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Results in Mathematics

Fueter's Theorem for Monogenic Functions in Biaxial Symmetric Domains

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Abstract. Fueter's theorem discloses a remarkable connection existing between holomorphic functions and monogenic functions in $\mathbb{R}^{m+1}$ when m is odd. It states that $\Delta_{m+1}^{k+\frac{m-1}{2}}\left[\left(u(x_0,|\underline{x}|)+\frac{\underline{x}}{|\underline{x}|}\,v(x_0,|\underline{x}|)\right)P_k(\underline{x})\right]$ is monogenic if $u+iv$ is holomorphic and $P_k(\underline{x})$ is a homogeneous monogenic polynomial in $\mathbb{R}^m$. Eelbode et al. (AIP Conf Proc 1479:340–343, 2012) proved that this statement is still valid if the monogenicity condition on $P_k(\underline{x})$ is dropped. To obtain this result, the authors used representation theory methods but their result also follows from a direct calculus we established in our paper Peña Peña and Sommen (J Math Anal Appl 365:29–35, 2010). In this paper we generalize the result from Eelbode et al. (2012) to the case of monogenic functions in biaxially symmetric domains. In order to achieve this goal we first generalize Peña Peña and Sommen (2010) to the biaxial case and then derive the main result from that.

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1. Introduction

Let $\mathbb{R}_m$ be the real Clifford algebra generated by the standard basis $\{e_1,\ldots,e_m\}$ of the Euclidean space $\mathbb{R}^m$ (see [2,17]). The multiplication in this associative algebra is determined by the relations: $e_j^2=-1$, $e_je_k+e_ke_j=0$, $1\leq j\neq k\leq m$. Any Clifford number $a\in\mathbb{R}_m$ may thus be written as

$$a=\sum_A a_Ae_A,\quad a_A\in\mathbb{R},$$

where the basis elements $e_A=e_{j_1}\ldots e_{j_k}$ are defined for every subset $A=\{j_1,\ldots,j_k\}$ of $\{1,\ldots,m\}$ with $j_1<\cdots<j_k$ (for $A=\emptyset$ one puts $e_\emptyset=1$).

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Observe that $\mathbb{R}^{m+1}$ may be naturally embedded in $\mathbb{R}_m$ by associating to any element $(X_0, X_1, \ldots, X_m) \in \mathbb{R}^{m+1}$ the paravector $X_0 + \underline{X} = X_0 + \sum_{j=1}^{m} X_j e_j$. Furthermore, by the above multiplication rules it follows that $\underline{X}^2 = -|\underline{X}|^2 = -\sum_{j=1}^{m} X_j^2$.

The even and odd subspaces $\mathbb{R}_m^+$, $\mathbb{R}_m^-$ are defined as

$$\mathbb{R}_m^+ = \left\{ a \in \mathbb{R}_m : \; a = \sum_{|A| \text{ even}} a_A e_A \right\}, \;\; \mathbb{R}_m^- = \left\{ a \in \mathbb{R}_m : \; a = \sum_{|A| \text{ odd}} a_A e_A \right\},$$

where $|A| = j_1 + \cdots + j_k$. The subspace $\mathbb{R}_m^+$ is also a subalgebra and we have that

$$\mathbb{R}_m = \mathbb{R}_m^+ \oplus \mathbb{R}_m^-.$$

Consider the Dirac operator $\partial_{\underline{X}}$ in $\mathbb{R}^m$ given by

$$\partial_{\underline{X}} = \sum_{j=1}^{m} e_j \partial_{X_j},$$

which provides a factorization of the Laplacian, i.e. $\partial_{\underline{X}}^2 = -\Delta_{\underline{X}} = -\sum_{j=1}^{m} \partial_{X_j}^2$. Functions in the kernel of $\partial_{\underline{X}}$ are known as monogenic functions (see [1,7,10,13,14]).

Definition 1. A function $F : \Omega \to \mathbb{R}_m$ defined and continuously differentiable in an open set $\Omega \subset \mathbb{R}^m$ is said to be (left) monogenic in Ω if $\partial_{\underline{X}} F(\underline{X}) = 0$, $\underline{X} \in \Omega$. In a similar way one defines monogenicity with respect to the generalized Cauchy-Riemann operator $\partial_{X_0} + \partial_{\underline{X}}$ in $\mathbb{R}^{m+1}$.

It is clear that monogenic functions are harmonic. Furthermore, for the particular case $m = 1$ the equation $(\partial_{X_0} + \partial_{\underline{X}})F(X_0, \underline{X}) = 0$ is nothing but the classical Cauchy-Riemann system for holomorphic functions. This is not the only connection existing between holomorphic and monogenic functions as the following result shows (see [28]).

Theorem 1 (Fueter's theorem). *Let $w(z) = u(x, y) + iv(x, y)$ be a holomorphic function in the open subset Ξ of the upper half-plane and assume that $P_K(\underline{X})$ is a homogeneous monogenic polynomial of degree K in $\mathbb{R}^m$. If m is odd, then the function*

$$\left(\partial_{X_0}^2 + \Delta_{\underline{X}}\right)^{K + \frac{m-1}{2}} \left[\left(u(X_0, |\underline{X}|) + \frac{\underline{X}}{|\underline{X}|} v(X_0, |\underline{X}|) \right) P_K(\underline{X}) \right] \qquad (1)$$

is monogenic in $\Omega = \left\{ (X_0, \underline{X}) \in \mathbb{R}^{m+1} : \; (X_0, |\underline{X}|) \in \Xi \right\}$.

The idea of using holomorphic functions to construct monogenic functions was first presented by Fueter [12] in the setting of quaternionic analysis ($m = 3$, $K = 0$) and for that reason Theorem 1 bears his name. In 1957 Sce [26] extended Fueter's idea to Clifford analysis by proving the validity of the above result for the case $K = 0$, m odd. Forty years later Qian [23] showed that a similar result holds when m is even. In the last years several articles have been published on this topic (see e.g. [3–6, 9, 11, 16, 20, 25]). For more information we refer the reader to the survey article [24].

Consider the biaxial decomposition $\mathbb{R}^m = \mathbb{R}^p \oplus \mathbb{R}^q$, $p + q = m$. In this way, for any $\underline{X} \in \mathbb{R}^m$ we may write

$$\underline{X} = \underline{x} + \underline{y},$$

where $\underline{x} = \sum_{j=1}^{p} x_j e_j$ and $\underline{y} = \sum_{j=1}^{q} x_{p+j} e_{p+j}$. We shall denote by $\mathbb{R}_p$ and $\mathbb{R}_q$ the real Clifford algebras constructed over $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, i.e.

$$\mathbb{R}_p = \mathrm{Alg}_{\mathbb{R}}\{e_1, \dots, e_p\}, \quad \mathbb{R}_q = \mathrm{Alg}_{\mathbb{R}}\{e_{p+1}, \dots, e_m\}.$$

In this paper we further investigate the following generalization of Fueter's theorem to the biaxial case (see [19, 21, 25]). We note that in this setting there is a slight change regarding the initial function w. Namely, w will be assumed to be antiholomorphic, i.e. a solution of $\partial_z w = 0$, where $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$.

Theorem 2. *Let $w(\overline{z}) = u(x, y) + iv(x, y)$ be an antiholomorphic function in an open subset of $\{(x, y) \in \mathbb{R}^2 : \ x, y > 0\}$. Suppose that $P_k(\underline{x}) : \mathbb{R}^p \to \mathbb{R}_p$ and $P_\ell(\underline{y}) : \mathbb{R}^q \to \mathbb{R}_q$ are homogeneous monogenic polynomials. If p and q are odd, then the functions*

$$\begin{aligned}
&\mathsf{Ft}^{+}_{p,q}\left[w(\overline{z}), P_k(\underline{x}), P_\ell(\underline{y})\right](\underline{X}) \\
&\quad = \Delta_{\underline{X}}^{k+\ell+\frac{m-2}{2}}\left[\left(u(|\underline{x}|, |\underline{y}|) + \frac{\underline{x}\,\underline{y}}{|\underline{x}||\underline{y}|}\, v(|\underline{x}|, |\underline{y}|)\right) P_k(\underline{x}) P_\ell(\underline{y})\right] \\
&\mathsf{Ft}^{-}_{p,q}\left[w(\overline{z}), P_k(\underline{x}), P_\ell(\underline{y})\right](\underline{X}) \\
&\quad = \Delta_{\underline{X}}^{k+\ell+\frac{m-2}{2}}\left[\left(\frac{\underline{x}}{|\underline{x}|}\, u(|\underline{x}|, |\underline{y}|) + \frac{\underline{y}}{|\underline{y}|}\, v(|\underline{x}|, |\underline{y}|)\right) P_k(\underline{x}) P_\ell(\underline{y})\right]
\end{aligned}$$

are monogenic.

It is remarkable that Theorem 1 is still true if $P_K(\underline{X})$ is replaced by a homogeneous monogenic polynomial $P_K(X_0, \underline{X})$ in $\mathbb{R}^{m+1}$ (see [22]), or if the monogenicity condition on $P_K(\underline{X})$ is dropped. The latter result was proved in [11] with the help of representation theory, but it can also be derived using the results obtained in [22].

Motivated by [11] and using similar methods as in [22], we prove in this paper that Theorem 2 also holds if $P_k(\underline{x})$ and $P_\ell(\underline{y})$ are assumed to be only homogeneous polynomials.

2. A Higher Order Version of Theorem 2

The goal in this section is to generalize Theorem 2 to a larger class of initial functions. More precisely, we shall assume that $w(z, \overline{z}) = u(x, y) + iv(x, y)$ is a solution of the equation

$$\partial_z \Delta^{\mu}_{x,y} w(z,\overline{z}) = 0, \quad \Delta_{x,y} = \partial_x^2 + \partial_y^2, \quad \mu \in \mathbb{N}_0. \tag{2}$$

In particular, poly-antiholomorphic functions of order $\mu + 1$ (i.e. solutions of $\partial_z^{\mu+1} w(z,\overline{z}) = 0$) clearly satisfy Eq. (2).

It is possible to compute in explicit form the monogenic function produced by Theorem 1 using the differential operators

$$\left(x^{-1}\frac{d}{dx}\right)^n, \qquad \left(\frac{d}{dx}\,x^{-1}\right)^n, \quad n \geq 0. \tag{3}$$

Namely, function (1) equals

$$(2K+m-1)!!\left(\left(R^{-1}\partial_R\right)^{K+\frac{m-1}{2}} u(X_0,R) + \frac{\underline{X}}{R}\left(\partial_R\, R^{-1}\right)^{K+\frac{m-1}{2}} v(X_0,R)\right) P_K(\underline{X}),$$

where $R = |\underline{X}|$ (see [19,20]).

The differential operators in (3) possess interesting properties (see [9,19, 20]) and in this paper we shall use the following.

Lemma 1. *If $f : \mathbb{R} \to \mathbb{R}$ is a infinitely differentiable function, then*

(i) $\displaystyle \frac{d^2}{dx^2}\left(x^{-1}\frac{d}{dx}\right)^n f(x) = \left(x^{-1}\frac{d}{dx}\right)^n \frac{d^2}{dx^2} f(x) - 2n\left(x^{-1}\frac{d}{dx}\right)^{n+1} f(x),$

(ii) $\displaystyle \frac{d^2}{dx^2}\left(\frac{d}{dx}\,x^{-1}\right)^n f(x) = \left(\frac{d}{dx}\,x^{-1}\right)^n \frac{d^2}{dx^2} f(x) - 2n\left(\frac{d}{dx}\,x^{-1}\right)^{n+1} f(x),$

(iii) $\displaystyle \left(\frac{d}{dx}\,x^{-1}\right)^n \frac{d}{dx} f(x) = \frac{d}{dx}\left(x^{-1}\frac{d}{dx}\right)^n f(x),$

(iv) $\displaystyle \left(x^{-1}\frac{d}{dx}\right)^n \frac{d}{dx} f(x) - \frac{d}{dx}\left(\frac{d}{dx}\,x^{-1}\right)^n f(x) = 2nx^{-1}\left(\frac{d}{dx}\,x^{-1}\right)^n f(x).$

Due to the decomposition $\mathbb{R}^m = \mathbb{R}^p \oplus \mathbb{R}^q$ it is convenient to split $\partial_{\underline{X}}$ and $\Delta_{\underline{X}}$ as

$$\partial_{\underline{X}} = \partial_{\underline{x}} + \partial_{\underline{y}} = \sum_{j=1}^{p} e_j \partial_{x_j} + \sum_{j=1}^{q} e_{p+j}\partial_{x_{p+j}},$$

$$\Delta_{\underline{X}} = \Delta_{\underline{x}} + \Delta_{\underline{y}} = \sum_{j=1}^{p} \partial^2_{x_j} + \sum_{j=1}^{q} \partial^2_{x_{p+j}}.$$

Furthermore, for any $\underline{x} \in \mathbb{R}^p$ and $\underline{y} \in \mathbb{R}^q$ we put

$$\underline{\omega} = \underline{x}/r, \quad r = |\underline{x}|,$$
$$\underline{\nu} = \underline{y}/\rho, \quad \rho = |\underline{y}|.$$

In this section, like in Theorem 2, we assume that $P_k(\underline{x}) : \mathbb{R}^p \to \mathbb{R}_p$ and $P_\ell(\underline{y}) : \mathbb{R}^q \to \mathbb{R}_q$ are homogeneous monogenic polynomials. It is convenient to make a few observations about these polynomials.

Remark 1. First, note that $P_k(\underline{x})$ can be uniquely written in the form $P_k(\underline{x}) = P_k^+(\underline{x}) + P_k^-(\underline{x})$, where $P_k^+(\underline{x})$, $P_k^-(\underline{x})$ take values in $\mathbb{R}_p^+$, $\mathbb{R}_p^-$ respectively. Since $\partial_{\underline{x}} P_k^+(\underline{x}) \in \mathbb{R}_p^-$, $\partial_{\underline{x}} P_k^-(\underline{x}) \in \mathbb{R}_p^+$ for $\underline{x} \in \mathbb{R}^p$, one can conclude that $P_k(\underline{x})$ is monogenic if and only if both components $P_k^+(\underline{x})$ and $P_k^-(\underline{x})$ are monogenic. Of course, a similar remark holds for $P_\ell(\underline{y})$.

Let $\Delta_2 = \partial_r^2 + \partial_\rho^2$ be the two-dimensional Laplacian in the variables (r, ρ) and recall the definition of a multinomial coefficient

$$\binom{n}{j_1, j_2, \ldots, j_s} = \frac{n!}{j_1!\, j_2! \cdots j_s!}.$$

Consider the function $D : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{Z}$ satisfying

$$D(0,0) = 1, \quad D(j_1, j_2) = D(j_1, 0) D(0, j_2), \;\; j_1, j_2 \geq 1$$

$$D(j, 0) = \prod_{s=1}^{j} \big(2k + p - (2s-1)\big), \quad D(0, j) = \prod_{s=1}^{j} \big(2\ell + q - (2s-1)\big), \quad j \geq 1.$$

Lemma 2. *Suppose that $h : \mathbb{R}^2 \to \mathbb{R}$ is an infinitely differentiable function in an open subset of $\{(x,y) \in \mathbb{R}^2 : \; x, y > 0\}$. Then for $n \in \mathbb{N}$ and $s_1, s_2 \in \{0,1\}$ it holds that*

$$\begin{aligned} &\Delta_{\underline{X}}^n \Big(h(r,\rho)\, \underline{\omega}^{s_1} \underline{\nu}^{s_2} P_k(\underline{x}) P_\ell(\underline{y}) \Big) \\ &= \left(\sum_{\substack{j_1 + j_2 \leq n \\ j_1, j_2 \geq 0}} \binom{n}{j_1, j_2, n - j_1 - j_2} D(j_1, j_2) W_{j_1, j_2}^{s_1, s_2}(r, \rho) \right) \underline{\omega}^{s_1} \underline{\nu}^{s_2} P_k(\underline{x}) P_\ell(\underline{y}), \end{aligned} \tag{4}$$

where

$$\begin{aligned} W_{j_1,j_2}^{0,0}(r,\rho) &= \big(r^{-1}\partial_r\big)^{j_1} \big(\rho^{-1}\partial_\rho\big)^{j_2} \Delta_2^{n-j_1-j_2} h(r,\rho), \\ W_{j_1,j_2}^{1,0}(r,\rho) &= \big(\partial_r\, r^{-1}\big)^{j_1} \big(\rho^{-1}\partial_\rho\big)^{j_2} \Delta_2^{n-j_1-j_2} h(r,\rho), \\ W_{j_1,j_2}^{0,1}(r,\rho) &= \big(r^{-1}\partial_r\big)^{j_1} \big(\partial_\rho\, \rho^{-1}\big)^{j_2} \Delta_2^{n-j_1-j_2} h(r,\rho), \\ W_{j_1,j_2}^{1,1}(r,\rho) &= \big(\partial_r\, r^{-1}\big)^{j_1} \big(\partial_\rho\, \rho^{-1}\big)^{j_2} \Delta_2^{n-j_1-j_2} h(r,\rho). \end{aligned}$$

Proof. We shall prove the case $s_1 = s_2 = 0$ using induction. The other cases can be proved similarly. First, note that

$$\partial_{\underline{x}} h = \sum_{j=1}^{p} e_j \partial_{x_j} h = \sum_{j=1}^{p} e_j \big(\partial_r h\big)\big(\partial_{x_j} r\big) = \underline{\omega}\, \partial_r h$$

and hence

$$\begin{aligned} \Delta_{\underline{x}} h = -\partial_{\underline{x}}^2 h = -\partial_{\underline{x}}\big(\underline{\omega}\, \partial_r h\big) &= -\underline{\omega}^2 \partial_r^2 h - \big(\partial_{\underline{x}}\, \underline{\omega}\big)\big(\partial_r h\big) \\ &= \partial_r^2 h + \frac{p-1}{r}\, \partial_r h. \end{aligned}$$

From this equality and using Euler's theorem for homogeneous functions we obtain

$$\begin{aligned}\Delta_{\underline{x}}\big(hP_k\big) &= \big(\Delta_{\underline{x}}h\big)P_k + 2\sum_{j=1}^{p}\big(\partial_{x_j}h\big)\big(\partial_{x_j}P_k\big) + h\big(\Delta_{\underline{x}}P_k\big)\\ &= \left(\partial_r^2 h + \frac{p-1}{r}\,\partial_r h\right)P_k + 2\frac{\partial_r h}{r}\sum_{j=1}^{p}x_j\partial_{x_j}P_k\\ &= \left(\partial_r^2 h + \frac{2k+p-1}{r}\,\partial_r h\right)P_k.\end{aligned}$$

In a similar way one also get

$$\Delta_{\underline{y}}\big(hP_\ell\big) = \left(\partial_\rho^2 h + \frac{2\ell+q-1}{\rho}\,\partial_\rho h\right)P_\ell.$$

These equalities then yield

$$\Delta_{\underline{X}}\big(hP_kP_\ell\big) = \left(\Delta_2 h + \frac{2k+p-1}{r}\,\partial_r h + \frac{2\ell+q-1}{\rho}\,\partial_\rho h\right)P_kP_\ell. \tag{5}$$

It is clear that the assertion is true in the case $n=1$. Assume now that the identity holds for some natural number n. We thus get

$$\begin{aligned}\Delta_{\underline{X}}^{n+1}\big(hP_kP_\ell\big) = &\sum_{\substack{j_1+j_2\le n\\ j_1,j_2\ge 0}}\binom{n}{j_1,j_2,n-j_1-j_2}D(j_1,j_2)\\ &\times\Delta_{\underline{X}}\left(\left(r^{-1}\partial_r\right)^{j_1}\left(\rho^{-1}\partial_\rho\right)^{j_2}\Delta_2^{n-j_1-j_2}hP_kP_\ell\right).\end{aligned}$$

By statement (i) of Lemma 1 we obtain

$$\begin{aligned}&\Delta_2\left(r^{-1}\partial_r\right)^{j_1}\left(\rho^{-1}\partial_\rho\right)^{j_2}\Delta_2^{n-j_1-j_2}h = \left(r^{-1}\partial_r\right)^{j_1}\left(\rho^{-1}\partial_\rho\right)^{j_2}\Delta_2^{n+1-j_1-j_2}h\\ &-2j_1\left(r^{-1}\partial_r\right)^{j_1+1}\left(\rho^{-1}\partial_\rho\right)^{j_2}\Delta_2^{n-j_1-j_2}h - 2j_2\left(r^{-1}\partial_r\right)^{j_1}\left(\rho^{-1}\partial_\rho\right)^{j_2+1}\Delta_2^{n-j_1-j_2}h.\end{aligned}$$

This equality and (5) imply that

$$\begin{aligned}&D(j_1,j_2)\Delta_{\underline{X}}\left(\left(r^{-1}\partial_r\right)^{j_1}\left(\rho^{-1}\partial_\rho\right)^{j_2}\Delta_2^{n-j_1-j_2}hP_kP_\ell\right)\\ &= \Big(D(j_1,j_2)\left(r^{-1}\partial_r\right)^{j_1}\left(\rho^{-1}\partial_\rho\right)^{j_2}\Delta_2^{n+1-j_1-j_2}h\\ &\quad + D(j_1+1,j_2)\left(r^{-1}\partial_r\right)^{j_1+1}\left(\rho^{-1}\partial_\rho\right)^{j_2}\Delta_2^{n-j_1-j_2}h\\ &\quad + D(j_1,j_2+1)\left(r^{-1}\partial_r\right)^{j_1}\left(\rho^{-1}\partial_\rho\right)^{j_2+1}\Delta_2^{n-j_1-j_2}h\Big)P_kP_\ell.\end{aligned}$$

Therefore

$$\Delta_{\underline{X}}^{n+1}\big(hP_kP_\ell\big) = \big(T_1+T_2+T_3\big)P_kP_\ell, \tag{6}$$

where

$$T_1 = \sum_{\substack{j_1+j_2\leq n\\ j_1,\, j_2\geq 0}} \binom{n}{j_1, j_2, n-j_1-j_2} D(j_1,j_2)\left(r^{-1}\partial_r\right)^{j_1}\left(\rho^{-1}\partial_\rho\right)^{j_2}\Delta_2^{n+1-j_1-j_2} h,$$

$$T_2 = \sum_{\substack{j_1+j_2\leq n+1\\ j_1\geq 1,\, j_2\geq 0}} \binom{n}{j_1-1, j_2, n+1-j_1-j_2} D(j_1,j_2)\left(r^{-1}\partial_r\right)^{j_1}\left(\rho^{-1}\partial_\rho\right)^{j_2}\Delta_2^{n+1-j_1-j_2} h,$$

$$T_3 = \sum_{\substack{j_1+j_2\leq n+1\\ j_1\geq 0,\, j_2\geq 1}} \binom{n}{j_1, j_2-1, n+1-j_1-j_2} D(j_1,j_2)\left(r^{-1}\partial_r\right)^{j_1}\left(\rho^{-1}\partial_\rho\right)^{j_2}\Delta_2^{n+1-j_1-j_2} h.$$

Observe that set $\{(j_1,j_2): j_1+j_2\leq n+1,\ j_1,j_2\geq 0\}$ can be expressed as the union of the disjoint sets $\{(0,0)\}$, $\{(n+1,0)\}$, $\{(0,n+1)\}$, $\{(j,0): 1\leq j\leq n\}$, $\{(0,j): 1\leq j\leq n\}$, $\{(j,n+1-j): 1\leq j\leq n\}$ and $\{(j_1,j_2): j_1+j_2\leq n,\ j_1,j_2\geq 1\}$. Taking this into account it is easy to verify that (6) equals

$$\left(\sum_{\substack{j_1+j_2\leq n+1\\ j_1,j_2\geq 0}} \binom{n+1}{j_1, j_2, n+1-j_1-j_2} D(j_1,j_2)\left(r^{-1}\partial_r\right)^{j_1}\left(\rho^{-1}\partial_\rho\right)^{j_2}\Delta_2^{n+1-j_1-j_2} h\right) P_k P_\ell.$$

Thus proving the assertion for $n+1$. □

Remark 2. Theorem 2 yields biaxial monogenic functions, i.e. monogenic functions of the form

$$\big(A(r,\rho)+\underline{\omega}\,\underline{\nu}\,B(r,\rho)\big)P_k(\underline{x})P_\ell(\underline{y})$$

or

$$\big(\underline{\omega}\,C(r,\rho)+\underline{\nu}\,D(r,\rho)\big)P_k(\underline{x})P_\ell(\underline{y}),$$

where A, B, C, D are $\mathbb{R}$-valued continuously differentiable functions in $\mathbb{R}^2$ (see [8,15,18,27]). A direct computation shows that the pairs (A,B) and (C,D) satisfy the following Vekua-type systems

$$\begin{cases}\partial_r A+\partial_\rho B = -\dfrac{2\ell+q-1}{\rho}B\\[2mm] \partial_\rho A-\partial_r B=\dfrac{2k+p-1}{r}B,\end{cases}\qquad \begin{cases}\partial_r C+\partial_\rho D = -\dfrac{2k+p-1}{r}C-\dfrac{2\ell+q-1}{\rho}D\\[2mm] \partial_\rho C-\partial_r D=0.\end{cases}$$

We now come to our first main result that generalizes [22] to the biaxial case.

Theorem 3. *Suppose that $w(z,\overline{z})=u(x,y)+iv(x,y)$ is a $\mathbb{C}$-valued function satisfying the Eq. (2) in the open set $\Xi\subset\{(x,y)\in\mathbb{R}^2:\ x,y>0\}$. If p and q are odd, then the functions*

$$\begin{aligned}&\mathsf{Ft}^{\mu,+}_{p,q}\left[w(z,\overline{z}),P_k(\underline{x}),P_\ell(\underline{y})\right](\underline{X})\\ &\quad=\Delta_{\underline{X}}^{\mu+k+\ell+\frac{m-2}{2}}\left[\big(u(r,\rho)+\underline{\omega}\,\underline{\nu}\,v(r,\rho)\big)P_k(\underline{x})P_\ell(\underline{y})\right],\\ &\mathsf{Ft}^{\mu,-}_{p,q}\left[w(z,\overline{z}),P_k(\underline{x}),P_\ell(\underline{y})\right](\underline{X})\\ &\quad=\Delta_{\underline{X}}^{\mu+k+\ell+\frac{m-2}{2}}\left[\big(\underline{\omega}\,u(r,\rho)+\underline{\nu}\,v(r,\rho)\big)P_k(\underline{x})P_\ell(\underline{y})\right]\end{aligned}$$

are monogenic in $\Omega = \left\{\underline{X} = \left(\underline{x}, \underline{y}\right) \in \mathbb{R}^m : \ (r,\rho) \in \Xi\right\}$.

Proof. We use Lemma 2 to compute $\mathsf{Ft}_{p,q}^{\mu,+}$ in closed form. First, note that function w also satisfies the equation $\Delta_2^{\mu+1} w = 0$, thus

$$\Delta_2^{\mu+k+\ell+\frac{m-2}{2}-j_1-j_2} w = 0 \ \text{ for } \ j_1 + j_2 \leq k + \ell + (m-4)/2.$$

Since p and q are odd we also have that $D(j_1, j_2) = 0$ for $j_1 \geq k + (p+1)/2$ or $j_2 \geq \ell + (q+1)/2$. It follows that for $n = \mu + k + \ell + (m-2)/2$ the only term in (4) which does not vanish corresponds to the case $j_1 = k + (p-1)/2$, $j_2 = \ell + (q-1)/2$. Therefore

$$\begin{aligned} \mathsf{Ft}_{p,q}^{\mu,+}\left[w(z,\overline{z}), P_k(\underline{x}), P_\ell(\underline{y})\right](\underline{X}) &= (2k+p-1)!!(2\ell+q-1)!! \\ &\times \begin{pmatrix} \mu+k+\ell+\frac{m-2}{2} \\ k+\frac{p-1}{2}, \ell+\frac{q-1}{2}, \mu \end{pmatrix} \big(A(r,\rho) + \underline{\omega}\,\underline{\nu}\, B(r,\rho)\big) P_k(\underline{x}) P_\ell(\underline{y}), \end{aligned}$$

with

$$\begin{aligned} A &= \left(r^{-1}\partial_r\right)^{k+\frac{p-1}{2}} \left(\rho^{-1}\partial_\rho\right)^{\ell+\frac{q-1}{2}} \Delta_2^\mu u, \\ B &= \left(\partial_r\, r^{-1}\right)^{k+\frac{p-1}{2}} \left(\partial_\rho\, \rho^{-1}\right)^{\ell+\frac{q-1}{2}} \Delta_2^\mu v. \end{aligned}$$

It thus remains to prove that (A, B) fulfills the first system of Remark 2. Using statement (iii) of Lemma 1 and the fact that w satisfies (2) we obtain

$$\begin{aligned} \partial_r A &= \left(\partial_r\, r^{-1}\right)^{k+\frac{p-1}{2}} \left(\rho^{-1}\partial_\rho\right)^{\ell+\frac{q-1}{2}} \partial_r \Delta_2^\mu u \\ &= -\left(\partial_r\, r^{-1}\right)^{k+\frac{p-1}{2}} \left(\rho^{-1}\partial_\rho\right)^{\ell+\frac{q-1}{2}} \partial_\rho \Delta_2^\mu v. \end{aligned}$$

Hence we get

$$\begin{aligned} \partial_r A + \partial_\rho B &= -\left(\partial_r\, r^{-1}\right)^{k+\frac{p-1}{2}} \Bigg(\left(\rho^{-1}\partial_\rho\right)^{\ell+\frac{q-1}{2}} \partial_\rho \Delta_2^\mu v \\ &\quad - \partial_\rho \left(\partial_\rho\, \rho^{-1}\right)^{\ell+\frac{q-1}{2}} \Delta_2^\mu v \Bigg) \\ &= -\frac{2\ell+q-1}{\rho} \left(\partial_r\, r^{-1}\right)^{k+\frac{p-1}{2}} \left(\partial_\rho\, \rho^{-1}\right)^{\ell+\frac{q-1}{2}} \Delta_2^\mu v, \end{aligned}$$

where we have also used statement (iv) of Lemma 1. In a similar fashion, it can be shown that

$$\partial_\rho A - \partial_r B = \frac{2k+p-1}{r} \left(\partial_r\, r^{-1}\right)^{k+\frac{p-1}{2}} \left(\partial_\rho\, \rho^{-1}\right)^{\ell+\frac{q-1}{2}} \Delta_2^\mu v.$$

The proof of $\mathsf{Ft}_{p,q}^{\mu,-}$ goes along the same lines as that of $\mathsf{Ft}_{p,q}^{\mu,+}$. Indeed, it follows from Lemma 2 that

$$\begin{aligned} \mathsf{Ft}_{p,q}^{\mu,-}\left[w(z,\overline{z}), P_k(\underline{x}), P_\ell(\underline{y})\right](\underline{X}) &= (2k+p-1)!!(2\ell+q-1)!! \\ &\times \begin{pmatrix} \mu+k+\ell+\frac{m-2}{2} \\ k+\frac{p-1}{2}, \ell+\frac{q-1}{2}, \mu \end{pmatrix} \big(\underline{\omega}\, C(r,\rho) + \underline{\nu}\, D(r,\rho)\big) P_k(\underline{x}) P_\ell(\underline{y}), \end{aligned}$$

with

$$C = \left(\partial_r\, r^{-1}\right)^{k+\frac{p-1}{2}} \left(\rho^{-1}\partial_\rho\right)^{\ell+\frac{q-1}{2}} \Delta_2^\mu u,$$

$$D = \left(r^{-1}\partial_r\right)^{k+\frac{p-1}{2}} \left(\partial_\rho\, \rho^{-1}\right)^{\ell+\frac{q-1}{2}} \Delta_2^\mu v.$$

One can check, using statements (iii) and (iv) of Lemma 1 as well as equation (2), that (C, D) satisfies the second system of Remark 2. □

3. Fueter's Theorem with General Homogeneous Factors

We arrive at the third and last section of the paper where we shall prove our main result. In the proof we use Theorem 3 and the well-known Fischer decomposition (see [10]):

Theorem 4 (Fischer decomposition). *Every homogeneous polynomial $H_K(\underline{X})$ of degree K in $\mathbb{R}^m$ admits the following decomposition*

$$H_K(\underline{X}) = \sum_{n=0}^{K} \underline{X}^n P_{K-n}(\underline{X}),$$

where each $P_{K-n}(\underline{X})$ is a homogeneous monogenic polynomial.

Theorem 5. *Let $w(\overline{z}) = u(x, y) + iv(x, y)$ be an antiholomorphic function in the open set $\Xi \subset \{(x, y) \in \mathbb{R}^2 :\ x, y > 0\}$. Suppose that $H_k(\underline{x}) : \mathbb{R}^p \to \mathbb{R}_p$ and $H_\ell(\underline{y}) : \mathbb{R}^q \to \mathbb{R}_q$ are homogeneous polynomials. If p and q are odd, then the functions*

$$\mathsf{Ft}^+_{p,q}\left[w(\overline{z}), H_k(\underline{x}), H_\ell(\underline{y})\right](\underline{X}) \quad \text{and} \quad \mathsf{Ft}^-_{p,q}\left[w(\overline{z}), H_k(\underline{x}), H_\ell(\underline{y})\right](\underline{X})$$

are monogenic in $\Omega = \left\{\underline{X} = \left(\underline{x}, \underline{y}\right) \in \mathbb{R}^m :\ (r, \rho) \in \Xi\right\}$.

Proof. We only prove the statement for function $\mathsf{Ft}^+_{p,q}$ since the proof for $\mathsf{Ft}^-_{p,q}$ is similar. Note that the Fischer decomposition ensures the existence of homogeneous monogenic polynomials $P_{k-n_1}(\underline{x})$ and $P_{\ell-n_2}(\underline{y})$ such that

$$H_k(\underline{x}) H_\ell(\underline{y}) = \sum_{n_1=0}^{k} \sum_{n_2=0}^{\ell} \underline{x}^{n_1} P_{k-n_1}(\underline{x})\, \underline{y}^{n_2} P_{\ell-n_2}(\underline{y}).$$

This gives

$$\mathsf{Ft}^+_{p,q}\left[w(\overline{z}), H_k(\underline{x}), H_\ell(\underline{y})\right](\underline{X})$$
$$= \sum_{n_1=0}^{k} \sum_{n_2=0}^{\ell} \Delta_{\underline{X}}^{k+\ell+\frac{m-2}{2}} \Big[\big(u(r, \rho) + \underline{\omega}\, \underline{\nu}\, v(r, \rho)\big) \underline{x}^{n_1} P_{k-n_1}(\underline{x})\, \underline{y}^{n_2} P_{\ell-n_2}(\underline{y})\Big].$$

It will thus be sufficient to prove the monogenicity of each term in the previous sum. On account of Remark 1 we may assume without loss of generality that $P_{k-n_1}(\underline{x})$ takes values in $\mathbb{R}_p^+$ and hence

$$\underline{x}^{n_1} P_{k-n_1}(\underline{x})\, \underline{y}^{n_2} P_{\ell-n_2}(\underline{y}) = \underline{x}^{n_1} \underline{y}^{n_2} P_{k-n_1}(\underline{x}) P_{\ell-n_2}(\underline{y}).$$

It is easy to verify that if n_1+n_2 is even, then

$$\Delta_{\underline{X}}^{k+\ell+\frac{m-2}{2}}\Big[\big(u(r,\rho)+\underline{\omega}\,\underline{\nu}\,v(r,\rho)\big)\underline{x}^{n_1}\underline{y}^{n_2}P_{k-n_1}(\underline{x})P_{\ell-n_2}(\underline{y})\Big]$$
$$=\mathsf{Ft}^{n_1+n_2,+}_{p,q}\left[w(\overline{z})h^+(x,y),P_{k-n_1}(\underline{x}),P_{\ell-n_2}(\underline{y})\right](\underline{X}),$$

where $h^+(x,y)=\begin{cases}(-1)^{\frac{n_1+n_2}{2}}x^{n_1}y^{n_2} & \text{for } n_1,n_2 \text{ even}\\ (-1)^{\frac{n_1+n_2-2}{2}}i\,x^{n_1}y^{n_2} & \text{for } n_1,n_2 \text{ odd.}\end{cases}$

Similarly, if n_1+n_2 is odd, then

$$\Delta_{\underline{X}}^{k+\ell+\frac{m-2}{2}}\Big[\big(u(r,\rho)+\underline{\omega}\,\underline{\nu}\,v(r,\rho)\big)\underline{x}^{n_1}\underline{y}^{n_2}P_{k-n_1}(\underline{x})P_{\ell-n_2}(\underline{y})\Big]$$
$$=\mathsf{Ft}^{n_1+n_2,-}_{p,q}\left[w(\overline{z})h^-(x,y),P_{k-n_1}(\underline{x}),P_{\ell-n_2}(\underline{y})\right](\underline{X}),$$

where $h^-(x,y)=\begin{cases}(-1)^{\frac{n_1+n_2-1}{2}}x^{n_1}y^{n_2} & \text{for } n_1 \text{ odd, } n_2 \text{ even}\\ (-1)^{\frac{n_1+n_2-1}{2}}i\,x^{n_1}y^{n_2} & \text{for } n_1 \text{ even, } n_2 \text{ odd.}\end{cases}$

Clearly, $h^\pm$ satisfies $\partial_z^{n_1+n_2+1}h^\pm=0$ and for that reason

$$\partial_z^{n_1+n_2+1}\big(w(\overline{z})h^\pm(x,y)\big)=w(\overline{z})\partial_z^{n_1+n_2+1}h^\pm(x,y)=0.$$

Consequently, the functions $w(\overline{z})h^\pm(x,y)$ are solutions of (2) for $\mu=n_1+n_2$. The result now follows from Theorem 3. □

We conclude with some examples involving the homogeneous polynomials $H_k(\underline{x})=\langle\underline{x},\underline{t}\rangle^k$, $H_\ell(\underline{y})=\langle\underline{y},\underline{s}\rangle^\ell$, where $\underline{t}\in\mathbb{R}^p$ and $\underline{s}\in\mathbb{R}^q$ are arbitrary fixed vectors. In order to avoid too long computations we have chosen the cases $p=q=3$, $k=1,2$ and $\ell=1$.

$$\begin{aligned}
\mathsf{Ft}^{+}_{3,3}\left[\overline{z}^5,\langle\underline{x},\underline{t}\rangle,\langle\underline{y},\underline{s}\rangle\right](\underline{X}) &= \frac{10}{r^3}\langle\underline{x},\underline{t}\rangle\langle\underline{y},\underline{s}\rangle+\frac{6\,\underline{x}\,\underline{y}}{r^5}\langle\underline{x},\underline{t}\rangle\langle\underline{y},\underline{s}\rangle-\frac{2\,\underline{t}\,\underline{y}}{r^3}\langle\underline{y},\underline{s}\rangle\\
&\quad+\frac{(5r^2+3\rho^2)\underline{x}\,\underline{s}}{r^5}\langle\underline{x},\underline{t}\rangle-\frac{(5r^2+\rho^2)\underline{t}\,\underline{s}}{r^3}\\
\mathsf{Ft}^{+}_{3,3}\left[\overline{z}^8,\langle\underline{x},\underline{t}\rangle,\langle\underline{y},\underline{s}\rangle\right](\underline{X}) &= 10\langle\underline{x},\underline{t}\rangle\langle\underline{y},\underline{s}\rangle-2\,\underline{t}\,\underline{y}\langle\underline{y},\underline{s}\rangle+2\,\underline{x}\,\underline{s}\langle\underline{x},\underline{t}\rangle+(r^2-\rho^2)\underline{t}\,\underline{s}\\
\mathsf{Ft}^{+}_{3,3}\left[\overline{z}^{10},\langle\underline{x},\underline{t}\rangle,\langle\underline{y},\underline{s}\rangle\right](\underline{X}) &= 140(r^2-\rho^2)\langle\underline{x},\underline{t}\rangle\langle\underline{y},\underline{s}\rangle-56\,\underline{x}\,\underline{y}\langle\underline{x},\underline{t}\rangle\langle\underline{y},\underline{s}\rangle\\
&\quad-4(7r^2-5\rho^2)\underline{t}\,\underline{y}\langle\underline{y},\underline{s}\rangle+4(5r^2-7\rho^2)\underline{x}\,\underline{s}\langle\underline{x},\underline{t}\rangle\\
&\quad+(5r^4-14r^2\rho^2+5\rho^4)\underline{t}\,\underline{s}\\
\mathsf{Ft}^{-}_{3,3}\left[i\overline{z}^6,\langle\underline{x},\underline{t}\rangle,\langle\underline{y},\underline{s}\rangle\right](\underline{X}) &= \frac{2\,\underline{x}}{\rho^3}\langle\underline{x},\underline{t}\rangle\langle\underline{y},\underline{s}\rangle-\frac{(3r^2+5\rho^2)\underline{y}}{\rho^5}\langle\underline{x},\underline{t}\rangle\langle\underline{y},\underline{s}\rangle\\
&\quad+\frac{(r^2+5\rho^2)}{\rho^3}\left(\underline{t}\langle\underline{y},\underline{s}\rangle+\underline{s}\langle\underline{x},\underline{t}\rangle\right)\\
\mathsf{Ft}^{-}_{3,3}\left[\overline{z}^9,\langle\underline{x},\underline{t}\rangle,\langle\underline{y},\underline{s}\rangle\right](\underline{X}) &= 2(\underline{x}-\underline{y})\langle\underline{x},\underline{t}\rangle\langle\underline{y},\underline{s}\rangle+(r^2-\rho^2)\left(\underline{t}\langle\underline{y},\underline{s}\rangle+\underline{s}\langle\underline{x},\underline{t}\rangle\right)\\
\mathsf{Ft}^{-}_{3,3}\left[\overline{z}^{11},\langle\underline{x},\underline{t}\rangle^2,\langle\underline{y},\underline{s}\rangle\right](\underline{X}) &= 8\left(5\underline{x}-7\underline{y}\right)\langle\underline{x},\underline{t}\rangle^2\langle\underline{y},\underline{s}\rangle-4(7r^2-5\rho^2)|\underline{t}|^2\underline{y}\langle\underline{y},\underline{s}\rangle\\
&\quad+4(5r^2-7\rho^2)\left(2\underline{t}\langle\underline{x},\underline{t}\rangle\langle\underline{y},\underline{s}\rangle+|\underline{t}|^2\underline{x}\langle\underline{y},\underline{s}\rangle+\underline{s}\langle\underline{x},\underline{t}\rangle^2\right)\\
&\quad+(5r^4-14r^2\rho^2+5\rho^4)|\underline{t}|^2\underline{s}
\end{aligned}$$

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