Results Math (2018) 73:2

1422-6383/18/010001-26
published online January 30, 2018
https://doi.org/10.1007/s00025-018-0764-2

Results in Mathematics

Applications of the Shorgin Identity to Bernstein Type Operators

Ron Goldman, Xiao-Wei Xu, and Xiao-Ming Zeng

Abstract. By establishing an identity between a sequence of Bernstein-type operators and a sequence of Szász–Mirakyan operators, we prove that the convergence of Bernstein-type operators is related to convergence with respect to Szàsz–Mirakyan operators. As one application of this identity, we prove that whenever the parameters are conveniently chosen, if $f \in C[0,\infty)$ satisfies a growth condition of the form $|f(t)| \leq Ce^{\alpha t}(C, \alpha \in \mathbb{R}^+)$, then the classical Bernstein operators $B_{mn}(f(nu), x/n)$ converge to the Szàsz–Mirakyan operator $S_m(f,x)$. This convergence generalizes the classical result of De la Cal and Liquin to unbounded functions; moreover, the rth derivative of $B_{mn}(f(nu), x/n)$ converges to the rth derivative of $S_m(f,x)$. As another application of this identity, we derive Voronowskaja type result for the general Lototsky–Bernstein operators.

Mathematics Subject Classification. 41A10, 41A25, 41A36.

Keywords. Bernstein-type operators, operator semigroup, Shorgin formula, asymptotic formula, rate of convergence.

1. Introduction

The Bernstein approximation $B_n(f)$ to a function $f : [0,1] \to \mathbb{R}$ is the polynomial

$$B_n(f(u);x) := \sum_{k=0}^{n} b_k^n(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where $b_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}$ denotes the Bernstein basis of the space of polynomials of degree at most n. These operators B_n are now quite classical and have been the object of intensive research (see [2,5]).

Birkhäuser

In [10,21,29] independently the authors studied the so-called Szász–Mirakjan operators defined as

$$S_n(f(u);x) := \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1.2}$$

for all functions $f : [0,\infty) \to \mathbb{R}$ for which the series at the right-hand side is absolutely convergent. In particular, the operators S_n are well defined on the space $E_\alpha[0,\infty)$ of functions $f \in C[0,\infty)$ satisfying $|f(t)| \le Ce^{\alpha t}(C,\alpha \in \mathbb{R}^+)$. It is known (see [17,20]) that $\{S_n\}_{n\in\mathbb{N}}$ is a sequence of positive linear operators from $E_\alpha[0,\infty)$ to $C[0,\infty)$ such that for all $f \in E_\alpha[0,\infty)$

$$\lim_{n\to\infty} S_n(f;x) = f(x), \tag{1.3}$$

uniformly on every interval $[x_1, x_2], 0 \le x_1 < x_2 < \infty$.

However, the Szàsz–Mirakjan operators are non-polynomial operators which are represented by infinite series. In applications, a truncated version of the operators $S_n(f)$ is commonly used, and there are lots of papers that calculate the truncation error (see [16,20,22,28,32,36]). Therefore, it is natural to ask whether it is possible to approximate functions defined on the non-negative x-axis by using Bernstein operators (polynomial operators)?

It is well known [26,31] that by taking the limit of the binomial distribution with the parameter nx, we get the Poisson distribution. Therefore, the Bernstein operators B_n are naturally related to the Szàsz–Mirakjan operators S_n constructed by using the Poisson distribution. We can construct Bernstein-type operators from B_n by choosing suitable parameters that can approximate functions defined on the non-negative x-axis. Explicitly, for $f \in C[0,\infty)$, $m \in \mathbb{N}^+$, $n \ge x \ge 0$

$$V_{m,n}(f;x) := B_{mn}\left(f(nu);x/n\right) = \sum_{k=0}^{mn} \binom{mn}{k} \left(\frac{x}{n}\right)^k \left(1-\frac{x}{n}\right)^{mn-k} f\left(\frac{k}{m}\right), \tag{1.4}$$

where $n \ge x \ge 0$ guarantees the positivity of $B_{mn}\left(f(nu), x/n\right)$. De la Cal and Liquin [8] show that if f is a continuous and bounded real-valued function on $[0,\infty)$, then

$$\lim_{n\to\infty} B_{mn}(f(nu);x/n) = S_m(f(u);x). \tag{1.5}$$

However, the requirement of this boundedness constraint on f in (1.5) restricts the computational usefulness of Eq. (1.5). In this paper, we shall remove the boundedness constraint on f. We shall also discuss the convergence of the r $(r \ge 1)$th derivative of $B_{mn}(f(nu), x/n)$.

On the other hand, from the binomial theorem, we can derive a generating function for the classical Bernstein basis functions (see [11], and see also [[12], Chapter 5, p. 299–306])

$$(xw + (1-x))^n = \sum_{k=0}^{n} b_k^n(x) w^k. \tag{1.6}$$

The Lototsky–Bernstein basis functions $b_{n,k}(x)$ $(0 \le k \le n)$ are genealizations of (1.6). Let $\{p_i(x), 0 \le i \le n\}$ denote a sequence of real-valued functions defined on $[0,1]$. Set

$$b_{0,0}(x) = 1, b_{0,k}(x) = 0, k > 0, \tag{1.7}$$

$$\prod_{j=1}^{n} (wp_j(x) + 1 - p_j(x)) = \sum_{k=0}^{n} b_{n,k}(x) w^k. \tag{1.8}$$

By simple computations from (1.8) it is straightforward to confirm that

$$b_{n,k}(x) = \sum_{\substack{K \bigcup L = \{1,2,\cdots,n\} \\ |L|=n-k, |K|=k}} \prod_{m \in L} (1 - p_m(x)) \prod_{l \in K} p_l(x), \tag{1.9}$$

Throughout this paper, we always assume $p_i(x) \in C[0,1]$, $0 < p_i(x) < 1$ for $x \in (0,1)$ and $p_i(0) = 0, p_i(1) = 1$, for each $i = 1, \ldots, n$. If all the $p_i(x) = x$ $(1 \le i \le n)$, then $b_{n,k}(x)$ $(0 \le k \le n)$ reduce to the Bernstein basis functions $b_k^n(x)$. Note that the functions $b_{n,k}(x)$ form a generalized binomial probability distribution. The Lototsky–Bernstein operators L_n are defined for each function $f \in C[0,1]$ by

$$L_n(f(u); x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right). \tag{1.10}$$

Theses operators L_n were first introduced by King [18], and further studied by Eisenberg and Wood [9]. Many other authors studied these general Lototsky–Bernstein operators L_n under suitable restrictions on $p_i(x)$ $(1 \le i \le n)$. More precisely, whenever all the $p_i(x)$ are equal to a suitably chosen function $r_n^*(x)$ such that $\lim_{n\to\infty} r_n^*(x) = x$, then the operators $L_n(f)$ turn into King-type operators. For studies on King-type operators, see [1,3,4,13,14,19].

King [18] (see also Eisenberg and Wood [9]) derives the following convergence result. If $f \in C[0,1]$, and $\lim\limits_{n\to\infty} (p_1(x) + \cdots + p_n(x))/n = x$ uniformly on $[0,1]$, then

$$\lim_{n\to\infty} L_n(f; x) = f(x), \tag{1.11}$$

uniformly on $[0,1]$. But King does not derive the general convergence of $L_n(f,x)$ without any restriction on $p_i(x)$ $(1 \le i \le n)$. In this paper, we shall derive a general convergence theorem with respect to $p_i(x)$ $(1 \le i \le n)$ and even more a Voronowskaja type result for the Lototsky–Bernstein operators $L_n(f,x)$. Because the articles [9,18] (which are devoted to some basic convergence results for the operators $L_n(f,x)$) are the only two papers we can find with respect to the Lototsky–Bernstein operators, our present paper and our former paper [35] fill a gap in the investigation of Lototsky–Bernstein operators.

To study the operators $V_{n,m}(f,x)$ and $L_n(f,x)$ we shall adopt the framework of semigroups by constructing an identity of Shorgin [26] between Bernstein type operators and Szàsz–Mirakyan operators.

We proceed in the following fashion. In Sect. 2, we introduce some function spaces and some concepts concerning semigroups of operators. In particular, we shall give a forward difference representation and a semigroup representation for the operators B_n, S_n and $V_{n,m}$. In Sect. 3, we present some useful lemmas, which shall be used in the proof of the main results. In Sect. 4, we discuss the convergence of $V_{n,m}(f,x)$ and show that the classical result of De la Cal and Liquin [8] can be generalized to unbounded functions. We also prove that for $m \in \mathbb{N}^+$, the rth derivative of $V_{n,m}(f,x)$ converges to the rth derivative of $S_m(f,x)$. Finally, in Sect. 5, we derive a general convergence theorem and Voronowskaja type result for the Lototsky–Bernstein operators $L_n(f,x)$.

2. Semigroup Representations for Bernstein-Type Operators

In this section we introduce some function spaces that we shall deal with throughout this paper. We shall also recall the representation of the sequence of positive linear operators we quoted in the Introduction by use of difference operators and semigroups.

To fix our notation, let J be a bounded or unbounded interval whose left end point is 0. Denote by $C[J]$ the space of all real-valued continuous functions on J and by $C_B[J]$ the space of all real-valued bounded continuous functions on J. As usual, $C_B[J]$ is endowed with the sup-norm which, if no confusion arises, will be denoted by $\|\cdot\|$—that is, $\|f\| = \sup_{x\in J}|f(x)|$ $(f \in C_B[J])$. We also denote by

$$C_B^r[J] := \{f(x)|f^{(k)}(x) \in C_B[J], 0 \le k \le r\}.$$

In order to find a relationship between $B_n(f;x)$ (or $L_n(f;x)$) and $S_n(f;x)$, we shall introduce some concepts concerning C_0- semigroups. Consider the space of all bounded sequences

$$\Omega := \{c = (c_0, c_1, c_2, \ldots)|c_k \in \mathbb{R}, k \ge 0\}; \tag{2.1}$$

Ω is endowed with the norm: $\|c\| := \sup_{k\ge 0}|c_k|$ $(c \in \Omega)$.

The linear shift operator $B : \Omega \to \Omega$ is defined by $(Bc)_k = c_{k+1}$ $(k = 0, 1, 2, \ldots)$, i.e., $Bc = (c_1, c_2, \ldots)$. For any operator $D : \Omega \to \Omega$, define $\|D\| := \sup_{c\in\Omega} \dfrac{\|Dc\|}{\|c\|} = \sup_{\|c\|=1}\|Dc\|, c \in \Omega$. Notice, in particular, that $\|B\| = 1$. To define iteration for operators, let $I : \Omega \to \Omega$ denote the identity operator. For each operator $D : \Omega \to \Omega$, define $D^k : \Omega \to \Omega$ recursively by setting

$$D^0 = I, D^k = D(D^{k-1}), e^D := \sum_{k=0}^{\infty} \frac{D^k}{k!}.$$

It follows from the definition of the shift operator B that $(B^k c)_0 = c_k$.

Now define the difference operator $A := B - I$. Then the operator A is the generator of the C_0- convolution semigroup $T(t) := e^{tA}(t \geq 0)$, since for any sequence $c \in \Omega$

$$Ac = \lim_{t \to 0^+} \frac{e^{At}c - Ic}{t}. \tag{2.2}$$

For more details on semigroups, refer to [23].

Now for each continuous function we are going to introduce a sequence in Ω. For each $f(x) \in C[J]$, define $\psi(f, n) := f_n : C[J] \times \mathbb{N} \to \Omega$ by

$$f_n = (f(0/n), f(1/n), f(2/n), \ldots), \tag{2.3}$$

where $f(i/n) = 0$ if $i/n \notin J$.

Thus the Bernstein operators $B_n(f)$ can be represented by using the shift operator B and the difference operator A as follows (see [33])

$$B_n(f;x) = (((1-x)I + xB)^n f_n)_0 = ((I + xA)^n f_n)_0 =: (I + xA)^n f_n(0), \tag{2.4}$$

where $f_n(i) := (f_n)_i$ denotes the $(i+1)^{th}$ coordinate of the sequence f_n. Thus $B_n(f;x)$ is the first coordinate of the sequence $(I + xA)^n f_n$. Moreover, the Szàsz–Mirakjan operators $S_n(f;x)$ can be represented as follows,

$$S_n(f;x) = \left(e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} B^k f_n \right)_0 = (e^{nx(B-I)} f_n)_0 = (e^{nxA} f_n)_0 =: e^{nxA} f_n(0). \tag{2.5}$$

Similarly, $S_n(f;x)$ is the first coordinate of the sequence $e^{nxA} f_n$.

The representation (2.4) of Bernstein operators $B_n(f;x)$ can be generalized to the Lototsky–Bernstein operators. Indeed, by (1.8) with B in place of w we get

$$\prod_{i=1}^{n}(I + p_i(x)A) = \prod_{i=1}^{n}((1 - p_i(x))I + p_i(x)B) = \sum_{k=0}^{n} b_{n,k}(x)B^k.$$

Therefore since $(B^k f_n)_0 = f(\frac{k}{n})$, we have

$$\begin{aligned} L_n(f;x) &= \left(\prod_{i=1}^{n}((1 - p_i(x))I + p_i(x)B) f_n \right)_0 = \left(\prod_{i=1}^{n}(I + p_i(x)A) f_n \right)_0 \\ &=: \prod_{i=1}^{n}(I + p_i(x)A) f_n(0). \end{aligned} \tag{2.6}$$

Shorgin [26] derives an asymptotic expansion of the generalized binomial distribution (1.9) in terms of convergence to the Poisson distribution (Shorgin formula). Applying this formula, we can uncover the relationship between Bernstein-type operators and Szàsz–Mirakjan operators. This formula has been widely studied in probability theory and in the theory of semigroups (see [6,7,26,31]). However, here is the first time, to our knowledge, for this formula to be used to study positive linear operators. Therefore here we offer a detailed introduction.

For any $\beta_1, \beta_2, \ldots, \beta_n$ such that $0 \le \beta_k \le 1$ $(1 \le k \le n)$, set

$$\lambda = \sum_{j=1}^{n} \beta_j, \qquad \lambda_r = \sum_{j=1}^{n} \beta_j^r, r \ge 2. \tag{2.7}$$

The Bernoulli operator is defined by $\prod_{i=1}^{n}(I + \beta_i A)$, and the Poisson convolution semigroup operator is defined by $e^{\lambda A} = e^{\lambda(B-I)} = e^{-\lambda} \sum_{k=0}^{\infty} \dfrac{\lambda^k}{k!} B^k$. The Shorgin formula is (see [6,7,26,31])

$$e^{\lambda A} - \prod_{k=1}^{n}(I + \beta_k A) = \frac{1}{2}\lambda_2 A^2 e^{\lambda A} - \sum_{k=3}^{\infty} a_k (-A)^k e^{\lambda A}, \tag{2.8}$$

where a_k is defined recursively by

$$a_k = -\frac{1}{k}\left(\lambda_k + \sum_{i=2}^{k-2} a_i \lambda_{k-i}\right), k \ge 4, \tag{2.9}$$

with $a_2 = -\dfrac{1}{2}\lambda_2, a_3 = -\dfrac{1}{3}\lambda_3$.

If $\beta_k = x$ $(1 \le k \le n)$, then $\lambda(x) := \lambda = nx, \lambda_r(x) := \lambda_r = nx^r$ $(r \ge 2)$ and $a_k = b_k(n)x^k$, where $b_k(n)$ is a polynomial of n. In this case the Bernoulli operator reduces to the Bernstein operator, i.e.,

$$\begin{aligned} B_n(f;x) &= (I + xA)^n f_n(0) \\ &= e^{nxA} f_n(0) - \frac{1}{2} nx^2 (e^{nxA} A^2) f_n(0) + \sum_{k=3}^{\infty} (-1)^k b_k(n) x^k (e^{nxA} A^k) f_n(0), \end{aligned} \tag{2.10}$$

$$= e^{nxA} f_n(0) + \sum_{k=2}^{\infty} (-1)^k b_k(n) x^k (e^{nxA} A^k) f_n(0), \tag{2.11}$$

where (2.11) holds since $b_2(n) = -\dfrac{1}{2}n$.

Moreover, it is straightforward to deduce that

$$\begin{aligned} B_{mn}\left(f(nu); x/n\right) &= \left(I + \frac{x}{n}A\right)^{mn} f_m(0) \\ &= e^{mxA} f_m(0) - \frac{1}{2n} mx^2 e^{mxA} A^2 f_m(0) \\ &\quad + \sum_{k=3}^{\infty} (-1)^k \frac{b_k(mn)}{n^k} x^k e^{mxA} A^k f_m(0), \end{aligned} \tag{2.12}$$

$$= e^{mxA} f_m(0) + \sum_{k=2}^{\infty} (-1)^k \frac{b_k(mn)}{n^k} x^k e^{mxA} A^k f_m(0). \tag{2.13}$$

Let $\Delta_h^q(f,x)$ be qth forward difference with step h defined by

$$\Delta_h^q(f,x) = \sum_{j=0}^{q}(-1)^{q-j}\binom{q}{j}f(x+jh), \quad h>0, q\in\mathbb{N}. \tag{2.14}$$

It follows by induction on k that

$$A^k f_m(i) := (A^k f_m)_i = \Delta_{1/m}^k(f, i/m), \quad i \in \mathbb{N}, \tag{2.15}$$

where $f_m \in \Omega$. Therefore by (2.5),(2.12), (2.13) and (2.14)

$$\begin{aligned} B_{mn}\left(f(nu); x/n\right) &= S_m(f(u),x) - \frac{1}{2n}mx^2 S_m\left(\Delta_{1/m}^2(f,u),x\right) \\ &\quad + \sum_{k=3}^{\infty}(-1)^k x^k \frac{b_k(mn)}{n^k} S_m\left(\Delta_{1/m}^k(f,u),x\right) \qquad (2.16) \\ &= S_m(f(u),x) + \sum_{k=2}^{\infty}(-1)^k x^k \frac{b_k(mn)}{n^k} S_m\left(\Delta_{1/m}^k(f,u),x\right). \qquad (2.17) \end{aligned}$$

On the other hand, if $\beta_k = p_k(x)$ $(1 \le k \le n)$, then $\lambda(x) := \lambda = \sum_{k=1}^n p_k(x), \lambda_r(x) := \lambda_r = \sum_{k=1}^n p_k^r(x)$ $(r \ge 2)$. Now the Lototsky–Bernstein operators $L_n(f;x)$ can be represented in the following form

$$L_n(f;x) = e^{\lambda(x)A} f_n(0) + \sum_{k=2}^{\infty} a_k(-A)^k e^{\lambda(x)A} f_n(0), \tag{2.18}$$

where $e^{\lambda(x)A} f_n(0) = e^{-\sum_{i=1}^n p_i(x)} \cdot \sum_{k=0}^{\infty}[\sum_{i=1}^n p_i(x)]^k/k! f(k/n) = S_n(f; \sum_{i=1}^n p_i(x)/n)$.

3. Preliminary Lemmas

The proofs of our main results are based on a number of lemmas.

First, we mention a basic fact about the norm of $e^{\lambda A}$, i.e., $\|e^{\lambda A}\| \le 1$. Indeed, since $\|e^{-\lambda I}\| = \|\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}I\| = e^{-\lambda}$ and $\|e^{\lambda B}\| \le \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\|B\|^k \le e^{\lambda}$, it follows that

$$\|e^{\lambda A}\| = \|e^{-\lambda I}\cdot e^{\lambda B}\| \le \|e^{-\lambda I}\|\cdot\|e^{\lambda B}\| \le 1.$$

Now we give two estimates concerning the norm of A in (2.8) and of the coefficients in (2.9). Such estimates are derived by Shorgin [26], see also [6,7,31].

Lemma 3.1. *For $k \in \mathbb{N}^+$, and with $a_k, \lambda(x), \lambda_2(x)$ as in (2.18).*

$$|a_k| \le \left(\frac{e\lambda_2}{k}\right)^{k/2}, \quad k \ge 2, \tag{3.1}$$

$$\|A^k e^{\lambda A}\| \le \frac{\sqrt{e}(1+\sqrt{\pi/2})}{2}\left(\frac{k}{e\lambda}\right)^{(k+1)/2}, \quad k \ge 1. \tag{3.2}$$

Remark 3.2. *In some cases, we also use the estimate* $\|A\| = \|B - I\| \le \|B\| + \|I\| \le 2$. *And if all the* $p_k(x) = x$ $(1 \le k \le n)$, *then* $a_k = b_k(n)x^k$ *(see (2.10)). Thus,* $|b_k(n)x^k| \le \left(\frac{enx^2}{k}\right)^{k/2} \Rightarrow |b_k(n)| \le \left(\frac{en}{k}\right)^{k/2}$.

The general Taylor expansion of the semigroup operator stated in the following result is well-known (see Lemma 4.1, [24], p. 276).

Lemma 3.3. *Let* $T(v) = e^{vA}, v \ge 0$. *For* $c \in \Omega$ *with* $\|A^l c\| < \infty, l \ge 1$, *and arbitrary* $s, t \ge 0$

$$T(t)c - T(s)c = \sum_{k=1}^{l-1} \frac{(t-s)^k}{k!} T(s)A^k c + \int_s^t \frac{(t-u)^{l-1}}{(l-1)!} T(u)A^l c du. \tag{3.3}$$

The proof of the following Lemma can be found in [33,34]. For the completeness of this paper and for later applications, we give an outline of the proof.

Lemma 3.4. *Let* B *be a linear contraction operator on a Banach space* X. *The linear operator* $A = B - I$ *generates a contraction semigroup* $T(\xi) = e^{\xi A}$ $(\xi \ge 0)$. *Moreover for* $f \in X$ *and* $n > \xi$

$$\|T(m\xi)f - (I + \xi A/n)^{mn} f\| \le \frac{m\xi^2}{2n}\|A^2 f\|. \tag{3.4}$$

Proof. It is well-known that $A = B - I$ generates a contraction semigroup. Therefore, for $\xi \ge 0$, both $T(\xi)$ and $(I + \xi A/n) = (1 - \xi/n)I + \xi B/n$ are contractive operators. To prove (3.4), observe that for commuting contraction operators U, V on the space X, one has

$$\begin{aligned} \|U^n f - V^n f\| &= \|\sum_{k=0}^{n-1}(U^{n-k}V^k f - U^{n-k-1}V^{k+1}f)\| \\ &\le \sum_{k=0}^{n-1}\|U^{n-k-1}(U-V)V^k f\| \le n\|Uf - Vf\|. \end{aligned} \tag{3.5}$$

Using inequality (3.5) and the identity $T(m\xi) = (T(\xi/n))^{mn}$ yields

$$\|T(m\xi)f - (I + \xi A/n)^{mn} f\| \le mn \cdot \|T(\xi/n)f - (I + \xi A/n)f\|. \tag{3.6}$$

Now note that $T'(s) = T(s)A$. Therefore invoking integration by parts yields

$$\begin{aligned} \int_0^{\xi/n} (\xi/n - s)T(s)A^2 ds &= \int_0^{\xi/n} (\xi/n - s)A dT(s) \\ &= (\xi/n - s)AT(s)\Big|_0^{\xi/n} + \int_0^{\xi/n} T(s)A ds \end{aligned}$$

$$= -\xi A/n + T(s)\big|_0^{\xi/n} = T(\xi/n) - (I + \xi A/n).$$

Hence

$$T(\xi/n) - (I + \xi A/n) = \int_0^{\xi/n} (\xi/n - s)T(s)A^2 ds. \tag{3.7}$$

Therefore, by (3.7) and the fact that $\|T(s)\| \le 1$

$$\begin{aligned} \|T(\xi/n)f - (I + \xi A/n)f\| &\le \int_0^{\xi/n} (\xi/n - s)\|T(s)A^2 f\| ds \\ &\le \|A^2 f\| \int_0^{\xi/n} (\xi/n - s) ds \\ &= \frac{\xi^2}{2n^2}\|A^2 f\|. \end{aligned} \tag{3.8}$$

Combining (3.6) and (3.8) yields (3.4). □

The proof of the following result can be found in (Corollary 3.4.4 and (2.15), [5]).

Lemma 3.5. *Let $r, i, m \in \mathbb{N}^+$, and $f \in C^r[0, \infty)$. Then*

$$A^r f_m(i) = f^{(r)}(\xi)/m^r, \tag{3.9}$$

where $i/m < \xi < (r + i)/m$.

4. Convergence of $B_{mn}(f(nu), x/n)$

The Szász–Mirakjan operator $S_m(f; x)$ is the limit of Bernstein operators, whenever the parameters are conveniently chosen. De la Cal and Liquin [8] showed the following estimate between the Bernstein and the Szàsz–Mirakjan operators. For $m \in \mathbb{N}^+, f \in C_B[0, \infty)$, the error bound

$$|B_{mn}(f(nu); x/n) - S_m(f(u); x)| \le \frac{2mx^2}{n}\|f\|, \quad 0 \le x \le n, \tag{4.1}$$

given in [8] shows that the rate of convergence is at least 1/n.

On the other hand, the asymptotic formula

$$\lim_{n\to\infty} n\left(B_{mn}(f(nu); x/n) - S_m(f(u); x)\right) = -\frac{1}{2}mx^2 S_m(\Delta^2_{1/m}(f, u), x), \tag{4.2}$$

due to the first author [33], shows that the rate of convergence is precisely $1/n$ when the right hand side of (4.2) is non-zero.

Now we investigate two questions. First, may the restriction $f \in C_B[0, \infty)$ be relaxed to an unbounded function such as $f \in E_\alpha[0, \infty)$? Second, what is the rate of convergence of the derivatives $(B_{mn}(f(nu), x/n))^{(r)}$ $(r \ge 1)$ as n tends to infinity?

To answer the first question, we have the following

Theorem 4.1. *Let $m \in \mathbb{N}^+$, $f \in E_\alpha[0,\infty)$. Then for $n \geq x \geq 0$,*

$$|B_{mn}(f(nu);x/n) - S_m(f(u);x)| \leq C\frac{2e^{mx(e^{\alpha/m}-1)+1+2\alpha/m} \cdot mx^2}{n} \times \sum_{k=2}^{\infty}\left(\frac{4e^{1+2\alpha/m}mx^2}{nk}\right)^{(k-2)/2}. \quad (4.3)$$

Moreover,

$$\left|n((B_{mn}(f(nu);x/n) - S_m(f(u);x)) + \frac{1}{2}mx^2 S_m(\Delta_{1/m}^2(f,u);x)\right| \leq C\frac{8e^{mx(e^{\alpha/m}-1)+3/2+3\alpha/m} \cdot m\sqrt{m}x^3}{3\sqrt{3n}}\sum_{k=3}^{\infty}\left(\frac{4e^{1+2\alpha/m}mx^2}{nk}\right)^{(k-3)/2}. \quad (4.4)$$

Proof. By the assumption that $f \in E_\alpha[0,\infty)$, then $|f(t)| \leq Ce^{\alpha t}(C,\alpha \in \mathbb{R}^+)$. First we estimate the following quantity for $m \geq 1, j \geq 0$

$$\begin{aligned}|\Delta_{1/m}^k(f,j/m)| &= \left|\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(j/m+i/m)\right| \\ &\leq 2^k \max\{|f(j/m)|, |f((j+1)/m)|, \ldots, |f((j+k)/m)|\} \\ &\leq C2^k \max\{e^{\alpha j/m}, e^{\alpha(j+1)/m}, \ldots, e^{\alpha(j+k)/m}\} = C2^k e^{\alpha(j+k)/m}. \end{aligned} \quad (4.5)$$

Therefore by (4.5)

$$\begin{aligned}\left|S_m\left(\Delta_{1/m}^k(f,u);x\right)\right| &= \left|e^{-mx}\sum_{j=0}^{\infty}\frac{(mx)^j}{j!}\Delta_{1/m}^k(f,j/m)\right| \\ &\leq e^{-mx}\sum_{j=0}^{\infty}\frac{(mx)^j}{j!}\left|\Delta_{1/m}^k(f,j/m)\right| \\ &\leq C2^k e^{\alpha k/m}e^{-mx}\sum_{j=0}^{\infty}\frac{(mx)^j}{j!}e^{\alpha j/m} = C2^k e^{\alpha k/m}S_m(e^{\alpha u};x). \end{aligned} \quad (4.6)$$

Moreover

$$S_m(e^{\alpha u};x) = e^{-mx}\sum_{j=0}^{\infty}\frac{(mx)^j}{j!}e^{\alpha j/m} = e^{-mx+e^{\alpha/m}mx}. \quad (4.7)$$

Now using (2.17), Remark 3.2 with $a_k = b_k(mn)x^k$ and (4.6), (4.7), we bound the remainder term

$$\begin{aligned}&|B_{mn}(f(nu);x/n) - S_m(f(u);x)| \\ &= \left|\sum_{k=2}^{\infty}(-1)^k x^k \frac{b_k(mn)}{n^k}S_m\left(\Delta_{1/m}^k(f,u);x\right)\right|\end{aligned}$$

$$\leq \sum_{k=2}^{\infty} \left|(-1)^k x^k/n^k\right| \cdot |b_k(mn)| \cdot \left|S_m\left(\Delta_{1/m}^k(f,u);x\right)\right|$$
$$\leq \sum_{k=2}^{\infty} \frac{x^k}{n^k}\left(\frac{emn}{k}\right)^{k/2} \cdot C2^k e^{\alpha k/m} \cdot e^{-mx+e^{\alpha/m}mx}$$
$$= Ce^{(e^{\alpha/m}-1)mx}\sum_{k=2}^{\infty}\left(\frac{4e^{1+2\alpha/m}mx^2}{nk}\right)^{(k-2)/2}\left(\frac{4e^{1+2\alpha/m}mx^2}{nk}\right)$$
$$\leq C\frac{4e^{(e^{\alpha/m}-1)mx+1+2\alpha/m}mx^2}{2n}\sum_{k=2}^{\infty}\left(\frac{4e^{1+2\alpha/m}mx^2}{nk}\right)^{(k-2)/2},$$

which is precisely (4.3).

Along the same lines, we can prove (4.4). Indeed, by (2.16)

$$\left|n((B_{mn}(f(nu);x/n) - S_m(f(u);x)) + \frac{1}{2}mx^2 S_m(\Delta_{1/m}^2(f,u);x)\right|$$
$$= \left|\sum_{k=3}^{\infty}(-1)^k x^k b_k(mn)/n^{k-1} S_m\left(\Delta_{1/m}^k(f,u);x\right)\right|$$
$$\leq \sum_{k=3}^{\infty}\frac{x^k}{n^{k-1}}\left(\frac{emn}{k}\right)^{k/2} C2^k e^{\alpha k/m} \cdot e^{-mx+e^{\alpha/m}mx}$$
$$= Cne^{(e^{\alpha/m}-1)mx}\sum_{k=3}^{\infty}\left(\frac{4e^{1+2\alpha/m}mx^2}{nk}\right)^{(k-3)/2}\left(\frac{4e^{1+2\alpha/m}mx^2}{nk}\right)^{3/2}$$
$$\leq C\frac{8e^{mx(e^{\alpha/m}-1)+3/2+3\alpha/m}\cdot m\sqrt{m}x^3}{3\sqrt{3n}}\sum_{k=3}^{\infty}\left(\frac{4e^{1+2\alpha/m}mx^2}{nk}\right)^{(k-3)/2}, \tag{4.8}$$

which is precisely (4.4). □

It is straightforward from (4.3) and (4.4) that

Corollary 4.2. *Let $m \in \mathbb{N}^+$, $f \in E_\alpha[0,\infty)$. Then*

$$\lim_{n\to\infty} B_{mn}(f(nu);x/n) = S_m(f(u);x), \tag{4.9}$$

and

$$\lim_{n\to\infty} n\left(B_{mn}(f(nu);x/n) - S_m(f(u);x)\right) = -\frac{1}{2}mx^2 S_m(\Delta_{1/m}^2(f,u);x). \tag{4.10}$$

uniformly on every interval $[x_1,x_2], 0 \leq x_1 < x_2 < \infty$.

To answer the second question, we have the following two theorems.

Theorem 4.3. *Let m be a fixed integer and suppose that $n \geq x \geq 0$. Then for every $r \geq 1$ and $mn > r$*

(i). *If* $f \in C_B[0,\infty)$, *then*

$$\left|B_{mn}^{(r)}\left(f(nu);x/n\right)-S_m^{(r)}(f(u);x)\right| \leq \frac{(2m)^{r+1}}{n}\left(x^2+\frac{rx}{m-r/n}+\frac{r^2x^2}{(mn-r)^2}\right)\|f\| + \frac{(2m)^{r-1}r(r-1)}{n}\|f\|. \tag{4.11}$$

(ii). *If* $f \in C_B^r[0,\infty)$, *then*

$$\left|B_{mn}^{(r)}\left(f(nu);x/n\right)-S_m^{(r)}(f(u);x)\right| \leq \frac{2mx}{n}\left(x+\frac{r}{m-r/n}+\frac{r^2x}{(mn-r)^2}\right)\|f^{(r)}\| + \frac{r(r-1)}{2mn}\|f^{(r)}\|. \tag{4.12}$$

(iii). *If* $f \in E_\alpha[0,\infty)$, *then*

$$\left|B_{mn}^{(r)}\left(f(nu);x/n\right)-S_m^{(r)}(f(u);x)\right| \leq C\frac{e^{1+(2+r)\alpha/m+(e^{\alpha/m}-1)mx}(4m)^{r+1}\max\{1,x^2\}}{2n} \times\sum_{k=2}^{\infty}\left(\frac{4e^{1+2\alpha/m}m\max\{1,x^2\}}{nk}\right)^{(k-2)/2}\cdot k^r. \tag{4.13}$$

In addition, Voronovskaya's formula can be differentiated:

Theorem 4.4. *Let* $m, r \in \mathbb{N}^+$ *and* $f \in E_\alpha[0,\infty)$. *Then*

$$\lim_{n\to\infty} n\left(B_{mn}^{(r)}\left(f(nu);\frac{x}{n}\right)-S_m^{(r)}(f(u);x)\right) = -\frac{1}{2}\frac{d^r}{dx^r}\{mx^2S_m(\Delta_{1/m}^2(f,u);x)\}, \tag{4.14}$$

uniformly on every interval $[x_1,x_2], 0 \leq x_1 < x_2 < \infty$. *Thus the* r*th derivative of* $B_{mn}(f(nu);x/n)$ *converges at the rate* $1/n$ *when the right hand side of* (4.14) *is non-zero.*

To prove Theorems 4.3 and 4.4, we shall need the following lemma.

Lemma 4.5. *Let* $r \in \mathbb{N}^+$. *Then*

$$S_m^{(r)}(f(u);x) = m^r e^{mxA}A^r f_m(0), \tag{4.15}$$

and

$$B_{mn}^{(r)}\left(f(nu);x/n\right) = m(m-\frac{1}{n})\cdots(m-\frac{r-1}{n})\left(I+\frac{x}{n}A\right)^{mn-r}A^r f_m(0) \tag{4.16}$$

$$= m^r e^{mxA}A^r f_m(0) + \sum_{k=2}^{\infty}(-1)^k\frac{b_k(mn)}{n^k}\sum_{j=0}^{r}\binom{r}{j}(k)_j x^{k-j}$$

$$\mathbb{I}_{\{k\geq j\}} m^{r-j} e^{mxA} A^{k+r-j} f_m(0) \tag{4.17}$$

$$= S_m^{(r)}(f(u);x) + \sum_{k=2}^{\infty}(-1)^k \frac{b_k(mn)}{n^k} \sum_{j=0}^{r} \binom{r}{j} (k)_j x^{k-j}$$

$$\mathbb{I}_{\{k\geq j\}} m^{r-j} S_m \left(\Delta_{1/m}^{k+r-j}(f,u);x\right) \tag{4.18}$$

$$= S_m^{(r)}(f(u);x) - \frac{1}{2n}\frac{d^r}{dx^r}\left(mx^2 S_m(\Delta_{1/m}^2(f,u);x)\right)$$

$$+ \sum_{k=3}^{\infty}(-1)^k \frac{b_k(mn)}{n^k} \sum_{j=0}^{r} \binom{r}{j} (k)_j x^{k-j} \mathbb{I}_{\{k\geq j\}} m^{r-j} S_m$$

$$\left(\Delta_{1/m}^{k+r-j}(f,u);x\right). \tag{4.19}$$

where $(k)_j = k(k-1)\cdots(k-j+1)$ *is the falling factorial power, and* $\mathbb{I}_{\{k\geq j\}}$ *denotes the indicator function for* $\{k \geq j\}$.

Proof. The proof of (4.15) is straightforward by differentiating (2.5). We can prove (4.16)–(4.19) in the following fashion. From the proof of (4.3) and the fact that $n \geq x \geq 0$ for $k \geq 2$ it follows that

$$\left|(-1)^k x^k \frac{b_k(mn)}{n^k} S_m\left(\Delta_{1/m}^k(f,u);x\right)\right|$$

$$\leq C\frac{2e^{mx(e^{\alpha/m}-1)+1+2\alpha/m}\cdot mx^2}{n}\left(\frac{4e^{1+2\alpha/m}mx^2}{nk}\right)^{(k-2)/2}$$

$$\leq C2e^{mx(e^{\alpha/m}-1)+1+2\alpha/m}\cdot mn\left(\frac{4e^{1+2\alpha/m}mn}{k}\right)^{(k-2)/2}. \tag{4.20}$$

Since it is immediate that the infinite series $\sum_{k=2}^{\infty}\left(\frac{4e^{1+2\alpha/m}mn}{k}\right)^{(k-2)/2}$ converges, the infinite series (2.13) converges uniformly. Thus by differentiating Eq. (2.13) [notice also (2.16) and (2.17)] r times and using Leibniz's rule together with (2.15), we can deduce (4.16)–(4.19). $\square$

Proof of Theorem 4.3. To prove (4.11) and (4.12), begin by observing that [see (4.15) and (4.16)]

$$B_{mn}^{(r)}(f(nu);x/n) - S_m^{(r)}(f(u);x)$$

$$= m\left(m-\frac{1}{n}\right)\cdots\left(m-\frac{r-1}{n}\right)\left(I+\frac{x}{n}A\right)^{mn-r} A^r f_m(0) - m^r e^{mxA} A^r f_m(0)$$

$$= m\left(m-\frac{1}{n}\right)\cdots\left(m-\frac{r-1}{n}\right)\left[\left(I+\frac{x}{n}A\right)^{mn-r} - e^{mxA}\right] A^r f_m(0) \tag{4.21}$$

$$+\left[m\left(m-\frac{1}{n}\right)\cdots\left(m-\frac{r-1}{n}\right)-m^r\right] e^{mxA} A^r f_m(0). \tag{4.22}$$

First, we handle the term in (4.21) by using Lemma 3.3. Setting $g_m := A^r f_m$ and $T(t) = e^{tA}$, we find that

$$\begin{aligned}
&\left|\left(I+\frac{x}{n}A\right)^{mn-r} g_m(0) - e^{mxA} g_m(0)\right| \\
&\quad \le (mn-r)\left\|\left(I+\frac{x}{n}A\right) g_m - T\left(\frac{mx}{mn-r}\right) g_m\right\| \\
&\quad \le (mn-r)\left\|T\left(\frac{x}{n}\right) g_m - \left(I+\frac{x}{n}A\right) g_m\right\| + (mn-r) \\
&\qquad \left\|T\left(\frac{mx}{mn-r}\right) g_m - T\left(\frac{x}{n}\right) g_m\right\| \\
&\quad \le \frac{(mn-r)x^2}{2n^2}\left\|A^2 g_m\right\| + (mn-r) \\
&\qquad \left\|\frac{rx}{n(mn-r)} T\left(\frac{x}{n}\right) A g_m + \int_{x/n}^{mx/(mn-r)} \left(\frac{mx}{mn-r} - u\right) T(u) A^2 g_m \mathrm{d}u\right\| \\
&\quad \le (mn-r)\left(\frac{x^2}{2n^2}\left\|A^2 g_m\right\| + \frac{rx}{n(mn-r)}\left\|A g_m\right\| + \frac{r^2x^2}{2n^2(mn-r)^2}\left\|A^2 g_m\right\|\right) \\
&\quad = (mn-r)\left(\frac{x^2}{2n^2}\left\|A^{r+2} f_m\right\| + \frac{rx}{n(mn-r)}\left\|A^{r+1} f_m\right\|\right. \\
&\qquad \left. + \frac{r^2x^2}{2n^2(mn-r)^2}\left\|A^{r+2} f_m\right\|\right), \qquad (4.23)
\end{aligned}$$

where the first inequality follows by (3.5). The third inequality follows by using (3.8) and applying the general Taylor's expansion $T(t)-T(s) = (t-s)T(s)A + \int_s^t (t-u)T(u)A^2 \mathrm{d}u$ (see Lemma 3.3 with $l = 2$). The fourth inequality can be derived from $\|T(x/n)\| \le 1$ and by an almost verbatim extension of the reasoning given in deriving (3.8).

Moreover, the second term, i.e., the term in (4.22), can be dealt with as follows:

$$\begin{aligned}
&\left|\left[m(m-\frac{1}{n})\cdots(m-\frac{r-1}{n}) - m^r\right] e^{mxA} A^r f_m(0)\right| \\
&\quad \le m^r \left|(1-\frac{1}{mn})\cdots(1-\frac{r-1}{mn}) - 1\right| \cdot \left|e^{mxA} A^r f_m(0)\right| \qquad (4.24) \\
&\quad \le m^r \frac{r(r-1)}{2mn}\left\|A^r f_m\right\| = \frac{m^{r-1} r(r-1)}{2n}\left\|A^r f_m\right\|, \qquad (4.25)
\end{aligned}$$

where the last inequality follows from the generalized Bernoulli inequality: $1-(1-x_1)(1-x_2)\cdots(1-x_n) \le x_1 + x_2 + \cdots + x_n, 0 < x_i < 1\ (1 \le i \le n)$.

Summing (4.21), (4.22) together with (4.23), (4.25) and noting that $m(m-1/n)\cdots(m-r/n) \le m^{r+1}$ yields

$$\begin{aligned}&\left|B_{mn}^{(r)}\left(f(nu);x/n\right)-S_m^{(r)}(f(u);x)\right|\\&\quad\le\frac{m^{r+1}}{2n}\left(x^2\left\|A^{r+2}f_m\right\|+\frac{2rx}{m-r/n}\left\|A^{r+1}f_m\right\|+\frac{r^2x^2}{(mn-r)^2}\left\|A^{r+2}f_m\right\|\right)\\&\qquad+\frac{m^{r-1}r(r-1)}{2n}\left\|A^rf_m\right\|.\end{aligned}\tag{4.26}$$

By Remark 3.2, $\|A\|\le 2$, so $\|A^k\|\le 2^k$ $(k\in\mathbb{N})$. Therefore, (4.11) follows from (4.26). To derive (4.12), observe from (3.9) that $\|A^rf_m\|\le\|f^{(r)}\|/m^r$. Now (4.12) follows since $\|A^{r+2}f_m\|\le\|A^2\|\cdot\|A^rf_m\|\le 4\|f^{(r)}\|/m^r$ and $\|A^{r+1}f_m\|\le\|A\|\cdot\|A^rf_m\|\le 2\|f^{(r)}\|/m^r$.

Next, we shall prove (4.13). Note that by (2.5) and the definition of divided difference together with (4.6) and (4.7), for $0\le j\le r$

$$\begin{aligned}\left|e^{mxA}A^{k+r-j}f_m(0)\right|&=\left|S_m\left(\Delta_{1/m}^{k+r-j}(f,u);x\right)\right|\\&\le C2^{k+r-j}e^{\alpha(k+r-j)/m}S_m(e^{\alpha u};x)\\&\le C2^{k+r}e^{\alpha(k+r)/m}e^{(e^{\alpha/m}-1)mx}.\end{aligned}\tag{4.27}$$

By invoking (4.18), Lemmas 3.1 and (4.27), we estimate that

$$\begin{aligned}&\left|B_{mn}^{(r)}\left(f(nu);x/n\right)-S_m^{(r)}(f(u);x)\right|\\&\quad=\left|\sum_{k=2}^{\infty}(-1)^k\frac{b_k(mn)}{n^k}\sum_{j=0}^{r}\binom{r}{j}(k)_jx^{k-j}\mathbb{I}_{\{k\ge j\}}m^{r-j}S_m\left(\Delta_{1/m}^{k+r-j}(f,u);x\right)\right|\\&\quad\le\sum_{k=2}^{\infty}\frac{1}{n^k}\left|b_k(mn)\right|\sum_{j=0}^{r}\binom{r}{j}(k)_jx^{k-j}\mathbb{I}_{\{k\ge j\}}m^{r-j}\left|S_m\left(\Delta_{1/m}^{k+r-j}(f,u);x\right)\right|\\&\quad\le\sum_{k=2}^{\infty}\frac{1}{n^k}\left(\frac{emn}{k}\right)^{k/2}\cdot k^r\cdot\max\{1,x\}^k\cdot 2^r\cdot m^r\cdot C2^{k+r}e^{\alpha(k+r)/m}e^{(e^{\alpha/m}-1)mx}\\&\quad=C(4m)^re^{\alpha r/m}e^{(e^{\alpha/m}-1)mx}\sum_{k=2}^{\infty}\left(\frac{4e^{1+2\alpha/m}m\max\{1,x^2\}}{nk}\right)^{k/2}\cdot k^r\\&\quad\le C\frac{e^{1+(2+r)\alpha/m+(e^{\alpha/m}-1)mx}(4m)^{r+1}\max\{1,x^2\}}{2n}\\&\qquad\sum_{k=2}^{\infty}\left(\frac{4e^{1+2\alpha/m}m\max\{1,x^2\}}{nk}\right)^{(k-2)/2}\cdot k^r,\end{aligned}$$

where in the second inequality we apply these inequalities: for $0\le j\le r$, $m^{r-j}\le m^r,(k)_j\le k^r,x^{k-j}\le\max\{1,x\}^k$ $(x\ge 0)$ and $\sum_{j=0}^{r}\binom{r}{j}=2^r$. This complete the proof of (iii). □

Remark 4.6. *Originally, we proved inequality* (4.12). *But we notice that if higher smoothness of f is assumed, then* (4.26) *implies stronger estimates* *than* (4.12). *Next we present two cases. Note that* $\|A^{r+1}f_m\| \le \|f^{r+1}\|/m^{r+1}$, $\|A^{r+2}f_m\| \le \|A\| \cdot \|A^{r+1}f_m\| \le 2\|f^{r+1}\|/m^{r+1}$ *for* $f \in C^{r+1}[0,\infty)$ *and* $\|A^{r+2}f_m\| \le \|f^{r+2}\|/m^{r+2}$ *for* $f \in C^{r+2}[0,\infty)$. *Therefore it follows from* (4.26) *that*

(ii)-a. *If* $f \in C_B^{r+1}[0,\infty)$, *then*

$$\left|B_{mn}^{(r)}(f(nu);x/n) - S_m^{(r)}(f(u);x)\right| \le \frac{x}{n}\left(x+\frac{r}{m-r/n} + \frac{r^2x}{(mn-r)^2}\right)\|f^{(r+1)}\| + \frac{r(r-1)}{2mn}\|f^{(r)}\|. \tag{4.28}$$

(ii)-b. *If* $f \in C_B^{r+2}[0,\infty)$, *then*

$$\begin{aligned}&\left|B_{mn}^{(r)}(f(nu);x/n) - S_m^{(r)}(f(u);x)\right| \\ &\le \frac{x}{2mn}\left(x\|f^{(r+2)}\| + \frac{2r}{1-r/mn}\|f^{(r+1)}\| + \frac{r^2x}{(mn-r)^2}\|f^{(r+2)}\|\right) \\ &+ \frac{r(r-1)}{2mn}\|f^{(r)}\|.\end{aligned} \tag{4.29}$$

Proof of Theorem 4.4. By reasoning as in the proof of Theorem 4.3 and taking (4.19) into account, we have

$$\begin{aligned}&\left|n\left(B_{mn}^{(r)}\left(f(nu);\frac{x}{n}\right) - S_m^{(r)}(f(u);x)\right) + \frac{1}{2}\frac{d^r}{dx^r}\left(mx^2S_m(\Delta_{1/m}^2(f,u);x)\right)\right| \\ &= \left|\sum_{k=3}^{\infty}(-1)^k\frac{b_k(mn)}{n^{k-1}}\sum_{j=0}^{r}\binom{r}{j}(k)_j x^{k-j}\mathbb{I}_{\{k\ge j\}}m^{r-j}S_m\left(\Delta_{1/m}^{k+r-j}(f,u);x\right)\right| \\ &\le Cn(4m)^r e^{\alpha r/m}e^{(e^{\alpha/m}-1)mx}\sum_{k=3}^{\infty}\left(\frac{4e^{1+2\alpha/m}m\max\{1,x^2\}}{nk}\right)^{k/2}\cdot k^r \\ &\le C\frac{2^{2r+3}m^{r+3/2}e^{3/2+(3+r)\alpha/m+(e^{\alpha/m}-1)mx}\max\{1,x^3\}}{3\sqrt{3n}} \\ &\quad \sum_{k=3}^{\infty}\left(\frac{4e^{1+2\alpha/m}m\max\{1,x^2\}}{nk}\right)^{(k-3)/2}\cdot k^r.\end{aligned}$$

□

The actual computation of the operators $S_m(f,x)$ requires estimating infinite series, which restricts in practice the usefulness of these operators. Many authors suggest using a partial sum of $S_m(f,x)$ to approximate $f(x)$ (see [16,20,22,28,32,36]). For example, Grof [16] introduces and investigates the operator

$$S_{n,m}(f;x) = e^{-nx}\sum_{k=0}^{m}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \tag{4.30}$$

for which one has

Theorem 4.7. *Let $m(n)$ be a sequence of positive integers with* $\lim\limits_{n\to\infty} m(n)/n = \infty$*, and let $f \in E_\alpha[0,\infty)$. Then*

$$\lim_{n\to\infty} S_{n,m(n)}(f;x) = f(x). \tag{4.31}$$

Later, Lehnhoff proves that if $m(n) = n$, then $\lim_{n\to\infty} S_{n,n}(f,x) = f(x), f \in C[0,1]$ ([20] Theorem 5). He also asks what happens if the sequence $m(n)/n$ does not tend to infinity. Lehnhoff proves the following theorem [20]

Theorem 4.8. *Let $m = [n(x+\delta)]$, where $\delta = \delta(n)$ is a sequence of positive numbers with $\lim_{n\to\infty} n^{1/2}\delta(n) = \infty$. Then $\{S_{n,m}\}_{n\in\mathbb{N}}$ is a sequence of positive linear operators from $E_\alpha[0,\infty)$ to $C[0,\infty)$ such that for $f \in E_\alpha[0,\infty)$*

$$\lim_{n\to\infty} S_{n,m}(f;x) = f(x), \tag{4.32}$$

uniformly on every interval $[x_1,x_2], 0 \le x_1 < x_2 < \infty$.

Theorem 4.8 was later refined by Omey [22] and Xie [32]. However, from Theorems 4.3 and 4.4 we have seen that the operators $B_{mn}(f(nu),x/n)$ can be used as another finite sum replacement for $S_m(f,x)$. Moreover $B_{mn}(f(nu),x/n)$ is easier to handle than $S_{m,n}(f,x)$. Indeed, by using the Korovkin's theorem it is straightforward to prove that $\lim_{m\to\infty} B_{mn}(f(nu),x/n) = f(x), f(x) \in C[0,n]$. Thus, we have derived the following results:

$$B_{mn}(f(nu);x/n) \to \begin{cases} S_m(f;x), & n\to\infty, f(x)\in E_\alpha[0,\infty), \\ f(x), & m\to\infty, f(x)\in C[0,n]. \end{cases}$$

Moreover,

Corollary 4.9. *Let $m := m(n)$ be a sequence of positive integers with $\lim_{n\to\infty} m(n) = \infty$ and $\lim_{n\to\infty} m(n)/n = 0$. Then for $f \in E_\alpha[0,\infty)$*

$$\lim_{n\to\infty} B_{m(n)n}(f(nu);x/n) = f(x), \tag{4.33}$$

uniformly on every interval $[x_1,x_2], 0 \le x_1 < x_2 < \infty$.

Proof. Clearly

$$\begin{aligned} B_{m(n)n}(f(nu);x/n) - f(x) &= B_{m(n)n}(f(nu);x/n) - S_{m(n)}(f(u);x) \\ &\quad + S_{m(n)}(f(u);x) - f(x), \end{aligned} \tag{4.34}$$

Taking the limit of both sides of (4.34) as $n\to\infty$, and involving (4.3) and (1.3) yields the desired result. □

Moreover, Sun [27] (corollary 1) derives

Theorem 4.10 [27]. *Let $r \in \mathbb{N}^+$. Then for $f \in E_\alpha[0,\infty)$*

$$\lim_{n\to\infty} S_n^{(r)}(f;x) = f^{(r)}(x), \tag{4.35}$$

uniformly on every interval $[x_1,x_2], 0 \le x_1 < x_2 < \infty$.

Therefore, by Theorem 4.3 [see (4.13)] and Theorem 4.10, we conclude that

Corollary 4.11. *Let* $r \in \mathbb{N}^+$ *and let* $m := m(n)$ *be a sequence of positive integers with* $\lim_{n\to\infty} m(n) = \infty$ *and* $\lim_{n\to\infty} m^{r+1}(n)/n = 0$. *Then for* $f \in E_\alpha[0,\infty)$

$$\lim_{n\to\infty} B^{(r)}_{m(n)n}(f(nu); x/n) = f^{(r)}(x), \tag{4.36}$$

uniformly on every interval $[x_1, x_2], 0 \le x_1 < x_2 < \infty$.

5. Convergence of $L_n(f, x)$

Lototsky–Bernstein operators $L_n(f; x)$ were first introduced by King in [18], where he discusses conditions on the sequence of real-valued functions $p_j(x)$ which ensure that $L_n(f; x)$ converges uniformly to $f(x)$. Later, Eisenberg and Wood [9] discuss uniform approximation of analytic functions by means of the Lototsky–Bernstein operators $L_n(f, x)$. These articles [9,18] are the only two papers we can find that are devoted to the convergence of $L_n(f, x)$. Recently, the authors of this paper discuss shape preserving properties of the operators $L_n(f, x)$. In [35], many interesting properties, such as fixed points, iteration, and total positivity of Lototsky–Bernstein bases $\{b_{n,k}(x)\}$, as well as monotonicity preservation and convexity preservation, are systematically investigated. At the same time, we find that for any strictly increasing function $p(x) \in C[0, 1]$ with $p(0) = 0$ and $p(1) = 1$, there exist Lototsky–Bernstein operators $L_n(f, x)$ (by choosing suitable $p_j(x), 1 \le j \le n$) that satisfy the following two properties:

(i) $L_n(f, x)$ preserve the constant function 1 and $p(x)$,

(ii) $\lim_{n\to\infty} L_n(f, x) = f(x)$ uniformly on $[0, 1]$ for all $f \in C[0, 1]$.

In this section, we shall use the difference representation (2.6) and the semigroup representation (2.18) to further study the convergence of the operators $L_n(f, x)$. Our first theorem [35] concerns convergence of the operators $L_n(f, x)$ without any additional restrictions on $p_i(x)$ $(1 \le i \le n)$ except the assumptions that for all i, $0 < p_i(x) < 1$ for $x \in (0, 1)$ and $p_i(0) = 0, p_i(1) = 1$. In this Section, we fix once and for all the notation $\lambda(x) = \sum_{k=1}^n p_k(x), \lambda_2(x) = \sum_{k=1}^n p_k^2(x)$.

The following theorem has been proved in [35].

Theorem 5.1 [35]. *Suppose that* $p_i(x) \in C[0, 1]$, $0 < p_i(x) < 1$ *for* $x \in (0, 1)$ *and* $p_i(0) = 0, p_i(1) = 1$ $(i \ge 1)$. *If* $(p_1(x) + \cdots + p_n(x))/n$ *converges, then for* $f \in C[0, 1]$ *the positive sequence* $\{L_n(f; x)\}_{n=1}^\infty$ *converges to a bounded function* $L_\infty(f; x)$, *where*

$$L_\infty(f; x) = \lim_{n\to\infty} L_n(f; x) = f\left(\lim_{n\to\infty} \frac{p_1(x) + \cdots + p_n(x)}{n}\right). \tag{5.1}$$

Moreover, if $\lim_{n\to\infty}(p_1(x)+\cdots+p_n(x))/n$ *converges uniformly on* $[0, 1]$, *then the convergence in* (5.1) *is uniform on* $[0, 1]$.

Conversely, if for some strictly monotone function $f_0 \in C[0,1]$, the positive sequence $\{L_n(f_0;x)\}_{n=1}^{\infty}$ converges, then $(p_1(x)+\cdots+p_n(x))/n$ also converges.

Let $e_1(x) = x$ and $e_2(x) = x^2$. Then it is straightforward using (2.6) (see also [18]) to derive that

$$L_n(e_1;x) = \sum_{i=1}^{n} p_i(x)/n,$$
$$L_n(e_2;x) = \left(\sum_{i=1}^{n} p_i(x)\right)^2 /n^2 - \sum_{i=1}^{n} p_i^2(x)/n^2 + \sum_{i=1}^{n} p_i(x)/n^2. \tag{5.2}$$

Remark 5.2. *Let $\{L_n^*\}_{n\geq 1}$ be a sequence of positive linear operators on $C[0,1]$, and let $e_i(x) = x^i, i \geq 1$. Wang [30] (Theorem 2 in [30]) shows that if $\{L_n^*\}_{n\geq 1}$ satisfy the following two conditions:*

(a) *the sequence $\{L_n^*(e_2)\}$ converges to a function $L_\infty^*(e_2)$ in $C[0,1]$,*

(b) *$\{L_n^*\}_{n\geq 1}$ satisfies $L_n^*(f) \geq L_{n+1}^*(f)$ for every convex function f, then there exists a bounded function $L_\infty^*(f,x)$ such that $\lim_{n\to\infty} L_n^*(f,x) = L_\infty^*(f,x)$.*

Now if $(p_1(x)+\cdots+p_n(x))/n$ converges, then from (5.2) we find that $L_n(e_2) \to L_\infty(e_2) := (\lim_{n\to\infty}(p_1(x)+\cdots+p_n(x))/n)^2$. However, we can show by example that the inequality $L_n(f,x) \geq L_{n+1}(f)$ does not generally hold for every convex function f and every $x \in [0,1]$. Indeed, note that the inequality

$$L_n(e_1) - L_{n+1}(e_1) = \sum_{k=1}^{n} (p_k(x) - p_{n+1}(x))/(n(n+1)) \geq 0,$$

does not hold for $p_k(x) = x^k$ $(1 \leq k \leq n), p_{n+1}(x) = \sqrt{x}$. Therefore, Wang's Theorem [30] fails to guarantee convergence of the operators $L_n(f,x)$ for general $p_i(x)$ $(1 \leq i \leq n)$.

Corollary 5.3 [18]. *Under the same hypotheses and with the same notation as in Theorem 5.1: If $\lim_{n\to\infty}(p_1(x)+\cdots+p_n(x))/n = x$ uniformly on $[0,1]$, then*

$$\lim_{n\to\infty} L_n(f;x) = f(x) \tag{5.3}$$

uniformly on $[0,1]$.

Like the classical Bernstein operators, we have a Voronovskaja type theorem for the Lototsky–Bernstein operators.

In order to derive a Voronovskaja type theorem for the classical Bernstein-type operators, we usually follow Davis [5] (p. 117–118), beginning with the sums

$$T_{n,r}(x) = \sum_{k=0}^{n} (k - n\mu(x))^r b_{n,k}(x) = n^r \sum_{i=0}^{r} \binom{r}{i} (-\mu(x))^{r-i} L_n(e_i;x). \tag{5.4}$$

By (2.6)

$$L_n(e_i;x) = \prod_{k=1}^{n}(I + p_i(x)A)(e_i)_n(0) = \sum_{j=0}^{i}\sigma_j(p_1(x), \cdots, p_n(x))A^j(e_i)_n(0), \tag{5.5}$$

where $\sigma_j(u_1, \ldots, u_n)$ is the jth elementary symmetric function in the variables $u_1, \ldots, u_n$ (see [35]). Also by (2.14) and (2.15)

$$A^j(e_i)_n(0) = \Delta^j_{1/n}(e_i, 0) = \sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}(k/n)^i = j!S(i,j)/n^i, \tag{5.6}$$

where $S(i,j)$ is the Stirling number of the second kind. Combining (5.4), (5.5) and (5.6)

$$T_{n,r}(x) = \sum_{i=0}^{r}\sum_{j=0}^{i}\binom{r}{i}(-\mu(x))^{r-i}j!S(i,j)n^{r-i}\sigma_j(p_1(x), \ldots, p_n(x)). \tag{5.7}$$

Along Davis's line, we are concerned with the dependence of $T_{n,6}(x)$ on n. Davis's proof is valid only when we can prove $|T_{n,6}(x)| \leq cn^4$ for $x \in [0,1]$. However, since the $p_i(x)$ are mutually independent, it is quite hard to derive a bound on $T_{n,6}(x)$ in terms of powers of n. Instead, we will use (2.18) to achieve our purpose.

Recall the classical Voronovskaja type theorem for Szász–Mirakjan operators [15,17]

$$\lim_{n\to\infty} n(S_n(f;x) - f(x)) = \frac{1}{2}xf''(x), f \in E_\alpha[0,\infty). \tag{5.8}$$

This result will be used in the proof of the following theorem, where we will also use the modulus of continuity of f defined by

$$\omega(f,t) := \sup_{|x-y|\leq t, x,y\in[0,1]} |f(x) - f(y)|.$$

Theorem 5.4. *Suppose that $p_i(x) \in C[0,1]$, $0 < p_i(x) < 1$ for $x \in (0,1)$ and $p_i(0) = 0, p_i(1) = 1$ $(i \geq 1)$. If the following three limits exist*

$$\mu(x) := \lim_{n\to\infty} \lambda(x)/n, \tag{5.9}$$

$$\nu(x) =: \lim_{n\to\infty} \lambda_2(x)/n, \tag{5.10}$$

$$\eta(x) =: \lim_{n\to\infty} (\lambda(x) - n\mu(x)), \tag{5.11}$$

then for any $f \in C^2[0,1]$, the rate of convergence of the positive linear sequence of Lototsky–Bernstein operators is governed by

$$\lim_{n\to\infty} n(L_n(f;x) - f(\mu(x))) = \frac{1}{2}(\mu(x) - \nu(x))f''(\mu(x)) + \eta(x)f'(\mu(x)). \tag{5.12}$$

Proof. By (2.18)

$$
\begin{aligned}
&L_n(f;x) - f(\mu(x)) \\
&\quad = e^{\lambda(x)A} f_n(0) - f(\mu(x)) - \frac{1}{2}\lambda_2(x)e^{\lambda(x)A}A^2 f_n(0) + \sum_{k=3}^{\infty} a_k(-A)^k e^{\lambda(x)A} f_n(0) \\
&\quad = \left[e^{\lambda(x)A} f_n(0) - e^{n\mu(x)A} f_n(0)\right] + \left[e^{n\mu(x)A} f_n(0) - f(\mu(x))\right] \\
&\quad - \frac{1}{2}\lambda_2(x)e^{\lambda(x)A}A^2 f_n(0) + \sum_{k=3}^{\infty} a_k(-A)^k e^{\lambda(x)A} f_n(0) =: I_1 + I_2 + I_3 + I_4.
\end{aligned}
\tag{5.13}
$$

By Lemma 3.3, I_1 can be handled as follows:

$$
\begin{aligned}
I_1 &= e^{\lambda(x)A} f_n(0) - e^{n\mu(x)A} f_n(0) \\
&= (\lambda(x) - n\mu(x))e^{n\mu(x)A}Af_n(0) + \int_{n\mu(x)}^{\lambda(x)} (\lambda(x) - u)e^{uA}A^2 f_n(0)du.
\end{aligned}
\tag{5.14}
$$

Now we deal with the integral part in (5.14) by using (3.9) and the contraction of the operator $T(u) = e^{uA}$ ($\|e^{uA}\| \le 1$)

$$
\begin{aligned}
\left|\int_{n\mu(x)}^{\lambda(x)} (\lambda(x) - u)e^{uA}A^2 f_n(0)du\right| &\le \left\|\int_{n\mu(x)}^{\lambda(x)} (\lambda(x) - u)e^{uA}A^2 f_n du\right\| \\
&\le \int_{n\mu(x)}^{\lambda(x)} (\lambda(x) - u)\left\|e^{uA}A^2 f_n\right\| du \\
&\le \|A^2 f_n\| \int_{n\mu(x)}^{\lambda(x)} (\lambda(x) - u)du \le \frac{(\lambda(x) - n\mu(x))^2}{2n^2}\|f''\|.
\end{aligned}
\tag{5.15}
$$

To analyze the first term on the right-hand side of (5.14), consider Taylor expansion

$$
\begin{aligned}
Af_n(k) &= f((k+1)/n) - f(k/n) \\
&= f'(k/n)/n + f''(\xi_k)/(2n^2), k/n < \xi_k < (k+1)/n, k \ge 0.
\end{aligned}
\tag{5.16}
$$

Thus by (1.7) and (1.3) as $n \to \infty$

$$
ne^{n\mu(x)A}Af_n(0) = ne^{-n\mu(x)} \sum_{k=0}^{\infty} \frac{(n\mu(x))^k}{k!} \tag{5.17}
$$

$$
[f'(k/n)/n + f''(\xi_k)/(2n^2)] \to f'(\mu(x)). \tag{5.18}
$$

Combining (5.14)–(5.18), we conclude that

$$
\lim_{n\to\infty} nI_1 = \eta(x)f'(\mu(x)). \tag{5.19}
$$

By (5.8), it is obvious that

$$
\lim_{n\to\infty} nI_2 = \lim_{n\to\infty} n(S_n(f, \mu(x)) - f(\mu(x))) = \frac{1}{2}\mu(x)f''(\mu(x)). \tag{5.20}
$$

To analyze I_3, observe that as in (5.14) and (5.15), we have

$$n\left[e^{\lambda(x)A}A^2f_n(0)-e^{n\mu(x)A}A^2f_n(0)\right]=O(1/n). \tag{5.21}$$

Moreover, using (3.9) yields

$$\begin{aligned}A^2f_n(k)&=f((k+2)/n)-2f((k+1)/n)+f(k/n)\\&=f''(\varsigma_k)/n^2, k/n<\varsigma_k<(k+2)/n, k\geq 0,\end{aligned} \tag{5.22}$$

so

$$\begin{aligned}n^2e^{n\mu(x)A}A^2f_n(0)&=e^{-n\mu(x)}\sum_{k=0}^{\infty}\frac{(n\mu(x))^k}{k!}f''(\varsigma_k)\\&=e^{-n\mu(x)}\sum_{k=0}^{\infty}\frac{(n\mu(x))^k}{k!}f''(k/n)+e^{-n\mu(x)}\\&\quad\times\sum_{k=0}^{\infty}\frac{(n\mu(x))^k}{k!}[f''(\varsigma_k)-f''(k/n)].\end{aligned} \tag{5.23}$$

However, since $0<\varsigma_k-k/n<2/n$

$$\begin{aligned}&\left|e^{-n\mu(x)}\sum_{k=0}^{\infty}\frac{(n\mu(x))^k}{k!}[f''(\varsigma_k)-f''(k/n)]\right|\\&\quad\leq e^{-n\mu(x)}\sum_{k=0}^{\infty}\frac{(n\mu(x))^k}{k!}\left|f''(\varsigma_k)-f''(k/n)\right|\\&\quad\leq e^{-n\mu(x)}\sum_{k=0}^{\infty}\frac{(n\mu(x))^k}{k!}\omega(f'',2/n)=\omega(f'',2/n)\to 0, n\to\infty.\end{aligned} \tag{5.24}$$

Now (5.23) and (5.24) together with (1.7) and (1.3) yields

$$n^2e^{n\mu(x)A}A^2f_n(0)\to f''(\mu(x)). \tag{5.25}$$

Therefore, by (5.21) and (5.25)

$$\begin{aligned}nI_3&=-\frac{1}{2}\lambda_2(x)ne^{\lambda(x)A}A^2f_n(0)\\&=-\frac{1}{2}\lambda_2(x)n\left[e^{\lambda(x)A}A^2f_n(0)-e^{n\mu(x)A}A^2f_n(0)\right]\\&\quad-\frac{1}{2}\lambda_2(x)/n\cdot n^2e^{n\mu(x)A}A^2f_n(0)\\&\to-\frac{1}{2}\nu(x)f''(\mu(x)).\end{aligned} \tag{5.26}$$

Finally, we need to estimate the remainder term I_4. By (3.2) and (3.9), for $0\leq x<1$

$$\begin{aligned}
|I_4| &= \left|\sum_{k=3}^{\infty} a_k(-A)^k e^{\lambda(x)A} f_n(0)\right| = \left|\sum_{k=3}^{\infty} a_k(-A)^{k-2} e^{\lambda(x)A} A^2 f_n(0)\right| \\
&\le \sum_{k=3}^{\infty} |a_k| \cdot \left\|(-A)^{k-2} e^{\lambda(x)A}\right\| \cdot \|A^2 f_n\| \le c_0 \\
&\quad \sum_{k=3}^{\infty} \left(\frac{e\lambda_2(x)}{k}\right)^{k/2} \left(\frac{(k-2)}{e\lambda(x)}\right)^{(k-1)/2} \|f''\|/n^2 \\
&\le c_0 \sum_{k=3}^{\infty} \left(\frac{\lambda_2(x)}{\lambda(x)}\right)^{(k-1)/2} \left(\frac{e\lambda_2(x)}{nk}\right)^{1/2} \|f''\|/n^{3/2} \\
&\le \frac{c_0\|f''\|}{n^{3/2}} \left(\frac{e}{3}\right)^{1/2} \frac{\lambda_2(x)/\lambda(x)}{1-\sqrt{\lambda_2(x)/\lambda(x)}},
\end{aligned} \tag{5.27}$$

where $c_0 = \dfrac{\sqrt{e}(1+\sqrt{\pi/2})}{2}$. Thus, from (5.27), for $0 \le x < 1$

$$\lim_{n\to\infty} nI_4 = 0. \tag{5.28}$$

Combining (5.13),(5.19),(5.20),(5.26) and (5.28), we have derived the desired result (5.12) for $0 \le x < 1$. Since $L_n(f;1) = f(1)$, the case $x = 1$ is immediate. □

Corollary 5.5. *Under the same hypotheses and with the same notation as in Theorem 5.4: Let $\mu(x) = x$, then*

$$\lim_{n\to\infty} n\left(L_n(f;x) - f(x)\right) = \frac{1}{2}(x - \nu(x))f''(x) + \eta(x)f'(x). \tag{5.29}$$

Corollary 5.6. *Under the same hypotheses and with the same notation as in Theorem 5.4: If $p_i(x) = u_n(x)$ $(1 \le i \le n)$ and $\lim_{n\to\infty} u_n(x) = x$, then*

$$\lim_{n\to\infty} n(L_n(f;x) - f(x)) = \frac{1}{2}(x - x^2)f''(x) + \eta(x)f'(x). \tag{5.30}$$

Remark 5.7. *If $p_i(x) = u_n(x)$ $(1 \le i \le n)$, then the Lototsky–Bernstein operators $L_n(f,x)$ reduce to King-type operators (see [1,3,4,13,14,19]). Therefore, Theorem 5.4 is a generalization of the corresponding results for King-type operators. For more details, see Theorem 2 in [1], Section 3.3 in [3] and Theorem 5.1 in [14].*

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant Nos. 61572020,11601266).

References

[1] Birou, M.M.: Bernstein type operators with a better approximation for some functions. Appl. Math. Comput. **219**, 9493–9499 (2013)

[2] Bustamante, J.: Bernstein Operators and Their Properties. Birkhäuser, Basel (2017)

[3] Cárdenas-Morales, D., Garrancho, P., Muñoz-Delgado, F.J.: Shape preserving approximation by Bernstein-type operators which fix polynomials. Appl. Math. Comput. **182**, 1615–1622 (2006)

[4] Cárdenas-Morales, D., Garrancho, P., Rasa, I.: Bernstein-type operators which preserve polynomials. Comput. Math. Appl. **62**, 158–163 (2011)

[5] Davis, P.J.: Interpolation and Approximation. Dover, New York (1975)

[6] Deheuvels, P., Pfeifer, D.: A semigroup approach to Poisson approximation. Ann. Prob. **14**, 663–676 (1986)

[7] Deheuvels, P., Pfeifer, D., Puri, M.L.: A new semigroup technique in Poisson approximation. Semigroup Forum **38**, 189–201 (1989)

[8] De La Cal, J., Luquin, F.: A note on limiting properties of some Bernstein-type operators. J. Approx. Theory **68**, 322–329 (1992)

[9] Eisenberg, S., Wood, B.: Approximation of analytic functions by Bernstein-type operators. J. Approx. Theory **6**, 242–248 (1972)

[10] Favard, J.: Sur les multiplicateurs d'interpolation. J. Math. Pures Appl. **23**(9), 219–247 (1944)

[11] Goldman, R.: Identities for the Univariate and Bivariate Bernstein Basis Functions. In: Paeth, A. (ed.) Graphics Gems V, pp. 149–162. Academic Press, Cambridge (1995)

[12] Goldman, R.: Pyramid Algorithms: A Dynamic Programming Approach to Curves and Surfaces for Geometric Modeling. Morgan Kaufmann Publishers, San Diego (2002)

[13] Gonska, H., Pitul, P.: Remarks on an article of J. P. King. Comment. Math. Univ. Carolin. **46**(4), 645–652 (2005)

[14] Gonska, H., Pitul, P., Rasa, I.: General king-type operators. Results Math. **53**, 279–286 (2009)

[15] Grof, J.: A Szàsz Ottó-féle operátor approximácios tulajdonságairól. MTA III. Oszt. Közl. **20**, 35–44 (1971)

[16] Grof, J.: Über Approximation durch polynome mit Belegfunktionen. Acta Math. Acad. Sci. Hung. **35**, 109–116 (1980)

[17] Hermann, T.: Approximation of unbounded functions on unbounded interval. Acta Math. Acad. Sci. Hung. **29**, 393–398 (1977)

[18] King, J.P.: The Lototsky transform and Bernstein polynomials. Can. J. Math. **18**, 89–91 (1966)

[19] King, J.P.: Positive linear operators which preserve x^2. Acta Math. Hung. **99**, 203–208 (2003)

[20] Lehnhoff, H.G.: On a modified Szász–Mirakyan-operator. J. Approx. Theory **42**, 278–282 (1984)

[21] Mirakjan, G.M.: Approximation of continuous functions with the aid of polynomials (Russian). Dokl. Akad. Nauk SSSR **31**, 201–205 (1941)

[22] Omey, E.: Note on operators of Szász–Mirakyan type. J. Approx. Theory **47**, 246–254 (1986)

[23] Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, vol. 44. Springer, New York (1983)

[24] Pfeifer, D.: Approximation-theoretic aspects of probabilistic representations for operator semigroups. J. Approx. Theory **43**, 271–296 (1985)

[25] Shiryayev, A.N.: Probability. Springer, New York (1984)

[26] Shorgin, S.Y.: Approximation of a generalized binomial distribution. Theory Probab. Appl. **22**(4), 846–850 (1978)

[27] Sun, X.H.: On the simultaneous approximation of functions and their derivatives by the Szász–Mirakyan operator. J. Approx. Theory **55**, 279–288 (1988)

[28] Sun, X.H.: On the convergence of the modified Szász–Mirakyan operator. Approx. Theory Appl. **10**(1), 20–25 (1994)

[29] Szász, O.: Generalization of Bernstein' s polynomials to the infinite interval. J. Res. Nat. Bur. Stand. **45**, 239–245 (1950)

[30] Wang, H.P.: Korovkin-type theorem and application. J. Approx. Theory **132**, 258–264 (2005)

[31] Witte, H.J.: A unification of some approaches to Poisson approximation. J. Appl. Prob. **27**, 611–621 (1990)

[32] Xie, L.S., Xie, T.F.: Approximation theorems for locallized Szàsz–Mirakjan operators. J. Approx. Theory **152**, 125–134 (2008)

[33] Xu, X.W.: Semigroup structural form for Bernstein-type operators and its applications. Appl. Math. Comput. **271**, 923–934 (2015)

[34] Xu, X.W., Zeng, X.M.: Asymptotic properties of Bernstein–Durrmeyer operators. Results Math. **69**(3), 345–357 (2016)

[35] Xu, X.W., Zeng, X.M., Goldman, R.: Shape preserving properties of univariate Lototsky–Bernstein operators. J. Approx. Theory **224**, 13–42 (2017)

[36] Zhou, G.Z., Zhou, S.P.: A remark on a modified Szász–Mirakyan operator. Colloq. Math. **79**, 157–160 (1999)

Ron Goldman and Xiao-Wei Xu
Department of Computer Science
Rice University
Houston TX77005
USA
e-mail: rng@rice.edu

Xiao-Wei Xu and Xiao-Ming Zeng
School of Mathematical Sciences
Xiamen University
Xiamen 361005
China
e-mail:lampminket@263.net

Xiao-Ming Zeng
e-mail:xmzeng@xmu.edu.cn

Received: July 24, 2017.
Accepted: January 11, 2018.