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Results in Mathematics

Periodic Orbits for Radially Symmetric Systems with Singularities and Semilinear Growth

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Abstract. We are concerned with non-autonomous radially symmetric systems, with certain strong repulsive singularities near the origin and with some semilinear growth near infinity. By use of topological degree theory, we prove the existence of large-amplitude periodic solutions whose minimal period is an integer multiple of T, and these solutions rotate exactly once around the origin in their period time. The result in this paper shows that both of the antiperiodic and the periodic eigenvalues play the same role.

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1. Introduction

In recent few years, Fonda and his coworkers have studied the periodic, subharmonic and quasi-periodic orbits for the radially symmetric system

$$\ddot{x} + f(t, |x|)\frac{x}{|x|} = 0, \quad x \in \mathbb{R}^2\backslash\{0\}, \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}, \mathbb{R}_+ = (0, +\infty)$, is continuous L^1-Carathédory function and T-periodic in the first variable, and exhibits a repulsive singularity near the origin in the second variable. For details, see [13–18]. Setting $r(t) = |x(t)|$, the study of these solutions involves the scalar singular equation

$$\ddot{r} + f(t, r) = 0. \tag{1.2}$$

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In particular, in the paper [15], Fonda and Toader proved the following result.

Theorem 1.1. *Let the following two assumptions hold.*

(A_1) *There are an integer M and two constants α, β such that*

$$\left(\frac{M\pi}{T}\right)^2 < \alpha \leq \liminf_{r\to+\infty} \frac{f(t,r)}{r} \leq \limsup_{r\to+\infty} \frac{f(t,r)}{r} \leq \beta < \left(\frac{(M+1)\pi}{T}\right)^2 \quad (1.3)$$

holds uniformly for every $t \in [0,T]$;

(A_2) *There are a positive constant δ and a continuous function $h : (0,\delta] \to \mathbb{R}$ such that*

$$f(t,r) \leq h(r), \quad \text{for every } t \in [0,T] \text{ and every } r \in (0,\delta],$$

and

$$\lim_{r\to 0^+} h(r) = -\infty, \quad \int_0^\delta h(r)dr = -\infty.$$

Then, Eq. (1.2) has a T-periodic solution, and there exists a $k_1 \geq 1$ such that, for every integer $k \geq k_1$, Eq. (1.1) has a periodic solution $x_k(t)$ with a minimal period kT, which makes exactly one revolution around the origin in the period time kT. Moreover, there is a constant $R > 0$ such that, for every $k \geq k_1$

$$\frac{1}{R} < |x_k(t)| < R, \text{ for every } t \in \mathbb{R},$$

and, if μ_k denotes the angular momentum associated to $x_k(t)$ then

$$\lim_{k\to\infty} \mu_k = 0.$$

Roughly speaking, system (1.1) is singular at 0 means that $f(t,r)$ becomes unbounded when $r \to 0^+$. We say that (1.1) is of repulsive type (attractive type) if $f(t,r) \to -\infty$ (respectively $f(t,r) \to +\infty$) when $r \to 0^+$. As mentioned in [16], such a type of singular systems appears in many problems of applications. Such as, if we take $f(t,r) = c/r^2 (c > 0)$, it is the famous Newtonian equation

$$\ddot{x} + \frac{cx}{|x|^3} = 0, \quad x \in \mathbb{R}^2\backslash\{0\},$$

which describes the motion of a particle subjected to the gravitational attraction of a sun that lies at the origin. If we take $f(t,r) = c/r^2 (c < 0)$, (1.1) may be used to model Rutherford's scattering of α particles by heavy atomic nuclei.

The question of existence of non-collision periodic orbits for scalar equations and dynamical systems with singularities has attracted much attention of many researchers over many years, such as [3,5,11,19,25–28]. There are two main lines of research in this area. The first one is the variational approach [1,31,32]. More general systems is the type

$$\ddot{x}(t) + \nabla_x(t, x(t)) = 0,$$

where $V \in C^1(\mathbb{R} \times \mathbb{R}^N \backslash \{0\}, \mathbb{R})$ is a singular potential. In relation with the variational approach, a common hypothesis is the unboundedness of the action function near the singularity to guarantee that its critical points have no collision with the singularity. Usually, the proof requires some strong force condition, which was first introduced by Gordon in [21]. Roughly, it establishes a maximum rate of growth of the potential near the singularity. For example, if we consider the system

$$\ddot{x} = \frac{1}{|x|^\alpha} + f(t),$$

the strong force condition corresponds to the case $\alpha \geq 2$.

Besides the variational approach, topological methods have been widely applied, starting with the pioneering paper of Lazer and Solimini [22]. In particular, some classical tools have been used to study singular differential equations and dynamical systems in the literature, including the degree theory [13,23,33–35], the method of upper and lower solutions [2,30], Schauder's fixed point theorem [20], some fixed point theorems in cones for completely continuous operators [7,24] and a nonlinear Leray–Schauder alternative principle [6,8]. Contrasting with the variational setting, the strong force condition plays here a different role linked to repulsive singularities. A counterexample in the paper of Lazer and Solimini [22] shows that a strong force assumption (unboundedness of the potential near the singularity) is necessary in some sense for the existence of positive periodic solutions in the scalar case.

For the scalar singular equation (1.2), we recall the following results. When $f(t,r) = f(r) - g(t)$, where $f \in C(\mathbb{R}, \mathbb{R})$ is T-periodic and $g \in C(\mathbb{R}_+, \mathbb{R})$ satisfies the following strong force condition at $r = 0$,

$$\lim_{r\to 0^+} f(r) = -\infty \quad \text{and} \quad \lim_{r\to 0^+} \int_1^r f(r)dr = +\infty$$

and $f(r)$ is superlinear at $r = +\infty$,

$$\lim_{r\to +\infty} \frac{f(r)}{r} = +\infty,$$

Fonda et al. [12] used the Poincaré–Birkhoff theorem to obtain the existence of positive periodic solutions, including all subharmonics. Similarity, when $f(t,r)$ is superlinear at $r = +\infty$ and satisfies the following strong force condition at $r = 0$: There are positive constants c, c', υ such that $\upsilon \geq 1$ and

$$cr^{-\upsilon} \leq -f(t,r) \leq cr^{-\upsilon} \tag{1.4}$$

for every t and every r sufficiently small, del Pino and Manásevich proved in [9] the existence of infinitely many periodic solutions to (1.2).

When $f(t,r)$ is semilinear at $r=+\infty$, del Pino et al. [10] proved the existence of at least one positive T-periodic solution of (1.2) if $f(t,r)$ satisfies (1.4) near $r=0$, and the following nonresonance conditions at $r=+\infty$: There exists $k\in\mathbb{N}$ such that

$$\left(\frac{k\pi}{T}\right)^2<\liminf_{r\to+\infty}\frac{f(t,r)}{r}<\limsup_{r\to+\infty}\frac{f(t,r)}{r}\le\left(\frac{(k+1)\pi}{T}\right)^2. \tag{1.5}$$

Their result, later improved by Yan and Zhang [33], the conditions (1.5) is removed and the existence of at least one positive solutions under suitable nonresonance conditions is obtain by using the topological degree theory.

In [24,25], we considered the behaviour under the first eigenvalue. Our main motivation is to obtain by the above papers [14,15], we also prove the existence of large-amplitude periodic orbits whose minimal period is an integer multiple of T, and rotate exactly once around the origin in their period time. Compared with Theorem 1.1, the main novelty in the paper is represented by the conditions at infinity, which remind of a situation between the higher order eigenvalue, but are more general since the comparison involves the "weighted" eigenvalue associated with the functions controlling the ratio $f(t,r)/r$ (where r is the radial coordinate).

The rest of this paper is organized as follows. In Sect. 2, some preliminary results will be given. In Sect. 3, by use of topological degree theory, we will state and prove the main results.

2. Preliminaries

In this section, we present some results which will be needed in Sect. 3. Recall that the L^1- Carathéodory means

- $t\to f(t,r)$ is measurable and $T-$periodic, for every $r>0$;
- $r\to f(t,r)$ is continuous, for almost every $t\in[0,T]$;
- for every compact interval $[a,b]\subset(0,+\infty)$, there exists a $\ell_{a,b}\in L^1(0,T)$ such that

$$|f(t,r)|\le\ell_{a,b}(t)\quad\text{for a.e. } t\in[0,T] \text{ and every } r\in[a,b].$$

We look for solutions $x(t)\in\mathbb{R}^2$ which never attain the singularity, in the sense that

$$x(t)\ne0,\quad\text{for every } t\in\mathbb{R}.$$

Using the same idea in [15], we may write the solutions of (1.1) in polar coordinates

$$x(t)=r(t)(\cos\varphi(t),\sin\varphi(t)).$$

Then we have the collisionless orbits if $r(t) > 0$ for every t. Moreover, Eq. (1.1) is equivalent to the following system

$$\begin{cases} \ddot{r} + f(t,r) - \frac{\mu^2}{r^3} = 0, \\ r^2\dot{\varphi} = \mu, \end{cases} \tag{2.1}$$

where μ is the angular momentum of $x(t)$. Recall that μ is constant in time along any solution. In the following, when considering a solution of (2.1) we will always implicitly assume that $\mu \geq 0$ and $r > 0$.

If $x(t)$ is a T-radially periodic, then $r(t)$ must be T-periodic. We will look for solutions for which $r(t)$ is T-periodic. We thus consider the boundary value problem

$$\begin{cases} \ddot{r} + f(t,r) = \frac{\mu^2}{r^3}, \\ r(0) = r(T), \quad \dot{r}(0) = \dot{r}(T). \end{cases} \tag{2.2}$$

Let $\mu = 0$, (2.2) reduce to the T-periodic problem (1.2) and let X be a Banach space of functions such that $C^1([0,T]) \subseteq X \subseteq C([0,T])$, with continuous immersions, and set $X_* = \{r \in X : \min_t r(t) > 0\}$.

Define the space:

$$D(L) = \{r \in W^{2,1}(0,T) : r(0) = r(T), \dot{r}(0) = \dot{r}(T)\},$$

and the following two operators:

$$L : D(L) \subset X \to L^1(0,T), \quad (Lr)(t) = \ddot{r}(t),$$

$N : X_* \to L^1(0,T)$ is the Nemyskii operator associated with f:

$$(Nr)(t) = -f(t, r(t)).$$

The Carathéodory condition implies N is continuous. Taking $\sigma \in \mathbb{R}$ not belonging to the spectrum of L, the T-periodic for Eq. (1.2) is thus equivalent to the operator equation

$$Lr = Nr,$$

which is also can be translated to

$$r - (L - \sigma I)^{-1}(N - \sigma I)r = 0,$$

since $L - \sigma I$ is invertible.

Recall that a set $\Omega \subseteq X$ is uniformly positively bounded below if there is a constant $\delta > 0$ such that $\min r \geq \delta$ for every $r \in \Omega$. In order to prove the main result of this paper, we need the following lemma, which was proved by Fonda and Toader [13].

Lemma 2.1. *Let Ω be an open bounded subset of X, uniformly positively bounded below. Assume that there is no solution of (1.2) on the boundary $\partial\Omega$, and that*

$$\deg(I - (L - \sigma I)^{-1}(N - \sigma I), \Omega, 0) \neq 0.$$

Then, there exists a $k_2 \geq 1$ such that, for every integer $k \geq k_2$, system (1.1) has a periodic solution $x_k(t)$ with minimal period kT, which makes exactly one revolution around the origin in the period time kT. The function $|x_k(t)|$ is T-periodic and, when restricted to $[0,T]$, it belongs to Ω. Moreover, if μ_k denotes the angular momentum associated to $x_k(t)$, then

$$\lim_{k\to\infty} \mu_k = 0.$$

3. Main Results

This section is dedicated to the main results. Let us first introduce some known results on eigenvalues. Let $q(t)$ be a T-periodic potential such that $q \in L^1(\mathbb{R})$. Consider the eigenvalue problems of

$$r'' + (\lambda + q(t))r = 0 \tag{3.1}$$

subject to the periodic boundary condition:

$$r(0) - r(T) = r'(0) - r'(T) = 0, \tag{3.2}$$

or, to the antiperiodic boundary condition:

$$r(0) + r(T) = r'(0) + r'(T) = 0. \tag{3.3}$$

We use $\lambda_1^D(q) < \lambda_2^D(q) < \cdots < \lambda_n^D(q) < \cdots$ to denote all eigenvalues of (3.1) with the Dirichlet boundary condition:

$$r(0) = r(T) = 0. \tag{3.4}$$

These eigenvalues, as a whole, are called the characteristic values of (3.1), the following are the standard results. See, e.g. Reference [29].

There exist two sequences $\{\underline{\lambda}_n(q) : n \in \mathbb{Z}^+\}$ and $\{\overline{\lambda}_n(q) : n \in \mathbb{N}\}$ such that

- (E$_1$) $-\infty < \overline{\lambda}_0(q) < \underline{\lambda}_1(q) \leq \overline{\lambda}_1(q) < \underline{\lambda}_2(q) \leq \overline{\lambda}_2(q) < \cdots < \underline{\lambda}_n(q) \leq \overline{\lambda}_n(q) < \cdots$ and $\underline{\lambda}_n(q) \to +\infty, \overline{\lambda}_n(q) \to +\infty$ as $n \to +\infty$.
- (E$_2$) λ is an eigenvalue of (3.1)–(3.2) if and only if $\lambda = \underline{\lambda}_n(q)$ or $\overline{\lambda}_n(q)$ if n is even; and λ is an eigenvalue of (3.1)–(3.3) if and only if $\lambda = \underline{\lambda}_n(q)$ or $\overline{\lambda}_n(q)$ if n is odd.
- (E$_3$) The comparison results hold for all of these eigenvalues. If $q_1 \leq q_2$ then
$$\underline{\lambda}_n(q_1) \geq \underline{\lambda}_n(q_2), \overline{\lambda}_n(q_1) \geq \overline{\lambda}_n(q_2), \lambda_n^D(q_1) \geq \lambda_n^D(q_2)$$
for any $n \geq 1$.
- (E$_4$) For any $n \geq 1$,
$$\underline{\lambda}_n(q) = \min\{\lambda_n^D(q_{t_0}) : t_0 \in \mathbb{R}\}, \overline{\lambda}_n(q) = \max\{\lambda_n^D(q_{t_0}) : t_0 \in \mathbb{R}\}$$
where $q_{t_0}(t)$ denotes the translation of $q(t) : q_{t_0}(t) \equiv q(t + t_0)$.
- (E$_5$) $\overline{\lambda}_n(q), \underline{\lambda}_n(q)$, and $\lambda_n^D(q)$ are continuous in q in the L^1-topology of $L^1(0,T)$.

Remark 3.1. *When $q \equiv 0, \overline{\lambda}_0(0) = 0$ and $\underline{\lambda}_M(0) = \overline{\lambda}_M(0) = \frac{M\pi}{T}$ for all $M \in \mathbb{N}$. These eigenvalues coincide with the constants in condition (1.3).*

Now we present our main result.

Theorem 3.1. *Let the following assumptions hold.*

(H_1) There exist a constant $l_0 > 0$ and a function $h \in C((0,\infty),\mathbb{R})$ such that

$$f(t,r) \le h(r)$$

for all t and all $0 < r \le l_0$, where h satisfies

$$\lim_{r\to 0^+} h(r) = -\infty$$

and

$$\lim_{r\to 0^+} H(r) = +\infty,$$

with the primitive function $H(r) = \int^r h(s)ds$;

(H_2) There exist positive T-periodic continuous functions ϕ, Φ such that

$$\phi(t) \le \liminf_{r\to+\infty} \frac{f(t,r)}{r} \le \limsup_{r\to+\infty} \frac{f(t,r)}{r} \le \Phi(t)$$

uniformily in t. Moreover, $\phi(t)$ and $\Phi(t)$ satisfy

$$\phi(t) \le a \le \Phi(t),$$

for some constant a;

(H_3) There exist $k \in \mathbb{Z}^+$ such that

$$\bar{\lambda}_{k-1}(\phi) < 0, \quad \underline{\lambda}_k(\Phi) > 0.$$

Then, Eq. (1.2) has a T-periodic solution, and there exist a $K_1 \ge 1$ such that, for every integer $k \ge K_1$, system (1.1) has a periodic solution with minimal period kT, which makes exactly one revolution around the origin in the period time kT. Moreover, exist constant $\tilde{C} > 0$ (independent of μ and k) such that

$$\frac{1}{\tilde{C}} < |x_k(t)| < \tilde{C}, \text{ for every } t \in \mathbb{R} \text{ and every } k \ge K_1,$$

and, if μ_k denotes the angular momentum associated to $x_k(t)$ then

$$\lim_{k\to\infty} \mu_k = 0.$$

In order to apply Lemma 2.1, we consider the T-periodic problem (1.2). We will prove that $\deg(I - (L-\sigma I)^{-1}(N - \sigma I), \Omega, 0) = 1$ for some uniformly positively bounded open set Ω. To this end, we deform (1.2) to a simpler singular autonomous equation

$$r'' + ar = \frac{1}{r},$$

where $a > 0$ is as in the condition (H_2). Consider the following homotopy equation

$$r'' + f(t,r;\tau) = 0, \quad \tau \in [0,1], \tag{3.5}$$

where

$$f(t,r;\tau)=\tau f(t,r)+(1-\tau)\left(ar-\frac{1}{r}\right).$$

We need to find a priori estimates for the possible positive T-periodic solutions of (3.5).

Note that $f(t,r;\tau)$ satisfies the conditions (H_1) uniformly with respect to $\tau\in[0,1]$. Moreover, for each $\tau\in[0,1]$, $f(t,r;\tau)$ satisfies (H_2) with $\phi=\phi_\tau=\tau\phi(t)+(1-\tau)a$ and $\Phi=\Phi_\tau=\tau\Phi(t)+(1-\tau)a$. Since $\phi_\tau(t)\geq\phi(t)$ and $\Phi_\tau(t)\leq\Phi(t)$,

$$\bar{\lambda}_{k-1}(\phi_\tau)\leq\bar{\lambda}_{k-1}(\phi)<0,\ \underline{\lambda}_k(\Phi_\tau)\geq\underline{\lambda}_k(\Phi)>0,$$

thus (H_3) is satisfied uniformly in $\tau\in[0,1]$. In order to simplify the notation, we just give a priori estimates for all possible positive solutions to (1.2)–(3.2). The usual L^p-norm is denoted by $\|\cdot\|_p$, and the supremum norm of $C[0,T]$ is denoted by $\|\cdot\|_\infty$.

From (H_1), there is $l_1>0$ such that

$$f(t,s)<0,\quad\text{for all}\quad 0<s\leq l_1\leq l_0. \tag{3.6}$$

Since $\phi(t)>0$ for all t, by (H_2), there exists $L_1>l_1$ such that

$$f(t,s)\geq\phi(t)s>0,\quad\text{for all}\quad s\geq L_1. \tag{3.7}$$

Suppose that $r(t)$ is a positive solution of (1.2) and t_0 is a critical point of $r(t)$. By (3.6) and (3.7), $r''(t_0)=-f(t_0,r(t_0))$ will be positive if $r(t_0)<l_1$ and will be negative if $r(t_0)\geq L_1$. We thus conclude with the following.

Lemma 3.2. *Let l_1,L_1 be as above, assume that $f(t,r)$ satisfies* (H_1) *and* (H_2) *and t_0 is a critical point of $r(t)$.*

(i) *If $r(t_0)\leq l_1$, then $r(t_0)$ is necessarily an isolated local minimum.*
(ii) *If $r(t_0)\geq L_1$, then $r(t_0)$ is necessarily an isolated local maximum.*

From (H_2) *, we know that there are $\varepsilon_0,D>0$ such that*

$$f(t,s)\leq(\Phi(t)+\varepsilon_0)s+D<Cs+D\quad\textit{for all}\,t\,\textit{and all}\,s>0, \tag{3.8}$$

here $C:=\max_t\Phi(t)+\varepsilon_0$ is a positive constants.

The following result is called the elasticity property in literature.

Lemma 3.3. *Let $r(t)$ be a positive Tperiodic solution of Eq.* (1.2)*, assume that $f(t,r)$ satisfies* (H_1) *and* (H_2)*. Then there exist some constants $l_2\in(0,l_1],L_2\in[L_1,+\infty)$ and $B\geq A>0$ such that the following properties hold.*

(i) *If t_0 is a critical point of $r(t)$ such that $r(t_0)\leq l_2$, then the next critical point $s_0>t_0$ of $r(t)$ satisfies $r(s_0)\geq L_1$ and*

$$A\leq r^2(s_0)(H(r(t_0)))^{-1}\leq B. \tag{3.9}$$

(ii) *If* s_0 *is a critical point of* $r(t)$ *such that* $r(s_0) \geq L_2$*, then the next critical point* $t_1 > s_0$ *of* $r(t)$ *satisfies* $r(t_1) \leq l_1$ *and*

$$A \leq r^2(s_0)(H(r(t_1)))^{-1} \leq B.$$

Proof. Let us prove (i). We first assume that $r(t_0) < l_1$. By Lemma 3.2, t_0 is an isolated local minimum point. Since $s_0 > t_0$ is the next critical point of $r(t)$, then we have $r'(t) > 0$ for $t \in (t_0, s_0)$. The inverse function of $r(t)(t \in (t_0, s_0))$ is denoted by ξ. Thus there exists $t_* \in (t_0, s_0)$ such that $r(t_*) = l_0$. Multiplying (1.2) by $r'(t)$ and integrating from t_0 to s_0, we obtain

$$\begin{aligned} 0 &= \int_{s_0}^{t_0} -r''(t)r'(t)dt = \int_{s_0}^{t_0} f(t, r(t))r'(t)dt \\ &= \int_{t_0}^{t_*} f(t, r(t))r'(t)dt + \int_{t_*}^{s_0} f(t, r(t))r'(t)dt \\ &= \int_{r(t_0)}^{l_0} f(\xi(s), s)ds + \int_{l_0}^{r(s_0)} f(\xi(s), s)ds. \end{aligned} \tag{3.10}$$

Note from (H_1) that $f(t, s) \leq h(s)$ for all $s \in (0, l_0]$. Using the fact (3.8), we get from (3.10) that

$$\begin{aligned} 0 &\leq \int_{r(t_0)}^{l_0} h(s)ds + \int_{l_0}^{r(s_0)} (Cs + D)ds \\ &= H(l_0) - H(r(t_0)) + \frac{1}{2}C(r^2(s_0) - l_0^2) + D(r(s_0) - l_0) \\ &\leq Cr^2(s_0) - H(r(t_0)) + D'. \end{aligned}$$

for some $D' > 0$. Hence

$$Cr^2(s_0) \geq H(r(t_0)) - D'.$$

Since $H(r) \to +\infty$ as $r \to 0^+$, we can take some $l_2' \in (0, l_1]$ such that $D' \leq \frac{H(r)}{2}$ for all $r \in (0, l_2']$.

Now, if $r(t_0) \leq l_2'$, then

$$Cr^2(s_0) \geq \frac{H(r(t_0))}{2}$$

and

$$r^2(s_0) \geq \frac{H(r(t_0))}{2C} =: AH(r(t_0)).$$

Thus the left-hand inequality of (3.9) is proved.

By (3.7), one has some $C' > 0$ and $\tilde{D} > 0$ such that

$$f(t, r) > C'r - \tilde{D}$$

for all $r \geq l_0$.

Now we get from (3.10) and f is an L^1-Carathéodory function that

$$0 \geq \int_{r(t_0)}^{l_0} f(\xi(s), s)ds + \int_{l_0}^{r(s_0)} (C's - \tilde{D})ds$$
$$\geq \frac{1}{2}C'r^2(s_0) - D''$$

for some $D'' > 0$. Hence

$$\frac{1}{2}C'r^2(s_0) \leq D''.$$

Also since $H(r) \to +\infty$ as $r \to 0^+$, we can take some $l_2'' \in (0, l_1]$ such that $D'' \leq H(r)$ for all $r \in (0, l_2'']$.

Now, if $r(t_0) \leq l_2''$, then

$$\frac{1}{2}C'r^2(s_0) \leq H(r(t_0))$$

and

$$r^2(s_0) \leq \frac{2}{C'}H(r(t_0)) =: BH(r(t_0)).$$

Thus the right-hand inequality of (3.9) is proved.

We get from (3.9) that if $r(t_0) \leq l_2'$ then $r(s_0) \geq \sqrt{AH(r(t_0))}$. If $r(t_0)$ satisfies further that $r(t_0) \leq H^{-1}(L_1^2/A)$, then $r(s_0) \geq L_1$.

These show that if $r(t_0)$ satisfies

$$r(t_0) \leq \min(l_2', l_2'', H^{-1}(L_1^2/A)),$$

then all conclusions in (i) have been proved. The proof of (ii) is similar. □

Remark 3.2. *Let $r(t)$ be a positive T-periodic solution of (1.2), from the fact that $H(r) \to +\infty$ as $r \to 0^+$, we have, if $\max_t r(t)$ is large then $\min_t r(t)$ is small, and vice versa.*

Lemma 3.4. *Let $r(t)$ be any T-periodic solution of (1.2). There exist $l_3 \in (0, l_2]$ and $L_3 \in [L_2, +\infty)$. If either $\min_t r(t) \leq l_3$ or $\max_t r(t) \geq L_3$, then $r(t)$ has $2N (N \in \mathbb{N})$ critical points in $[t_0, t_0 + T)$:*

$$t_0 < s_0 < t_1 < s_1 < \cdots < t_{N-1} < s_{N-1},$$

where $t_0 \in [0, T)$, t_i's are local minimum points and s_i's $(i = 0, 1, \ldots, N-1)$ are local maximum points.

Proof. We start with the first $t_0 \in [0, T)$ such that $r(t_0) = m_0 := \min_t r(t)$ and will study all critical points of $r(t)$ within $[t_0, t_0 + T)$. We assume that $m_0 < l_2$. Let $s_0 > t_0$ be the next critical point of $r(t)$. By Lemmas 3.2 and 3.3, $r(s_0) \geq L_1$, and t_0, s_0 are local minimal and maximal points respectively.

We claim that $s_0 - t_0$ have a positive lower gap. To see this, we know that $r'(t) > 0$ for $t \in (t_0, s_0)$ and also let $t_* \in (t_0, s_0)$ be such that $r(t_*) = l_0$. From (3.8) we have, for $t \in (t_*, s_0)$,

$$-r''(t) = f(t, r(t)) < Cr(t) + D$$

and

$$-r''(t)r'(t) < (Cr(t) + D)r'(t).$$

Given $t \in (t_*, s_0)$, integrating the inequality above from t to s_0, we have

$$\begin{aligned}\frac{1}{2}(r'(t))^2 &= \int_t^{s_0} -r'(t)r''(t)dt < \int_t^{s_0} (Cr(t) + D)r'(t)dt \\ &= \frac{C}{2}(r^2(s_0) - r^2(t)) + D(r(s_0) - r(t)) \\ &\leq (Cr(s_0) + D)(r(s_0) - r(t)),\end{aligned}$$

where the $r(t) \leq r(s_0)(t \in (t_*, s_0))$ is used.

Let $M_0 = r(s_0)$. We have

$$r'(t) \leq \sqrt{2(CM_0 + D)}\sqrt{M_0 - r(t)}, \quad t \in (t_*, s_0).$$

Thus

$$\begin{aligned}s_0 - t_0 \geq s_0 - t_* &= \int_{t_*}^{s_0} dt = \int_{r_0}^{M_0} \frac{1}{r'(t)} dr(t) \\ &\geq \int_{l_0}^{M_0} \frac{dr(t)}{\sqrt{2(CM_0 + D)}\sqrt{M_0 - r(t)}} \\ &= \sqrt{\frac{2(M_0 - l_0)}{CM_0 + D}}.\end{aligned}$$

From

$$\lim_{y \to +\infty} \sqrt{\frac{2(y - l_0)}{Cy + D}} = \sqrt{\frac{2}{C}},$$

there exists $E_0 > L_2$ such that if

$$M_0 > E_0, \tag{3.11}$$

then

$$s_0 - t_0 \geq \sqrt{\frac{1}{C}} =: V_0.$$

Next, we consider the next critical point $t_1 > s_0$ of $r(t)$. By Lemma 3.3, $m_1 := r(t_1) < l_1$ again. Note that $r'(t) < 0$ on (s_0, t_1). Let $t^* \in (s_0, t_1)$ be such that $r(t^*) = l_0$. For $t \in (s_0, t^*)$, from (3.8) we have

$$-r''(t) = f(t, r(t)) < Cr(t) + D$$

and

$$-r''(t)r'(t) > (Cr(t)+D)r'(t).$$

Given $t \in (s_0, t^*)$, integrating the inequality above from s_0 to t, we have

$$\begin{aligned}-\frac{1}{2}(r'(t))^2 &> \int_{s_0}^{t}(Cr(t)+D)r'(t)dt\\ &\geq (r(t)-M_0)\left(\frac{C(r(t)+M_0)}{2}+D\right).\end{aligned}$$

Thus

$$-r'(t) < \sqrt{2(CM_0+D)}\sqrt{M_0-r(t)}, \quad t \in (s_0, t^*).$$

Similarly,

$$t_1 - s_0 \geq t^* - s_0 \geq \sqrt{\frac{2(M_0-r_0)}{CM_0+D}}.$$

Under (3.11), one has

$$t_1 - s_0 \geq V_0.$$

By the Lemma 3.3, these conditions will be verified when $m_0 \leq l_3$, where $l_3 \in (0, l_2)$ is sufficiently small. Processing as above, we can find the next critical point $s_1 - t_1 > V_0$ and $t_2 - s_1 > V_0$, this process will be terminated after $2N$ steps to reach at the critical point $t_0 + T$, where $N > T/V_0$. This completes the proof. □

Lemma 3.5. *Let $r_n(t)$ be a sequence of positive T-periodic solutions of $r_n'' + f(t, r_n) = 0$ such that $M_n := \|r_n\|_\infty \to +\infty$. Then $z_n = \frac{r_n}{M_n}$ has a subsequence converging in $C(\mathbb{R}/T\mathbb{Z})$ to some $z(t)$.*

Proof. From Remark 3.2 we know $N_n := \min_t r_n(t) \to 0$ as $n \to \infty$. By Lemma 3.4, one knows that if n is large, there exists $m_n \in \{1, 2, \dots, N\}$ such that r_n has finitely many critical points

$$t_0^n < s_0^n < t_1^n < s_0^n < \cdots < t_{m_n-1}^n < s_{m_n-1}^n,$$

and the next critical point will be $t_0^n + T$, where $t_0^n \in [0, T)$, t_i^n's are local minimum points and s_i^n's are local maximum points of r_n.

By passing to subsequence if necessary, we may assume that m_n are the same number $m \in \{1, 2, \dots, N\}$. Moreover, as $n \to \infty$,

$$t_i^n \to t_i, \quad s_i^n \to s_i$$

for each $i = 0, 1, \dots, m-1$. From Lemma 3.4, as $n \to \infty$, one has

$$\int^{r_n} -\frac{1}{s}ds = -\ln r_n(t_i^n) \approx -\ln N_n \text{ and } r_n(s_i^n) \approx M_n,\ i = 0, 1, \dots, m-1,$$

and

$$\ln N_n \approx M_n^2.$$

Note that for any n, $\|z_n\| = 1$ and z_n satisfies

$$z'' + f_n(t,z) = 0, \tag{3.12}$$

where

$$f_n(t,z) = \frac{f(t, M_n z)}{M_n}.$$

From (3.8)

$$f_n(t,z) \le \frac{CM_n z + D}{M_n} = Cz + \frac{D}{M_n}.$$

Multiplying (3.12) by $z_n(t)$ and integrating by parts, we get

$$\begin{aligned}\int_0^T \mid z_n' \mid^2 dt &= \int_0^T f_n(t, z_n(t)) z_n(t) dt \\ &\le \int_0^T \left(C z_n(t) + \frac{D}{M_n} \right) z_n(t) dt \\ &\le \int_0^T \left(C\|z_n\|^2 + \frac{D\|z_n\|}{M_n} \right) dt \\ &= T\left(C + \frac{D}{M_n} \right).\end{aligned}$$

Thus $\{z_n\}$ is bounded in $H^1(0,T)$. Passing to a subsequence if necessary, we can assume that $z_n \to z$ weakly in $H^1(0,T)$ and $z_n \to z$ strongly in $C(0,T)$. Obviously, $z(t) \ge 0$ for all t. As $\|z_n\| \equiv 1$, $\|z\| = 1$ and $z \ne 0$. We will prove that $z(t)$ has only the isolated zeros $t_i (i = 0, 1, 2, \ldots, m)$ in the interval $[t_0, t_0 + T]$. See Lemma 3.6 below.

Let $U = \{t \in \mathbb{R} : z(t) > 0\}$, which is an open subset of $\mathbb{R}$. Take any interval $I = (\alpha, \beta)$ from U. Let J be an arbitrary closed subinterval of I. Define

$$q_n(t) = \frac{f(t, r_n(t))}{r_n(t)}, \quad t \in J.$$

Then z_n satisfies

$$z_n'' + q_n(t) z_n = 0. \tag{3.13}$$

By (H_2), for any given $\varepsilon > 0$, we have

$$\phi(t) - \varepsilon < q_n(t) < \Phi(t) + \varepsilon \tag{3.14}$$

for all n sufficiently large and all $t \in J$. In particular the sequence $\{q_n(t)\}$ is bounded in $L^2(J)$. We can assume, passing to a subsequence if necessary, that $q_n(t) \to q_J(t)$ weakly in $L^2(J)$ to some function $q_J(t)$. From (3.14), one has

$$\phi(t) - \varepsilon < q_J(t) < \Phi(t) + \varepsilon$$

for $a.e. t \in J$. Since ε is arbitrary, we conclude that q_J satisfies

$$\phi(t) < q_J(t) < \Phi(t) \qquad \text{a.e.} \quad t \in J. \tag{3.15}$$

As J is arbitrary, let us define the function $q^I : I \to \mathbb{R}$ by

$$q^I(t) = q_J(t) \qquad \text{whenever} \quad t \in J.$$

Note that $q^I(t)$ is well defined on I by the uniqueness of weak limits.

For any C^1 function v whose support is contained in J, it follows from (3.13) that

$$\int_J z_n' v' dt = \int_J q_n(t) z_n(t) v(t) dt$$

because $v = 0$ on ∂J. Using the convergence results $z_n \to z$ weakly in $H^1(0, T)$, $q_n \to q^I$ weakly in $L^2(J)$ and $z_n \to z$ strongly in $C(0, T)$, we have

$$\int_J z' v' dt = \int_J q^I(t) z_n(t) v(t) dt$$

for all C^1 function v with supports in J. As J is an arbitrary closed interval in I, the regularity theory shows that $z(t)$ is a classical solution of the following linear equation

$$z'' + q^I(t) z = 0, \qquad t \in I. \tag{3.16}$$

Note that $z(t) > 0$ for $t \in I = (\alpha, \beta)$. It is easy to verify from (3.16) that both the limits $\lim_{t\to\alpha^+} z'(t)$ and $\lim_{t\to\beta^-} z'(t)$ exist and are nonzero. In fact, for any $t_1, t_2 \in I$, one has

$$|z'(t_2) - z'(t_1)| = \left| \int_{t_1}^{t_2} q^I(t) z(t) dt \right| \le \|\Phi\| |t_2 - t_1|.$$

Thus the existence of $\lim_{t\to\alpha^+} z'(t)$ and $\lim_{t\to\beta^-} z'(t)$ follows from the Cauchy theorem. Now (3.16) has a nonzero solution $z(t)$ on $[\alpha, \beta]$. Since $z(\alpha) = z(\beta) = 0$, it is necessary that $z'(\alpha) \neq 0$ and $z'(\beta) \neq 0$ by the uniqueness of solutions of linear ODEs.

As I is an arbitrary interval contained in U and $\mathbb{R}\backslash U$ contains only countable points, we can define a measurable $T-$periodic function $q(t)$ by

$$q(t) = q^I(t), \qquad \text{whenever } t \in I.$$

By (3.15), $q(t) \in (\phi, \Phi)$. Moreover, $z(t)$ satisfies for a.e. $t \in \mathbb{R}$ the following equation

$$z''(t) + q(t) z = 0. \tag{3.17}$$

By Lemma 3.4, if n is large, there exists $m \in \{1, 2, \ldots, N\}$ such that $r_n(t)$ has finitely many critical points

$$t_0^n < s_0^n < t_1^n < s_1^n < \cdots < t_{m-1}^n < s_{m-1}^n$$

and the next critical point will be $t_0^n + T$, where $t_0^n \in [0, T)$, t_i^n's are local minimum points and s_i^n's are local maximum points of $r_n(t)$. As $n \to +\infty, t_i^n \to t_i, s_i^n \to s_i$ for each $i = 0, 1, \ldots, m-1$, the zeros of $z(t)$ are $t_0 < t_1 < \cdots < t_{m-1} < t_0 + T =: t_m$. As all the limits $\lim_{t\to t_i\pm} z'(t)$ exist and are nonzero, one can then choose nonzero constants ξ_i so that the following function

$$\widetilde{z}(t) = \xi_i z(t), \qquad t \in [t_{i-1}, s_{i-1}],\ i = 1, 2, \ldots, m,$$

is C^1 on $[t_0, t_m]$. Obviously, $\widetilde{z}$ satisfies $z(t_0) = z(t_0 + T) = 0$. For such a choice of t_0 and $\phi(t) \le q(t) \le \Phi(t)$, $\widetilde{z}$ satisfies (3.17) for $t \in [t_0, t_m]$. Moreover, the function $\widetilde{z}$ can be extended to whole $\mathbb{R}$ according so that it satisfies (3.17) on $\mathbb{R}$. □

Lemma 3.6. *The limiting function $z(t)$ has only isolated zeros $t_0 < t_1 < \cdots < t_{m-1} < t_0 + T =: t_m$ in $[t_0, t_m]$.*

Proof. First we know from (3.6) and (3.7) that

$$f(t, s) = 0 \Rightarrow s \in [l_1, L_1]. \tag{3.18}$$

Now we study the changes of $r_n(t)$ for $t \in I_n := [s_0^n, s_1^n]$, where the interval is spanned by two consecutive local maximum points of $r_n(t)$.

Note that $r_n''(s_0^n) < 0$ and $r_n''(t_1^n) > 0$. Let $u^n \in (s_0^n, t_1^n)$ be the first point left to t_1^n such that $r_n''(u^n) = 0$. Similarly, let $v^n \in (t_1^n, s_1^n)$ be the first point right to t_1^n such that $r_n''(v^n) = 0$. Thus

$$f(t, r_n(t)) < 0 \quad \text{for all} \quad t \in (u^n, v^n). \tag{3.19}$$

Since $f(u^n, r_n(u^n)) = f(v^n, r_n(v^n)) = 0$, we have from (3.18) that

$$r_n(u^n), r_n(v^n) \in [l_1, L_1]. \tag{3.20}$$

We claim that $v^n - u^n \to 0$ as $n \to \infty$.

For $t \in (t_1^n, v^n)$, we have $r_n''(t) > 0$. As $r_n'(t_1^n) = 0$, $r_n'(t) > 0$ for all $t \in (t_1^n, v^n)$ and $r_n(t)$ is strictly increasing on $[t_1^n, v^n]$. We use η_n to denote the inverse function of r_n restricted to (t_1^n, v^n).

From condition (H_1), one has,

$$-f(t, s) \ge -h(s) \quad \text{for all} \quad 0 < s \le l_0.$$

From

$$r_n''(t) r_n'(t) = -f(t, r_n(t)) r_n'(t),$$

we integrate the equation from t_1^n to t, $(t \in (t_1^n, v^n))$, and obtain

$$(r_n'(t))^2 = -2 \int_{t_1^n}^{t} f(t, r_n(t)) r_n'(t) dt = -2 \int_{\overline{r}_n}^{r_n(t)} f(\eta_n(s), s) ds,$$

where $\overline{r}_n = r_n(t_1^n)$. Note that $\overline{r}_n \to 0$ as $n \to +\infty$.

For $t \in (t_1^n, r_n^{-1}(l_0)]$, one has

$$(r_n'(t))^2 \ge -2 \int_{\overline{r}_n}^{r_n(t)} h(s) ds = 2(H(\overline{r}_n) - H(r_n(t))).$$

For $t \in (r_n^{-1}(l_0), v^n]$, by noticing (3.19), we have

$$\begin{aligned}(r_n'(t))^2 &= -2\int_{l_0}^{r_n(t)} f(\eta_n(s),s)ds - 2\int_{\overline{r}_n}^{l_0} f(\eta_n(s),s)ds\\ &\geq -2\int_{\overline{r}_n}^{l_0} f(\eta_n(s),s)ds \geq -2\int_{\overline{r}_n}^{l_0} h(s)ds\\ &= 2(H(\overline{r}_n)-H(l_0)).\end{aligned}$$

Thus is,

$$r_n'(t) \geq \begin{cases} \sqrt{2(H(\overline{r}_n)-H(r_n(t)))}, & \text{if} \quad t \in (t_1^n, r_n^{-1}(l_0)],\\ \sqrt{2(H(\overline{r}_n)-H(l_0))}, & \text{if} \quad t \in (r_n^{-1}(l_0), v^n] \end{cases} \tag{3.21}$$

By the intermediate value theorem, there exists $\varsigma_n \in (\overline{r}^n, l_0)$ such that

$$\begin{aligned}v^n - t_1^n &= \int_{t_1^n}^{v^n} dt = \int_{\overline{r}_n}^{r_n(v^n)} \frac{1}{r'(t)}dr(t) = \int_{\overline{r}_n}^{l_0}\frac{1}{r'}dr + \int_{l_0}^{r_n(v^n)}\frac{1}{r'(t)}dr(t)\\ &\leq \int_{\overline{r}_n}^{l_0} \frac{1}{\sqrt{2(H(\overline{r}_n)-H(r_n(t)))}}dr(t) + \int_{l_0}^{r_n(v^n)} \frac{1}{\sqrt{2(H(\overline{r}_n)-H(l_0))}}dr(t)\\ &= \frac{l_0-\overline{r}_n}{\sqrt{2(H(\overline{r}_n)-H(\varsigma_n(t)))}} + \frac{r_n(v_n)-l_0}{\sqrt{\sqrt{2(H(\overline{r}_n)-H(l_0))}}}\\ &\leq \frac{l_0-\overline{r}_n}{\sqrt{2(H(\overline{r}_n)-H(\varsigma_n(t)))}} + \frac{L_1}{\sqrt{\sqrt{2(H(\overline{r}_n)-H(l_0))}}},\end{aligned}$$

where (3.20) is used. Since $\overline{r}_n \to 0, H(\overline{r}_n) \to +\infty$, it is easy to see that $v^n - t_1^n \to 0$, as $n \to +\infty$. Similarly $t_1^n - u^n \to 0$ as $n \to +\infty$. Thus the claim has been proved. It is obvious that $v^n, u^n \to t_1$.

From (3.21), for n large, we have some constant $k > 0$, $z_n'(v^n) \geq k$. Now for $t > t_1$, we have $v^n < t$ for n large,

$$\begin{aligned}z_n(t) &= z_n(v^n) + z_n'(v^n)(t-v^n) + \frac{z_n''(\zeta_n)}{2}(t-v^n)^2\\ &= z_n(v^n) + z_n'(v^n)(t-v^n) - \frac{f(\zeta_n, x_n(\zeta_n))}{2M_n}(t-v^n)^2,\end{aligned} \tag{3.22}$$

where $\zeta_n \in (v^n, t)$. Using the estimate (3.8), one has

$$-\frac{f(\zeta_n, x_n(\zeta_n))}{2M_n} \geq -\frac{Cr_n(\zeta_n)+D}{2M_n} \geq -C$$

for n large. By (3.22), if n is large,

$$z_n(t) \geq z_n(v^n) + z_n'(v^n)(t-v^n) - C(t-v^n)^2.$$

Taking the limit, we have

$$z(t) \geq k(t-t_1) - C(t-t_1)^2,$$

for all $t > t_1$. Similarly, we will have some constants $k' > 0$ and $C' > 0$ such that

$$z(t) \geq k'(t - t_1) - C'(t - t_1)^2$$

for all $t < t_1$. From these, we know that t_1 is an isolated zero of $z(t)$. The same argument shows all $t_i (i = 2, 3, \dots, m)$ are isolated zeros of $z(t)$.

As a final step, we prove that $z(t)$ will be positive for all $t \in (t_0, t_0 + T)$ except $t = t_i, i = 1, 2, \dots, m$. For definiteness, let $t \in (t_0, t_1)$. Note that $t_i^n \to t_i$ and $s_i^n \to s_i$ as $n \to \infty$. Moreover, from the proof of Lemma 3.4, we know that $s_0^n - t_0^n$ and $t_1^n - s_0^n$ have positive lower bounds. We can fix some interval $t \in [\alpha, \beta] \subset (t_0, t_1)$ such that $\alpha - t_0$ is small, $z(\alpha) > 0$ and $t_1 - \beta$ is small, $z(\beta) > 0$. Moreover, for n large, one has

$$t_0^n < \alpha < s_0^n < \beta < t_1^n.$$

Since $r_n(t)$ has only the critical points t_i^n, s_i^n on $[t_0^n, t_0^n + T)$, we know that $r_n(t)$ is increasing on $[t_0^n, s_0^n]$ and is decreasing on $[s_0^n, t_1^n]$. Hence, for n large,

$$r_n(t) \geq \min(r_n(\alpha), r_n(\beta)).$$

Thus

$$\begin{aligned} z(t) = \lim_{n \to \infty} \frac{r_n(t)}{M_n} &\geq \min\left(\frac{r_n(\alpha)}{M_n}, \frac{r_n(\beta)}{M_n}\right) \\ &= \min(z(\alpha), z(\beta)) > 0. \end{aligned}$$

We finish all proof of the lemma. □

For (3.1) with $\lambda = 0$, the Hill's equations

$$r'' + q(t)r = 0, \tag{3.23}$$

and the corresponding Poincaré matrix is

$$M_T = \begin{pmatrix} \psi_1(T) & \psi_2(T) \\ \psi_1'(T) & \psi_2'(T) \end{pmatrix},$$

where $\psi_i(t) (i = 1, 2)$ are solutions of (3.23) satisfying $\psi_1(0) = \psi_2'(0) = 1$ and $\psi_1'(0) = \psi_2(0) = 0$, respectively. The eigenvalues $\mu_{1,2}$ of M_T are then called the Floquet multipliers of (3.1). They satisfy $\mu_1 \cdot \mu_2 = 1$. Equation (3.23) is called elliptic if $\lambda_1 = \bar{\lambda}_2, |\lambda_1| = 1, \lambda_1 \neq \pm 1$. Let us clarify the meaning of (H_3) in Theorem 3.1.

Lemma 3.7 [33]. *Let $q(t)$ be an measurable T-periodic function satisfying $\phi(t) \leq q(t) \leq \Phi(t)$. The condition* (H_3) *is equivalent to each of the following assertions.*

(i) *For any $t_0 \in \mathbb{R}$ and $n \in \mathbb{N}$, $\lambda_n^D(q_{t_0}) \neq 0$.*
(ii) *Equation* (3.23) *is elliptic.*

Now we give a priori estimates on T-periodic solutions of (1.2).

Lemma 3.8. *Assume that* $(\mathrm{H}_1), (\mathrm{H}_2)$ *and* (H_3) *are satisfied. Then there exist constants* $C_1, C_2 > 0$ *such that any positive* T*-periodic solutions* $r(t)$ *of* (1.2) *satisfies*

$$C_1 < r(t) < C_2 \quad \textit{for all} \quad t. \tag{3.24}$$

Proof. We argue by contradiction. Assume that (1.2) has a sequence $\{r_n\}$ of positive T-periodic solutions such that $\|r_n\|_\infty \to \infty$. It follows from Lemma 3.5 that one has some t_0 and some non-zero function $\tilde{z}$ such that $z''(t) + q(t)z = 0$ and $z(t_0) = z(t_0 + T) = 0$ are satisfied. Let now

$$y(t) = \widetilde{z}(t + t_0), \quad t \in [0, T].$$

Then $y(t) \neq 0$ satisfies

$$y'' + q_{t_0}(t)y = 0, \quad t \in [0, T].$$

and the Dirichlet boundary condition (3.4). This implies that $\lambda_n^D(q_{t_0}) = 0$ for some n, a contradiction to Lemma 3.7. The Lemma is thus proved. □

Lemma 3.9. *There exist* $C_3 > 0$ *such that any positive* T*-periodic solution* $r(t)$ *of* (1.2)–(3.2) *satisfies*

$$\|r'(t)\|_\infty < C_3.$$

Proof. Integrating (1.2) from 0 to T, we get

$$\int_0^T r''(t)dt + \int_0^T f(t, r(t))dt = 0.$$

Thus $\int_0^T f(t, r(t))dt = 0$. From (3.8), we recall that there are $\varepsilon_0, D > 0$ such that

$$f(t, s) \leq (\Phi(t) + \varepsilon_0)s + D$$

for all t and $s > 0$.

As $\int_0^T f(t, r(t))dt = 0$, thus $\|f(t, r(t))\|_1 = 2\|f^+(t, r(t))\|_1$. Since $r(0) = r(T)$, there exists $t_f \in [0, T]$ such that $r'(t_f) = 0$. Therefore

$$\begin{aligned}
\|r'\|_\infty &= \max_{0 \leq t \leq T} |r'(t)| = \max_{0 \leq t \leq T} \left| \int_{t_f}^t r''(s)ds \right| \\
&\leq \int_0^T |f(s, r(s))|\, ds = 2 \int_0^T \left| f^+(s, r(s)) \right| ds \\
&\leq 2 \int_0^T \left| (\Phi^+(s) + \varepsilon_0)r(s) + D \right| ds \\
&\leq 2\big((\|\Phi^+\|_1 + T\varepsilon_0)C_2 + DT\big) := C_3.
\end{aligned}$$

where $\Phi^+(t) = \max\{\Phi(t), 0\}$, $f^+(t, r(t)) = \max\{f(t, r(t)), 0\}$. □

Now we give the proof of Theorem 3.1. Consider the homotopy equation (3.5), we can get a priori estimates as in Lemmas 3.8 and 3.9. That is, any positive T-periodic solution of (3.5) satisfies

$$C_1' < r(t) < C_2', \quad \|r'\|_\infty < C_3'$$

for some positive constants C_1', C_2', C_3'. Define $\tilde{C} = \max\{1/C_1', C_2', C_3'\}$ and let the open bounded in X be

$$\Omega = \{r \in X : \frac{1}{\tilde{C}} < r(t) < \tilde{C} \quad \text{and} \quad |r'(t)| < \tilde{C} \quad \text{for all} \quad t \in [0,T]\}.$$

By the homotopy invariance of degree and the result of Capietto et al. [4],

$$\deg(I - (L - \sigma I)^{-1}(N - \sigma I), \Omega, 0) = \deg(ar - 1/r, \Omega \cap \mathbb{R}, 0) = 1.$$

Thus (3.5), with $\tau = 1$, has at least one solution in Ω, which is a positive T-periodic solution of (1.2). By Lemma 2.1, the proof of Theorem 3.1 is thus completed.

As a direct consequence of Theorem 3.1, we consider the system

$$\ddot{x} + \left(6|x| - \frac{1}{|x|^2}\right)\frac{x}{|x|} = 0, \quad x \in \mathbb{R}^2 \backslash \{0\}, \tag{3.25}$$

take $h(r) = f(t,r) = 6r - 1/r^2, \phi(t) = \Phi(t) = 6$, and $\lambda_n^D(6) = (n\pi/T)^2 - 6 \neq 0$ for all $n \in \mathbb{N}$, then it is easy to see that $(\mathrm{H}_1) - (\mathrm{H}_3)$ are satisfied, system (3.25) has a family of periodic orbits $\{x_k\}$ with angular momentum $\{\mu_k\}$ satisfying $\lim_{k\to\infty} \mu_k = 0$.

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