Results Math 72 (2017), 1839–1848

1422-6383/17/041839-10
published online September 2, 2017
DOI 10.1007/s00025-017-0737-x

Results in Mathematics

Weingarten Affine Translation Surfaces in Euclidean 3-Space

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Abstract. By solving certain (partial) differential equations we give the total classification of Weingarten affine translation surfaces in three dimensional Euclidean space $\mathbb{E}^3$. Explicitly, a Weingarten affine translation surface in Euclidean 3-space is the minimal affine Scherk surface or the surface with flat metric.

Mathematics Subject Classification. 53A05, 53A10, 53C42.

Keywords. Differential equation, affine translation surface, Weingarten surface, mean curvature, Gauss curvature.

1. Introduction

Classifying surfaces with certain conditions are very interesting and important work even if in the classical theories of surfaces in three dimensional Euclidean space $\mathbb{E}^3$ (simply, Euclidean 3-space $\mathbb{E}^3$). Usually some special (partial) differential equations will be solved in such classification process. The theories and properties of the special curves and surfaces are widely applied in a lot of scientific fields, especially in computer science. For example, Weingarten surfaces have several properties which make them attractive for use in surface design where shape control is important. In general, they admit more direct methods of shape control and mitigate the computation of shape parameters such as principal curvatures [1].

H. Liu was supported by NSFC (No. 11371080); Joint Research of NSFC and NRF; partially supported by the Chern Institute of Mathematics and Northeastern University. S. D. Jung was supported by NRF-2015R1A2A2A01003491.

Birkhäuser

In the classical theories of the minimal surfaces in Euclidean 3-space $\mathbb{E}^3$, it is well known that the Scherk surface

$$z(x,y) = \frac{1}{c}\log\frac{\cos cx}{\cos cy} \tag{1.1}$$

is the minimal surface of the type

$$z(x,y) = f(x) + g(y) \tag{1.2}$$

in $\mathbb{E}^3$. Here using the standard coordinate system of Euclidean 3-space $\mathbb{E}^3$, a surface $r(u,v)$ in $\mathbb{E}^3$ will be written as $r(u,v) = (x(u,v), y(u,v), z(u,v))$. The surface which can be written as (1.2) is usually called translation surface in Euclidean 3-space $\mathbb{E}^3$ [2–4,9].

In [8] we define affine translation surfaces $z(x,y) = f(x) + g(y + ax)$ in Euclidean 3-space $\mathbb{E}^3$ and classify the affine translation surfaces with zero mean curvature. We get the generalized Scherk surface or called affine Scherk surface which is the generalization of classical Scherk surface (1.1).

Theorem A. (minimal affine translation surface) *Let $r(x,y) = (x, y, z(x,y))$ be a minimal affine translation surface in Euclidean 3-space $\mathbb{E}^3$. Then either $z(x,y)$ is a linear function or can be written as*

$$z(x,y) = \frac{1}{c}\log\frac{\cos(c\sqrt{1+a^2}x)}{\cos[c(y+ax)]}, \tag{1.3}$$

where a and c are constants and $ac \neq 0$ [8].

In [7], for the affine translation surfaces $z(x,y) = f(x) + g(y+ax)$ in Euclidean 3-space $\mathbb{E}^3$, we classify the affine translation surfaces with non-zero constant mean curvature.

Theorem B. (non-zero constant mean curvature affine translation surfaces) *Let $r(x,y) = (x, y, z(x,y))$ be an affine translation surface with non-zero constant mean curvature H. Then $z(x,y)$ can be written as*

$$\begin{aligned} z(x,y) &= f(x) + g(y+ax) \\ &= bx + \frac{\pm\sqrt{1+a^2+b^2}}{2H(1+a^2)}\sqrt{1+a^2-4H^2(y+ax)^2} \\ &\quad -\frac{ab}{1+a^2}(y+ax) + c_1; \end{aligned} \tag{1.4}$$

or

$$\begin{aligned} z(x,y) &= f(x) + g(y+ax) \\ &= \frac{\pm\sqrt{1+b^2}}{2H}\sqrt{1-4H^2x^2} - abx + b(y+ax) + c_2; \end{aligned} \tag{1.5}$$

where a, b, c_1 and c_2 are constants [7].

In this paper we classify Weingarten affine translation surfaces in Euclidean 3-space. To solve some (partial) differential equations we prove the following conclusions.

Theorem C. *Let $r(x,y) = (x,y,z(x,y))$ be a Weingarten affine translation surface in Euclidean 3-space $\mathbb{E}^3$. Then $z(x,y)$ can be written as the one of the following*

1. *minimal affine translation surface* (1.3)*;*
2. *non-zero constant mean curvature affine translation surface* (1.4) *or* (1.5)*;*
3. *flat affine translation surface*

$$r(x,y) = (x,y,bx + g(y+ax)) \tag{1.6}$$

or

$$r(x,y) = (x,y,f(x) + by). \tag{1.7}$$

Where $f(u)$ and $g(u)$ are any functions of the parameter u.

Remark 1.1. Since minimal surfaces and flat surfaces are trivially Weingarten surfaces, Theorem C means that there exist only trivial Weingarten affine translation surfaces in Euclidean 3-space.

Remark 1.2. According to the conclusion of Lie [9, p. 139, §148; or see Lie [6], in particular vol. 2, p. 136], a (real) minimal surface can be regarded as the translation surface of an isotropic complex curve and its complex conjugate, it is meaningful to study the translation surfaces with some more generalized parameterized cases [5,10].

2. Affine Translation Surfaces

We give some preliminaries as in [8]. Let $r(u,v)$ be a regular surface with arbitrary parameters (u,v) in Euclidean 3-space $\mathbb{E}^3$. Using the standard coordinate system of $\mathbb{E}^3$ we denote the parametric representation of the surface $r(u,v)$ by

$$r(u,v) = (x,y,z) = (x(u,v), y(u,v), z(u,v)). \tag{2.1}$$

Definition 2.1. An affine translation surface in Euclidean 3-space $\mathbb{E}^3$ is defined as a parametric surface $r(u,v)$ in $\mathbb{E}^3$ which can be written as

$$\begin{aligned} r(u,v) &= (x(u,v), y(u,v), z(u,v)) \\ &= (u, v, f(u) + g(v+au)) \\ &= (x, y, f(x) + g(y+ax)) \end{aligned} \tag{2.2}$$

for some non zero constant a and functions $f(x)$ and $g(y+ax)$.

By a direct calculation, the first fundamental form of $r(x,y)$ can be written as

$$\begin{cases} \mathrm{I} = E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2, \\ E = 1 + (f' + ag')^2, \\ F = g'(f' + ag'), \\ G = 1 + g'^2, \end{cases} \tag{2.3}$$

where

$$\begin{cases} f' = \dfrac{\mathrm{d}f(x)}{\mathrm{d}x}, \\ g' = \dfrac{\mathrm{d}g(v)}{\mathrm{d}v} = \dfrac{\mathrm{d}g(y+ax)}{\mathrm{d}(y+ax)}, \qquad v = y + ax. \end{cases} \tag{2.4}$$

The second fundamental form of $r(x, y)$ can be written as

$$\begin{cases} \mathrm{II} = L\mathrm{d}x^2 + 2M\mathrm{d}x\mathrm{d}y + N\mathrm{d}y^2, \\ L = (f'' + a^2 g'')D^{-1}, \\ M = ag''D^{-1}, \\ N = g''D^{-1}, \end{cases} \tag{2.5}$$

where

$$D^2 = EG - F^2 = 1 + (f' + ag')^2 + g'^2. \tag{2.6}$$

The Gauss curvature of $r(x, y)$ can be written as

$$K = f''g''D^{-4} = \frac{f''g''}{[1 + (f' + ag')^2 + g'^2]^2}. \tag{2.7}$$

The mean curvature of $r(x, y)$ can be written as

$$H = \frac{1}{2}\left[f''(1 + g'^2) + g''(1 + a^2 + f'^2)\right] D^{-3} = \frac{f''(1 + g'^2) + g''(1 + a^2 + f'^2)}{2\left[1 + (f' + ag')^2 + g'^2\right]^{\frac{3}{2}}}. \tag{2.8}$$

From (2.6) we have

$$DD_y = g'g'' + ag''(f' + ag') = af'g'' + g'g'' + a^2 g'g'' \tag{2.9}$$

$$= g''[af' + g'(1 + a^2)], \tag{2.10}$$

$$DD_x = aDD_y + f'f'' + af''g'$$

$$= f'f'' + af''g' + a^2 f'g'' + ag'g'' + a^3 g'g''. \tag{2.11}$$

Where and in the following we use $D_x = \frac{\partial D}{\partial x}$ and $D_y = \frac{\partial D}{\partial y}$ etc.

Proposition 2.1. *Any affine translation surface with constant Gauss curvature in Euclidean* 3*-space* $\mathbb{E}^3$ *is flat and can be written as*

$$r(x, y) = (x, y, bx + g(y + ax)); \tag{2.12}$$

or

$$r(x, y) = (x, y, f(x) + by). \tag{2.13}$$

Where a*,* b *are constants and* $f(u)$*,* $g(u)$ *are any functions of the parameter* u*.*

Proof. If the Gauss curvature K of the affine translation surface is constant, differentiating (2.7) with respect to y and x respectively, we have

$$f''\{g'''D^{-4} - 4g''^2 D^{-6}[af' + (a^2 + 1)g']\} = 0 \tag{2.14}$$

and

$$f'''g''D^{-4} + af''\left\{g'''D^{-4} - 4g''^2D^{-6}[af' + (a^2+1)g']\right\}$$
$$-4f''^2g''D^{-6}(f' + ag') = 0. \tag{2.15}$$

Then if $g'' \not\equiv 0$ we have

$$f'''D^2 - 4f''^2(f' + ag') = 0 \tag{2.16}$$

that is

$$f'''(a^2+1)g'^2 + (2af'f''' - 4af''^2)g' + [f'''(1+f'^2) - 4f'f''^2] = 0.$$

Therefore we get $f'' \equiv 0$. By a translation we obtain the conclusions of this proposition. □

3. Weingarten Affine Translation Surfaces in $\mathbb{E}^3$

A surface $r(x, y)$ in Euclidean 3-space $\mathbb{E}^3$ is called a *Weingarten surface* or a *W-surface* if the two principal curvatures of $r(x, y)$ are not independent of one another or, equivalently, if a certain relation $\phi(K, H) = 0$ is identically satisfied for the Gauss curvature K and mean curvature H of the surface [1,4]. Therefore the surfaces with constant Gauss curvature or constant mean curvature are Weingarten surface and the surface $r(x, y)$ is a Weingarten surface if and only if its Gauss curvature K and mean curvature H satisfy $K_xH_y \equiv K_yH_x$.

For the Weingarten affine translation surfaces with non constant mean curvature and non constant Gauss curvature in Euclidean 3-space $\mathbb{E}^3$, we will prove the following conclusion.

Theorem 3.1. *Weingarten affine translation surfaces with non constant mean curvature in Euclidean* 3*-space* $\mathbb{E}^3$ *are flat.*

Proof. If the surface $r(x, y)$ is neither with constant Gauss curvature nor with constant mean curvature, from (2.7) we have

$$K_xD^5 = \alpha_1g''' + \alpha_2 = f'''g''D + af''g'''D - 4f''g''D_x, \tag{3.1}$$
$$K_yD^5 = \beta_1g''' + \beta_2 = f''g'''D - 4f''g''D_y. \tag{3.2}$$

And from (2.8) we have

$$2H_xD^4 = \xi_1g''' + \xi_2 = \Big[f'''(1+g'^2) + 2af''g'g'' + ag'''(1+a^2+f'^2)$$
$$+2f'f''g''\Big]D - 3\left[f''(1+g'^2) + g''(1+a^2+f'^2)\right]D_x, \tag{3.3}$$
$$2H_yD^4 = \eta_1g''' + \eta_2 = \left[2f''g'g'' + g'''(1+a^2+f'^2)\right]D$$
$$-3\left[f''(1+g'^2) + g''(1+a^2+f'^2)\right]D_y. \tag{3.4}$$

Where

$$\begin{cases} \alpha_1 &= af''D, \\ \alpha_2 &= f'''g''D - 4f''g''D_x, \\ \beta_1 &= f''D, \\ \beta_2 &= -4f''g''D_y, \\ \xi_1 &= a(1+a^2+f'^2)D, \\ \xi_2 &= \left[f'''(1+g'^2) + 2af''g'g'' + 2f'f''g''\right] \\ & \quad D - 3\left[f''(1+g'^2) + g''(1+a^2+f'^2)\right] D_x, \\ \eta_1 &= (1+a^2+f'^2)D, \\ \eta_2 &= 2f''g'g''D - 3\left[f''(1+g'^2) + g''(1+a^2+f'^2)\right] D_y. \end{cases} \tag{3.5}$$

For the Weingarten affine translation surfaces, when

$$K_x H_y = K_y H_x,$$

by a direct calculation we have

$$g''' = F(f', f'', f''', g', g'') = \frac{\alpha_2\eta_2 - \beta_2\xi_2}{\beta_1\xi_2 + \beta_2\xi_1 - \alpha_1\eta_2 - \alpha_2\eta_1}, \tag{3.6}$$

where $F(x_1, x_2, x_3, x_4, x_5)$ is a rational function of x_1, x_2, x_3, x_4, x_5 and

$$\begin{aligned} &\alpha_2\eta_2 - \beta_2\xi_2 \\ &\quad = 2f''f'''g'g''^2D^2 - 8f''^2g'g''^2DD_x \\ &\qquad -3f'''g''DD_y\left[f''(1+g'^2) + g''(1+a^2+f'^2)\right] \\ &\qquad +12f''g''D_xD_y\left[f''(1+g'^2) + g''(1+a^2+f'^2)\right] \\ &\qquad +4f''g''DD_y\left[f'''(1+g'^2) + 2af''g'g'' + 2f'f''g''\right] \\ &\qquad -12f''g''D_xD_y\left[f''(1+g'^2) + g''(1+a^2+f'^2)\right] \\ &\quad = 2f''f'''g'g''^2D^2 - 8f''^2g'g''^2DD_x \\ &\qquad -3f'''g''DD_y\left[f''(1+g'^2) + g''(1+a^2+f'^2)\right] \\ &\qquad +4f''g''DD_y\left[f'''(1+g'^2) + 2af''g'g'' + 2f'f''g''\right] \\ &\quad = 2f''f'''g'g''^2D^2 - 8f''^2g'g''^2DD_x \\ &\qquad +g''DD_y\left[f''f'''(1+g'^2) + 8f''^2g''(ag'+f') - 3f'''g''(1+a^2+f'^2)\right] \\ &\quad = \left\{[af' + g'(1+a^2)][f' - 3f'''(1+a^2+f'^2)]\right\} g''^3 + O(g''^2), \\ &\beta_1\xi_2 + \beta_2\xi_1 - \alpha_1\eta_2 - \alpha_2\eta_1 \\ &\quad = f''D^2\left[f'''(1+g'^2) + 2af''g'g'' + 2f'f''g''\right] \\ &\qquad -3f''DD_x\left[f''(1+g'^2) + g''(1+a^2+f'^2)\right] \\ &\qquad -4af''g''DD_y(1+a^2+f'^2) \end{aligned}$$

$$
\begin{aligned}
&-2af''^2g'g''D^2+3af''DD_y\left[f''(1+g'^2)+g''(1+a^2+f'^2)\right]\\
&-f'''g''D^2(1+a^2+f'^2)+4f''g''DD_x(1+a^2+f'^2)\\
&=f''D^2\left[f'''(1+g'^2)+2f'f''g''\right]-3f''^2DD_x(1+g'^2)+f''g''DD_x(1+a^2+f'^2)\\
&-af''^2DD_y(1+g'^2)+3af''g''DD_y(1+a^2+f'^2)-f'''g''D^2(1+a^2+f'^2)\\
&=D^2\left[f''f'''(1+g'^2)+g''(2f'f''^2-f'''(1+a^2+f'^2))\right]\\
&+f''DD_x\left[g''(1+a^2+f'^2)-3f''(1+g'^2)\right]\\
&+af''DD_y\left[3g''(1+a^2+f'^2)-f''(1+g'^2)\right]\\
&=\Big\{D^2(2f'f''^2-f'''(1+a^2+f'^2))+4af''\\
&[af'+g'(1+a^2)](1+a^2+f'^2)\Big\}g''^2+O(g'').
\end{aligned}
$$

Differentiating (3.6) with respect to x we get

$$ag'''=f''\frac{\partial F}{\partial x_1}+f'''\frac{\partial F}{\partial x_2}+f''''\frac{\partial F}{\partial x_3}+ag''\frac{\partial F}{\partial x_4}+ag'''\frac{\partial F}{\partial x_5}. \tag{3.7}$$

Differentiating (3.6) with respect to y we get

$$g''''=g''\frac{\partial F}{\partial x_4}+g'''\frac{\partial F}{\partial x_5}. \tag{3.8}$$

Then (3.7) and (3.8) yield that

$$f''\frac{\partial F}{\partial x_1}+f'''\frac{\partial F}{\partial x_2}+f''''\frac{\partial F}{\partial x_3}=0. \tag{3.9}$$

Differentiating (3.9) with respect to x we get

$$
\begin{aligned}
0&=f'''\frac{\partial F}{\partial x_1}+f''''\frac{\partial F}{\partial x_2}+f'''''\frac{\partial F}{\partial x_3}\\
&+f''\frac{\partial}{\partial x_1}\left(f''\frac{\partial F}{\partial x_1}+f'''\frac{\partial F}{\partial x_2}+f''''\frac{\partial F}{\partial x_3}+ag''\frac{\partial F}{\partial x_4}+ag'''\frac{\partial F}{\partial x_5}\right)\\
&+f'''\frac{\partial}{\partial x_2}\left(f''\frac{\partial F}{\partial x_1}+f'''\frac{\partial F}{\partial x_2}+f''''\frac{\partial F}{\partial x_3}+ag''\frac{\partial F}{\partial x_4}+ag'''\frac{\partial F}{\partial x_5}\right)\\
&+f''''\frac{\partial}{\partial x_3}\left(f''\frac{\partial F}{\partial x_1}+f'''\frac{\partial F}{\partial x_2}+f''''\frac{\partial F}{\partial x_3}+ag''\frac{\partial F}{\partial x_4}+ag'''\frac{\partial F}{\partial x_5}\right)
\end{aligned} \tag{3.10}
$$

Differentiating (3.9) with respect to y we get

$$
\begin{aligned}
0&=f''\frac{\partial}{\partial x_1}\left(g''\frac{\partial F}{\partial x_4}+g'''\frac{\partial F}{\partial x_5}\right)\\
&+f'''\frac{\partial}{\partial x_2}\left(g''\frac{\partial F}{\partial x_4}+g'''\frac{\partial F}{\partial x_5}\right)\\
&+f''''\frac{\partial}{\partial x_3}\left(g''\frac{\partial F}{\partial x_4}+g'''\frac{\partial F}{\partial x_5}\right).
\end{aligned} \tag{3.11}
$$

Then (3.9), (3.10) and (3.11) yield that

$$f''' \frac{\partial F}{\partial x_1} + f'''' \frac{\partial F}{\partial x_2} + f''''' \frac{\partial F}{\partial x_3} = 0. \tag{3.12}$$

As the same as above we get also

$$f^{(4)} \frac{\partial F}{\partial x_1} + f^{(5)} \frac{\partial F}{\partial x_2} + f^{(6)} \frac{\partial F}{\partial x_3} = 0. \tag{3.13}$$

At first we consider the case

$$\left(\frac{\partial F}{\partial x_1}\right)^2 + \left(\frac{\partial F}{\partial x_2}\right)^2 + \left(\frac{\partial F}{\partial x_3}\right)^2 \not\equiv 0. \tag{3.14}$$

Then from (3.9), (3.12), (3.13) we know that the Wronski determinant $W(f'', f''', f'''') \equiv 0$. Therefore f'', f''', f'''' are linear dependent and solving $a_1 f'' + a_2 f''' + a_3 f'''' = 0$, a_1, a_2, a_3 are constants , we know that the function $f(x)$ can be written as $f''(x) = c \exp(\lambda x)$, where c and λ are constants (maybe complex number). If $c \neq 0$, the relation (3.9) can be written as

$$\frac{\partial F}{\partial x_1} + \lambda \frac{\partial F}{\partial x_2} + \lambda^2 \frac{\partial F}{\partial x_3} = 0. \tag{3.15}$$

Therefore whether in the case (3.14) or in the case

$$\left(\frac{\partial F}{\partial x_1}\right)\left(\frac{\partial F}{\partial x_2}\right)\left(\frac{\partial F}{\partial x_3}\right) \equiv 0, \tag{3.16}$$

we can find a rational function $G(x_1, x_2, x_3, x_4)$ (maybe with the coefficients of complex number) of x_1, x_2, x_3, x_4 (even if for x_4) such that

$$g'' = G(f', f'', f''', g'). \tag{3.17}$$

Differentiating (3.17) with respect to x and y, respectively, we can get

$$f'' \frac{\partial G}{\partial x_1} + f''' \frac{\partial G}{\partial x_2} + f'''' \frac{\partial G}{\partial x_3} = 0. \tag{3.18}$$

Therefore if $f'' \not\equiv 0$, as the same way as above, we can fine a function $H(x_1, x_2, x_3)$ of x_1, x_2, x_3 such that

$$g' = H(f', f'', f''') \tag{3.19}$$

and this means that g' is constant.

Therefore from (2.7) we know that $f'' \equiv 0$ or $g'' \equiv 0$ yields that $K \equiv 0$. Then we obtain that the surface $r(x, y)$ is flat and can be written as (2.12) or (2.13). □

Remark 3.1. From Theorem 3.1, Proposition 2.1, [7] and [8] we obtain the conclusion of Theorem C in section one.

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Received: February 2, 2017.
Accepted: August 11, 2017.