,\beta)\in\Lambda$. Also let $p=(p_k)$ and $q=(q_k)$ be two real valued sequences with the property (1.3). Assume that $b_k\to\infty$, $\frac{b_{k+1}}{b_k}\to 1$, $b_k\neq 0$ for all $k\in\mathbb{N}$ and $x\in S$. Then*

$$F^{\alpha,\beta}_{\Delta,\lambda}(stat)-\lim_m\ b_k\ J_k\left(\left(\frac{s}{1-s}-\frac{x}{1-x}\right)^2;x\right)=\frac{x}{1-x}. \tag{5.2}$$

Proof. For $x\in S$, we immediately get

$$J_k\left(\left(\frac{s}{1-s}-\frac{x}{1-x}\right);x\right)=(1+x_k)\left[L_k\left(\frac{s}{1-s};x\right)-\frac{x}{1-x}L_k(1;x)\right]=0.$$

Similarly, we compute the second moment as

$$J_k\left(\left(\frac{s}{1-s}-\frac{x}{1-x}\right)^2;x\right)=(1+x_k)\left[\left(\frac{x}{1-x}\right)^2\left(\frac{b_{k+1}}{b_k}-1\right)+\frac{1}{b_k}\left(\frac{x}{1-x}\right)\right].$$

Upon multiplying both sides by b_k $(b_k>0)$, we find that

$$b_kJ_k\left(\left(\frac{s}{1-s}-\frac{x}{1-x}\right)^2;x\right)-\left[\left(\frac{x}{1-x}\right)^2(b_{k+1}-b_k)+\frac{x}{1-x}\right]$$
$$=x_k\left[\left(\frac{x}{1-x}\right)^2(b_{k+1}-b_k)+\frac{x}{1-x}\right]\leq x_k\ (b_{k+1}-b_k+1)$$

for every $x\in S$. Since $F^{\alpha,\beta}_{\Delta,\lambda}(stat)-\lim x_k=0$ and so

$$F^{\alpha,\beta}_{\Delta,\lambda}(stat)-\lim_m\big(x_k(b_{k+1}-b_k+1)\big)=0,$$

then

$$F_{\Delta,\lambda}^{\alpha,\beta}(stat) - \lim_m b_k J_k \left(\left(\frac{s}{1-s} - \frac{x}{1-x} \right)^2 ; x \right) = \frac{x}{1-x},$$

which is desired. □

Corollary 1. *Let $b_k \to \infty$, $\frac{b_{k+1}}{b_k} \to 1$, $b_k \neq 0$ for all $k \in \mathbb{N}$ and $x \in S$. There is a positive constant $M_0(x)$ depending only on x such that*

$$F_{\Delta,\lambda}^{\alpha,\beta}(stat) - \lim_m \ b_k^2 \ J_k \left(\left(\frac{s}{1-s} - \frac{x}{1-x} \right)^4 ; x \right) \leq M_0(x). \tag{5.3}$$

Theorem 9. *Suppose that $b_k \to \infty$, $\frac{b_{k+1}}{b_k} \to 1$, $b_k \neq 0$ for all $k \in \mathbb{N}$ and $x \in S$. Then, for every $f \in C_B(S)$ such that $f', f'' \in C_B(S)$,*

$$F_{\Delta,\lambda}^{\alpha,\beta}(stat) - \lim_m \ b_k \ \left\{ J_k \left(f \left(\frac{s}{1-s} \right) ; x \right) - f \left(\frac{x}{1-x} \right) \right\} = \frac{1}{2} f'' \left(\frac{x}{1-x} \right) \frac{x}{1-x}.$$

Proof. Let $x \in S$ and $f', f'' \in C_B(S)$. Using the Taylor formula for $f \in C_B(S)$, we can write

$$\begin{aligned} f\left(\frac{s}{1-s}\right) &= f\left(\frac{x}{1-x}\right) + \left(\frac{s}{1-s} - \frac{x}{1-x}\right) f'\left(\frac{x}{1-x}\right) \\ &\quad + \frac{1}{2}\left(\frac{s}{1-s} - \frac{x}{1-x}\right)^2 f''\left(\frac{x}{1-x}\right) \\ &\quad + \eta_x\left(\frac{s}{1-s}\right)\left(\frac{s}{1-s} - \frac{x}{1-x}\right)^2 \end{aligned}$$

where the function $\eta_x(\frac{s}{1-s})$ is the remainder, $\eta_x(\cdot) \in C_B(S)$ and $\eta_x(\frac{s}{1-s}) \to 0$ as $s \to x$. We thus observe that the operator J_k is linear and that

$$\begin{aligned} & b_k \left[J_k \left(f \left(\frac{s}{1-s} \right) ; x \right) - f \left(\frac{x}{1-x} \right) \right] \\ &\quad = b_k x_k f \left(\frac{x}{1-x} \right) + \frac{1+x_k}{2} f'' \left(\frac{x}{1-x} \right) \left[\left(\frac{x}{1-x} \right)^2 (b_{k+1} - b_k) + \frac{x}{1-x} \right] \\ &\qquad + b_k J_k \left(\eta \left(\frac{s}{1-s} ; x \right) \left(\frac{s}{1-s} - \frac{x}{1-x} \right)^2 ; x \right) \end{aligned}$$

which yields that

$$\begin{aligned} & \left| b_k \left[J_k \left(f \left(\frac{s}{1-s} \right) ; x \right) - f \left(\frac{x}{1-x} \right) \right] \right. \\ &\qquad \left. - \frac{1}{2} \left[f'' \left(\frac{x}{1-x} \right) \left(\left(\frac{x}{1-x} \right)^2 (b_{k+1} - b_k) + \frac{x}{1-x} \right) \right] \right| \\ &\quad \leq b_k x_k N_0 + \frac{x_k}{2} N_1 |b_{k+1} - b_k + 1| \end{aligned}$$

$$+ b_k \left| J_k\left(\eta_x\left(\frac{s}{1-s}\right)\cdot\left(\frac{s}{1-s}-\frac{x}{1-x}\right)^2;x\right)\right| \tag{5.4}$$

where $N_0 = \|f\|_{C_B(S)}$ and $N_1 = \|f''\|_{C_B(S)}$. Now, we shall show that

$$F_{\Delta,\lambda}^{\alpha,\beta}(stat) - \lim\ b_k \left| J_k\left(\eta_x\left(\frac{s}{1-s}\right)\cdot\left(\frac{s}{1-s}-\frac{x}{1-x}\right)^2;x\right)\right| = 0. \tag{5.5}$$

Applying the Cauchy-Schwarz inequality, we have

$$b_k\ J_k\left(\eta_x\left(\frac{s}{1-s}\right)\cdot\left(\frac{s}{1-s}-\frac{x}{1-x}\right)^2;x\right)$$
$$\leq \sqrt{J_k\left(\eta_x^2\left(\frac{s}{1-s}\right);x\right)}\sqrt{b_k^2\ J_k\left(\left(\frac{s}{1-s}-\frac{x}{1-x}\right)^4;x\right)}.$$

Let us take $\eta_x^2\left(\frac{s}{1-s}\right) = \zeta_x\left(\frac{s}{1-s}\right)$. In this case, we see that $\zeta_x(\cdot) \in C_B(S)$. From Theorem 6, we observe that

$$F_{\Delta,\lambda}^{\alpha,\beta}(stat) \lim_{m\to\infty} -J_k\left(\eta_x^2\left(\frac{s}{1-s}\right);x\right) = \zeta_x\left(\frac{x}{1-x}\right) = 0.$$

By using Corollary 1, the inclusion (5.5) holds true. Hence, using the fact that $F_{\Delta,\lambda}^{\alpha,\beta}(stat) - \lim x_k = 0$ and letting $m \to \infty$ in (5.4), we obtain

$$F_{\Delta,\lambda}^{\alpha,\beta}(stat) - \lim_{m\to\infty}\left\{b_k x_k N_0 + \frac{x_k}{2}N_1|b_{k+1}-b_k+1|\right\} = 0$$

which leads us to the desired assertion of Theroem 9. □

6. Computational and Geometrical Interpretations

In this section, we provide the computational and geometrical interpretations of Theorem 6 with respect to the linear operator $T_k(f;x,y)$ given in (3.10) under different choices for the parameters.

Here, in our computations, we take

(1) $\varGamma_n(s,t;x,y) = \frac{1}{(1-x-y)^{n+1}}$ and $p_{i,j}^n(s,t) = \frac{(n+i+j)!}{n!\ i!\ j!}$;
(2) $a_{i,j,n} = i$, $c_{i,j,n} = j$ and $b_n = n$;
(3) $a = 1$, $b = c$ and $h = 1$;
(4) $\alpha(m) = 1$, $\beta(m) = m$ and $\lambda_n = n$ for all $m, n \in \mathbb{N}$;
(5) the sequence (x_n) is given as in Example 3.

Using the above choices, we may define the operator $t_{mn}^{\alpha,\beta}(f;x,y)$ by

$$t_{mn}^{\alpha,\beta}(Tf;x,y) = \frac{1}{m}\sum_{k=n+1}^{n+m}\Delta_{h,p,q}^{a,b,c}(\lambda_k T_k(f;x,y)) \tag{6.1}$$
$$= \frac{1}{m}\left[(n+m)(1+x_{n+m})M_{n+m}(f;x,y) - n(1+x_n)M_n(f;x,y)\right]$$

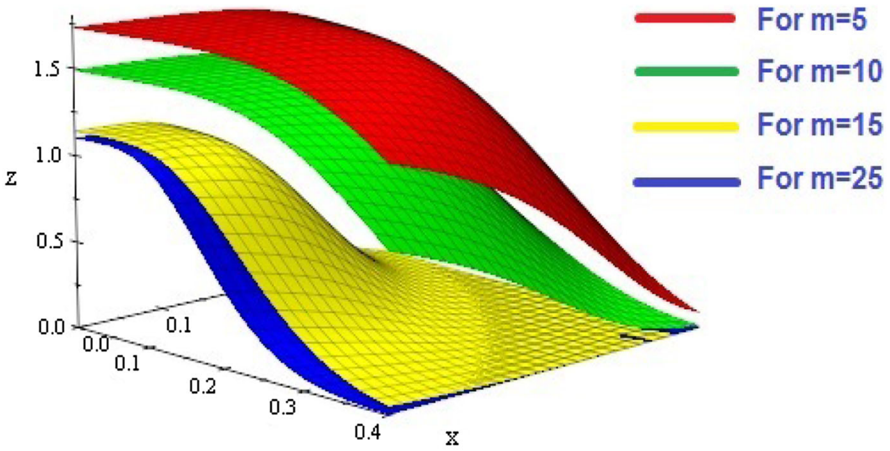

FIGURE 1. The convergence of $t_{mn}^{\alpha,\beta}(f_0;x,y)$ to $f_0(x,y)=1$

where

$$M_n(f;x,y)=(1-x-y)^{n+1}\sum_{i,j=0}^{\infty,\infty}\frac{(n+i+j)!}{n!\ i!\ j!}f\left(\frac{i}{i+j+n},\frac{j}{i+j+n}\right)x^iy^j. \tag{6.2}$$

It is noted that, we have investigated our double series in (6.2) only for finite sums. In a similar manner, the more complicated infinite series can be computed easily using more powerful equipments with higher speed.

If we interpret geometrically the convergence behavior of the operator in (6.1) for the test functions $f_i(s,t)$ defined in (3.1), it furnishes interesting and consistent results. In fact, in Fig. 1, we have plotted the convergence of $t_{mn}^{\alpha,\beta}(f_0;x,y)$ uniformly in n with different values of i and j run from $i,j=0$ to 50 for $m=5$, $m=10$, $m=15$ and $m=25$, respectively. Clearly, it is observed that the sequence $t_{mn}^{\alpha,\beta}(f_0;x,y)$ involving difference operator $\Delta_{h,p,q}^{a,b,c}$ converges towards (uniformly in n) to the function $f_0(x,y)=1$, as the value of m increases.

In Fig. 2, we have mentioned the convergence of $t_{mn}^{\alpha,\beta}(f_1;x,y)$ (uniformly in n) based above on values of i, j and m. Also, from this figure, it can be observed that, as the value of m increases, the sequence $t_{mn}^{\alpha,\beta}(f_1;x,y)$ converges (uniformly in n) to the function $f_1(x,y)=\frac{x}{1-x-y}$. Furthermore, the convergence of $t_{mn}^{\alpha,\beta}(f_2;x,y)$ to the function $f_2(x,y)=\frac{y}{1-x-y}$ can be obtained in a similar manner.

Similarly, for the test function given by

$$f_3(s,t)=\left(\frac{s}{1-s-t}\right)^2+\left(\frac{t}{1-s-t}\right)^2 \tag{6.3}$$

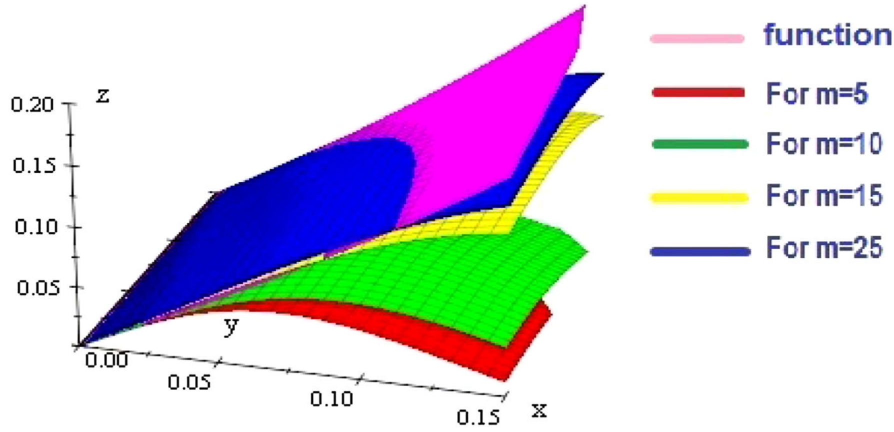

FIGURE 2. The convergence of $t_{mn}^{\alpha,\beta}(f_1;x,y)$ to $f_1(x,y) = \frac{x}{1-x-y}$

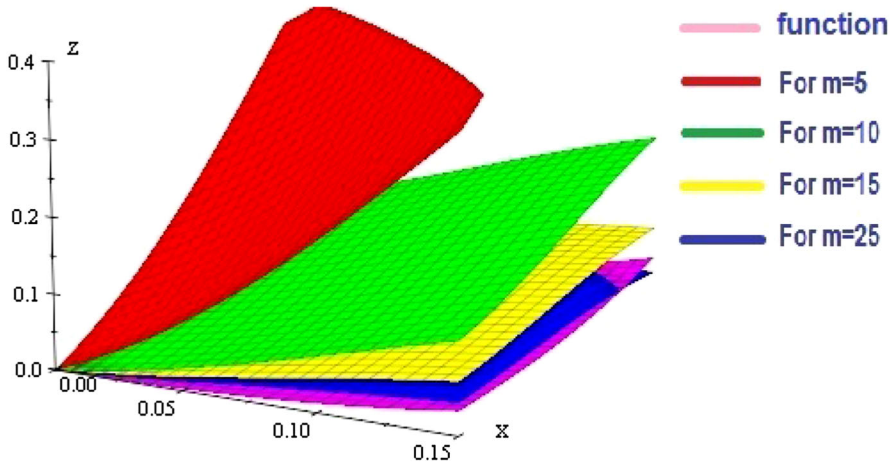

FIGURE 3. The convergence of $t_{mn}^{\alpha,\beta}(f_3;x,y)$ to $f_3(x,y) = \frac{x^2+y^2}{(1-x-y)^2}$

and the different values of m, it is also observe that $t_{mn}^{\alpha,\beta}(f_3;x,y)$ converges (uniformly in n) to the function $f_3(x,y)$ (see Fig. 3).

From the Figs. 1, 2 and 3, it is concluded that the conditions (3.3) of Theorem 6 are satisfied.

We observe from Fig. 4 that, as the value of m increases, the operators given by (6.1) converge (uniformly in n) towards to the function $f \in C(S_A)$

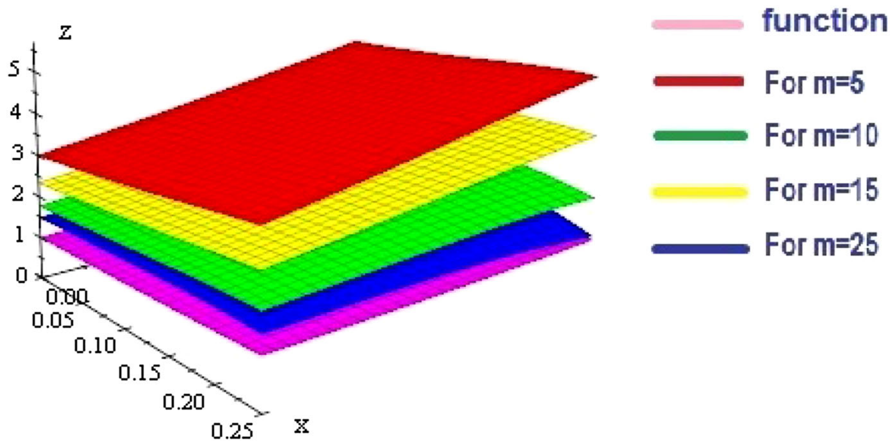

Figure 4. The convergence of $t_{mn}^{\alpha,\beta}(f;x,y)$ to $f(x,y) = \frac{\cos(3\pi xy)}{1-x-y}$

defined as

$$f(x,y) = \frac{\cos(3\pi xy)}{1-x-y}.$$

It is also observed that, this figure shows that the condition (3.2) holds true for the function f on $C(S_A)$.

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