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Results in Mathematics

Continuity and Structure of Generalized (ϕ, ψ)-Derivations

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Abstract. Let $\mathcal{X}$ be a Banach algebra, let ϕ, ψ be mappings on $\mathcal{X}$, let δ be a (ϕ, ψ)-derivation on $\mathcal{X}$ and let d be a generalized (ϕ, ψ)-derivation related to δ. If $\mathcal{X}$ is simple, we determine some sufficient conditions under which every generalized (ϕ, ψ)-derivation on $\mathcal{X}$ is continuous (without continuity of δ). In addition, we show that if d is inner on $\mathcal{F}_1(X)$ (the set of all rank one operators on $\mathcal{X}$) and $\phi, \psi : \mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ are homomorphisms and surjective on $\mathcal{F}_1(X)$ then d is inner on $\mathcal{B}(\mathcal{X})$. Finally, we characterize the linear mappings on $\mathcal{B}(\mathcal{X})$ which behave like generalized (ϕ, ψ)-derivations when acting on zero products.

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1. Introduction and Preliminaries

Suppose that $\mathcal{X}$ is a complex algebra and $\phi, \psi : \mathcal{X} \longrightarrow \mathcal{X}$ are two mappings. A linear mapping $\delta : \mathcal{X} \longrightarrow \mathcal{X}$ is called a (ϕ, ψ)-derivation, if for each $x, y \in \mathcal{X}$

$$\delta(xy) = \delta(x)\phi(y) + \psi(x)\delta(y). \tag{1.1}$$

Let $\phi, \psi : \mathcal{X} \longrightarrow \mathcal{X}$ be two mappings and let $\delta : \mathcal{X} \longrightarrow \mathcal{X}$ be a (ϕ, ψ)-derivation. A linear mapping $d : \mathcal{X} \longrightarrow \mathcal{X}$ is called a generalized (ϕ, ψ)-derivation related to δ, if

$$d(xy) = d(x)\phi(y) + \psi(x)\delta(y) \quad (x, y \in \mathcal{X}). \tag{1.2}$$

If ϕ, ψ are automorphisms of algebra $\mathcal{X}$ and there exist $x_0, y_0 \in \mathcal{X}$ such that $d(z) = x_0\phi(z) - \psi(z)y_0$, for all $z \in \mathcal{X}$ then d is a generalized (ϕ, ψ)-derivation which is called inner generalized (ϕ, ψ)-derivation.

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The theory of automatic continuity of derivations has a long history. Sakai [11] answered to the conjecture made by Kaplansky in [8]. He proved that every derivation on a C^*-algebra is automatically continuous. In [10], Ringrose generalized these results to derivations from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-module. Johnson and Sinclair in [6] proved the continuity of derivations on semisimple Banach algebras. In [9], it's shown that every (ϕ, ψ)-derivation on a C^*-algebra is automatically continuous, if ϕ, ψ are continuous *-linear mappings and in [3] the assumption of linearity of ϕ, ψ were deleted. Hou et al. [4] proved that every (ϕ, ψ)-derivation on $\mathcal{B}(\mathcal{X})$ is continuous, if $\mathcal{X}$ is simple and ϕ, ψ are surjective and continuous mappings on $\mathcal{B}(\mathcal{X})$. For more results concerning these subjects, we refer to [2,13].

In [7], Kadison showed that every derivation of a C^*-algebra on Hilbert space is spatial and every derivation of a von Neumann algebra is inner. Sakai [12] proved that every derivation of a W^*-algebra is inner. Innerness of derivations of a nest algebra has been proved by Christensen in [1]. Also Hou et al. [4] give some sufficient conditions on which every (ϕ, ψ)-derivation of $\mathcal{B}(\mathcal{X})$ is inner.

In [5], Jing et al. showed that an additive mapping on operator algebra is almost a derivation, if it satisfies the formula of derivations on pairs of elements with zero product. We discuss this issue in the last section.

In this paper, we extend some of the results concerning (ϕ, ψ)-derivations to generalized (ϕ, ψ)-derivations. In fact, by investigating of the notion of generalized (ϕ, ψ)-derivations, we deduce this results for the notion of derivations (if $\phi = \psi = id$ and $\delta = d$), the notion of generalized derivations (if $\phi = \psi = id$), the notion of (ϕ, ψ)-derivations (if $\delta = d$), and the notion of left centralizer (if $\phi = \psi = id$ and $\delta = 0$). So it's interesting to investigate details of this general notion of derivations.

This paper consists of three sections. After introducing the notion of generalized (ϕ, ψ)-derivations in the first section, we give some examples and theorems concerning continuity of generalized (ϕ, ψ)-derivations on a Banach algebra, in the second. In the third section, we give some sufficient conditions on which every (ϕ, ψ)-derivation on $\mathcal{B}(\mathcal{X})$ is inner. Also we characterize the linear mappings on $\mathcal{B}(\mathcal{X})$ which behave like derivations when acting on zero products.

For a complex Banach algebra $\mathcal{X}$ we denote the Banach dual space of $\mathcal{X}$ and the algebra of bounded linear operators on $\mathcal{X}$, by $\mathcal{X}^*$ and $\mathcal{B}(\mathcal{X})$. Denote by $\mathcal{F}_1(\mathcal{X})$, the set of all rank one operators on $\mathcal{X}$. We define rank one operators on $\mathcal{X}$ by $(x \otimes f)(y) = f(y)x$, for $x, y \in \mathcal{X}$ and $f \in \mathcal{X}^*$. We collect some properties of $x \otimes f$ in the following lemma;

Lemma 1. *Let $\mathcal{X}$ be a Banach algebra then for each $x, y \in \mathcal{X}$, $f, g \in \mathcal{X}^*$ and $T, S \in \mathcal{B}(\mathcal{X})$ we have;*

(1) *$T(x \otimes f)S = (Tx) \otimes (S'f)$, where S' denotes the adjoint of the operator S,*

(2) $(x \otimes f)(y \otimes g) = f(y)(x \otimes g)$,
(3) *If* $T(x \otimes f)=0$, *for each* $x \in \mathcal{X}$ *then* $T = 0$.

Recall that if $\mathcal{X}, \mathcal{Y}$ are normed spaces and $T : \mathcal{X} \longrightarrow \mathcal{Y}$ is a linear mapping, the separating space $\mathfrak{S}(T)$ of T is the set of all y such that there is a sequence $\{x_n\}$ in $\mathcal{X}$ that $x_n \longrightarrow 0$ and $Tx_n \longrightarrow y$. Clearly, $\mathfrak{S}(T)$ is a closed linear space. Also, if $\mathcal{X}, \mathcal{Y}$ are Banach spaces, by closed graph theorem, T is continuous if and only if $\mathfrak{S}(T) = \{0\}$.

Using assertions of [4], to investigate the continuity and structure of generalized (ϕ, ψ)-derivations of Banach algebras, we may assume that $\mathcal{X}$ is unital, $\delta(1) = 0$, $\phi(0) = \psi(0) = 0$, $\phi(1) = \psi(1) = 1$, where 1 is the identity of $\mathcal{X}$.

2. Automatic Continuity of Generalized (ϕ, ψ)-Derivations of Banach Algebras

We start this section with some examples.

Example 1. We use idea of Example 2.4 in [3] and we assume that γ, θ are two arbitrary mappings on $\mathcal{C}[0, 2]$ and take $f_1, f_2, h_0, h_1 \in \mathcal{C}[0, 2]$ such that $f_1 h_0 = 0 = f_2 h_0$ and $f_2 h_1 = 0$. Define ϕ, ψ, d, δ on $\mathcal{C}[0, 2]$ by

$$\psi(f) = \gamma(f) f_1, \quad \phi(f) = f + f_2 \theta(f)$$
$$\delta(f) = f h_0, \quad d(f) = f h_0 + f h_1.$$

Clearly, δ is a (ϕ, ψ)-derivation and d is a generalized (ϕ, ψ)-derivation related to δ. (We see that γ and θ can be nonlinear and discontinuous.)

Example 2. Let $\mathcal{X}$ be a Banach algebra, let γ, θ be continuous homomorphisms of $\mathcal{X}$, let f_1, f_2 be two functionals without linearity and continuity, and $x_1, y_1, z_1 \in \mathcal{X}$ with $x_1 z_1 = 0 = y_1 z_1$. Define the mappings ϕ, ψ, d on $\mathcal{X}$ for $u \in \mathcal{X}$ by

$$\phi(u) = \gamma(u) + f_1(u) z_1, \quad \psi(u) = \theta(u) + f_2(u) z_1$$
$$d(u) = x_1 \phi(u) - \psi(u) y_1.$$

So ϕ, ψ are nonlinear and discontinuous. Although, ϕ, ψ are not automorphism, we see d is an inner generalized (ϕ, ψ)-derivation of $\mathcal{X}$. Indeed, by definition of ϕ, ψ and the relations of between x_1, z_1 and y_1, z_1 we can show that d is an inner generalized (ϕ, ψ)-derivation.

Also, since we have $d(u) = x_1 \gamma(u) - \theta(u) y_1$ for each $u \in \mathcal{X}$, thus d is continuous.

Lemma 2. *Let* $\mathcal{X}$ *be a complex Banach algebra, let* $\phi, \psi : \mathcal{X} \longrightarrow \mathcal{X}$ *be two mappings with* ϕ *be continuous at* 0 *and let* d *be a generalized* (ϕ, ψ)*-derivation related to* δ*. If* $x_n \to 0$ *and* $d(x_n) \longrightarrow x$ *then* $\delta(x_n) \longrightarrow x$.

Proof. By Definition of generalized (ϕ, ψ)-derivations (1.2) we have

$$x = \lim_{n\to\infty} d(1x_n) = \lim_{n\to\infty}(d(1)\phi(x_n) + \psi(1)\delta(x_n)) = \lim_{n\to\infty} \delta(x_n).$$

□

Proposition 1. *Let $\mathcal{X}$ be a Banach algebra and let $\mathcal{M}$ be a Banach $\mathcal{X}$-bimodule. Let $\phi, \psi : \mathcal{X} \longrightarrow \mathcal{M}$ be two mappings with ϕ be continuous at 0. If $d : \mathcal{X} \longrightarrow \mathcal{M}$ is a generalized (ϕ, ψ)-derivation related to $\delta : \mathcal{X} \longrightarrow \mathcal{M}$ then d is continuous if and only if δ is continuous.*

Proof. Put $T := d - \delta$. Obviously, $T(xy) = T(x)\phi(y)$. So $T(y) = T(1y) = T(1)\phi(y)$. Therefore continuity of ϕ implies that T is continuous. Hence d is continuous if and only if δ is continuous. □

Now we are going to delete the assumption of the continuity of δ and obtain continuity of d. We present the first main result.

Theorem 1. *Let $\mathcal{X}$ be a simple complex Banach algebra and let $\phi, \psi : \mathcal{X} \longrightarrow \mathcal{X}$ be surjective and continuous at 0. If at least either ϕ or ψ is not homomorphism, then every generalized (ϕ, ψ)-derivation $d : \mathcal{X} \longrightarrow \mathcal{X}$ related to δ is continuous.*

Proof. At first, using assertions of [4], we can assume that $\phi^(0) = 0$ and $\psi(0) = 0$. Now, we show that $\mathfrak{S}(d)$ is two sided ideal in $\mathcal{X}$. For this, if $a \in \mathfrak{S}(d)$ then there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $x_n \to 0$ and $d(x_n) \to a$. Since ϕ and ψ are surjective, for every $x \in \mathcal{X}$ there are $y, z \in \mathcal{X}$ such that $\phi(y) = x$ and $\psi(z) = x$. So we have

$$\lim_{n\to\infty} d(x_n y) = \lim_{n\to\infty}(d(x_n)\phi(y) + \psi(x_n)\delta(y)) = \lim_{n\to\infty} d(x_n)\phi(y) = ax$$

with $x_n y \to 0$. Therefore $ax \in \mathfrak{S}(d)$. Similarly, by Lemma 2, we have $xa = \lim_{n\to\infty} d(zx_n)$ with $zx_n \to 0$, so $xa \in \mathfrak{S}(d)$. Hence $\mathfrak{S}$(d) is two sided ideal in $\mathcal{X}$. Since $\mathcal{X}$ is simple, either $\mathfrak{S}$(d)=$\{0\}$ or $\mathfrak{S}$(d)=$\mathcal{X}$.

If $\mathfrak{S}$(d)=$\mathcal{X}$, for arbitrary $y, z \in \mathcal{X}$ and $x \in \mathfrak{S}$(d), let $d(x_n) \to x$ which $x_n \to 0$. By using idea of Lemma 3.1 in [3], we have $d(x_n)(\phi(yz) - \phi(y)\phi(z)) = (\psi(x_n y) - \psi(x_n)\psi(y))\delta(z)$, which obtain

$$x(\phi(yz) - \phi(y)\phi(z)) = 0. \tag{2.1}$$

Similarly, we get to $d(z)(\phi(yx_n) - \phi(y)\phi(x_n)) = (\psi(zy) - \psi(z)\psi(y))\delta(x_n)$. So

$$(\psi(yz) - \psi(y)\psi(z))x = 0. \tag{2.2}$$

Similarly, by $d(x_n)(\phi(\lambda y + z) - \lambda\phi(y) - \phi(z)) = 0$, we can obtain that

$$x(\phi(\lambda y + z) - \lambda\phi(y) - \phi(z)) = 0. \tag{2.3}$$

And by $(\psi(\lambda y + z) - \lambda\psi(y) - \psi(z))\delta(x_n) = 0$, we have

$$(\psi(\lambda y + z) - \lambda\psi(y) - \psi(z))x = 0. \tag{2.4}$$

These four equations show that ϕ, ψ are homomorphisms, which is a contradiction. Hence $\mathfrak{S}$(d)=$\{0\}$. It means that d is continuous. □

Theorem 2. *Let $\mathcal{X}$ be a simple complex Banach algebra, let $\phi,\psi:\mathcal{X}\longrightarrow\mathcal{X}$ be surjective and continuous at 0 and let d be a generalized (ϕ,ψ)-derivation related to δ. If one of the following conditions holds:*

(1) $\mathcal{A}nn_l(\delta(\mathcal{X}))=\{y\in\mathcal{X}\mid y\delta(x)=0,\forall x\in\mathcal{X}\}\neq\{0\}$
(2) $\mathcal{A}nn_r(\delta(\mathcal{X}))=\{y\in\mathcal{X}\mid \delta(x)y=0,\forall x\in\mathcal{X}\}\neq\{0\}$
(3) *There exists a noninvertible and nonidempotent element $x_0\in\mathcal{Z}(\mathcal{X})$ (the center of $\mathcal{X}$) with $0\neq x_0^2=x_0^3$.*

Then d is continuous.

Proof. If at least either ϕ or ψ is not a homomorphism then by Theorem 1 we have done. Now, we suppose that ϕ,ψ are homomorphisms.

If (1) holds, then for $0\neq x_0\in\mathcal{A}nn_l(\delta(\mathcal{X}))$, we define $\psi_0:\mathcal{X}\longrightarrow\mathcal{X}$ with $\psi_0(x)=\psi(x)+x_0$. Since ψ is surjective and continuous at zero, ψ_0 is surjective and continuous at zero, too. However ψ_0 is not homomorphism. Clearly, δ is a (ϕ,ψ_0)-derivation and d is a generalized (ϕ,ψ_0)-derivation related to δ. Hence by Theorem 1, d is continuous.

If (2) holds, then for $0\neq x_0\in\mathcal{A}nn_r(\delta(\mathcal{X}))$, we define $\phi_0:\mathcal{X}\longrightarrow\mathcal{X}$ with $\phi_0(x)=\phi(x)+x_0$. Obviously, ϕ_0 is surjective and continuous at zero and δ is a (ϕ_0,ψ)-derivation. We can consider δ as a generalized (ϕ_0,ψ)-derivation related to δ so by Theorem 1, δ is continuous. Therefore by Proposition 1, d is continuous.

If (3) holds, then for $0\neq x_0\in\mathcal{Z}(\mathcal{X})$ with $x_0^2\neq x_0$ and $0\neq x_0^2=x_0^3$, by a simple calculating we have

$$\begin{aligned}x_0^2\delta(xy)&=x_0^2\delta(x)\phi(y)+x_0^2\psi(x)\delta(y)\\&=x_0^2\delta(x)\phi(y)+x_0^3\psi(x)\delta(y)\\&=x_0^2\delta(x)\phi(y)+x_0\psi(x)x_0^2\delta(y)\end{aligned}$$

So $x_0^2\delta$ is a $(\phi,x_0\psi)$-derivation and similarly, x_0^2d is a generalized $(\phi,x_0\psi)$-derivation related to $x_0^2\delta$. Since $(x_0^2-x_0)\psi(\mathcal{X})=(x_0^2-x_0)\mathcal{X}$ is an ideal of $\mathcal{X}$ containing $(x_0^2-x_0)$ and $\mathcal{X}$ is simple, so $(x_0^2-x_0)\psi(\mathcal{X})=\mathcal{X}$. Therefore $(x_0\psi)(xy)-(x_0\psi)(x)(x_0\psi)(y)=(x_0\psi)(xy)-(x_0^2\psi)(xy)\neq 0$ for some $x,y\in\mathcal{X}$. So $x_0\psi$ is not homomorphism. Also $(x_0\psi)(\mathcal{X})=x_0\mathcal{X}$ is an ideal of $\mathcal{X}$ so $x_0\psi(\mathcal{X})=\mathcal{X}$. Hence $x_0\psi$ is surjective. By Theorem 1, x_0^2d is continuous. Since $x_0^2\mathfrak{S}(d)\subseteq\mathfrak{S}(x_0^2d)$, we have $x_0^2\mathfrak{S}(d)=\{0\}$. We know $\mathfrak{S}(d)$ is an ideal of $\mathcal{X}$ (by first part of Theorem 1). Since $\mathcal{X}$ is simple, either $\mathfrak{S}(d)=\{0\}$ or $\mathfrak{S}(d)=\mathcal{X}$. If $\mathfrak{S}(d)=\mathcal{X}$ then $x_0^2\mathcal{X}=\{0\}$, so $x_0^2=x_0^3=0$ which is a contradiction. Therefore $\mathfrak{S}(d)=\{0\}$ and d is continuous. □

3. Characterizing Structure of Generalized (ϕ,ψ)-Derivations on $\mathcal{B}(\mathcal{X})$

The first theorem of this section states that innerness of generalized (ϕ,ψ)-derivations on $\mathcal{F}_1(\mathcal{X})$ completely decide innerness of generalized (ϕ,ψ)-derivations on $\mathcal{B}(\mathcal{X})$.

Theorem 3. *Let $\mathcal{X}$ be a complex Banach algebra and let two homomorphisms $\phi, \psi : \mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ be surjective on $\mathcal{F}_1(\mathcal{X})$. If $d : \mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ is a generalized (ϕ, ψ)-derivation related to δ which is inner on $\mathcal{F}_1(\mathcal{X})$ then d is inner on $\mathcal{B}(\mathcal{X})$.*

Proof. By the hypotheses on d, there exist $T, S \in \mathcal{B}(\mathcal{X})$ such that $d(x \otimes f) = S\phi(x \otimes f) - \psi(x \otimes f)T$, for every $x \otimes f \in \mathcal{F}_1(X)$. We define d' on $\mathcal{B}(\mathcal{X})$ by

$$d'(A) = d(A) - S\phi(A) + \psi(A)T \quad (A \in \mathcal{B}(\mathcal{X})).$$

We recal that $\phi(1) = 1$ and $\delta(1) = 0$. Now, the result is proved by four steps;

- At first, we define δ' on $\mathcal{B}(\mathcal{X})$ by

$$\delta'(B) = \delta(B) + \psi(B)T - T\phi(B) \quad (B \in \mathcal{B}(\mathcal{X})).$$

  We show that δ' is a (ϕ, ψ)-derivation. Indeed,

$$\begin{aligned}\delta'(AB) &= \delta(AB) + \psi(AB)T - T\phi(AB)\\ &= \delta(A)\phi(B) + \psi(A)\delta(B) + \psi(A)\psi(B)T - T\phi(A)\phi(B)\\ &\quad + (\psi(A)T\phi(B) - \psi(A)T\phi(B))\\ &= (\delta(A) + \psi(A)T - T\phi(A))\phi(B) + \psi(A)(\delta(B) - T\phi(B) + \psi(B)T)\\ &= \delta'(A)\phi(B) + \psi(A)\delta'(B).\end{aligned}$$

- Also we show that d' is a generalized (ϕ, ψ)-derivation related to δ'. Since

$$\begin{aligned}d'(AB) &= d(AB) - S\phi(AB) + \psi(AB)T\\ &= d(A)\phi(B) + \psi(A)\delta(B) - S\phi(A)\phi(B) + \psi(A)\psi(B)T\\ &\quad + (\psi(A)T\phi(B) - \psi(A)T\phi(B))\\ &= (d(A) - S\phi(A) + \psi(A)T)\phi(B) + \psi(A)(\delta(B) + \psi(B)T - T\phi(B))\\ &= d'(A)\phi(B) + \psi(A)\delta'(B).\end{aligned}$$

- Obviously, $d' |_{\mathcal{F}_1(X)} \equiv 0$. At this step we show $\delta' |_{\mathcal{F}_1(X)} \equiv 0$. For $x \otimes f, y \otimes g \in \mathcal{F}_1(X)$ we have

$$0 = d'((x \otimes f)(y \otimes g)) = d'(x \otimes f)\phi(y \otimes g) + \psi(x \otimes f)\delta'(y \otimes g).$$

  Since $d' |_{\mathcal{F}_1(X)} \equiv 0$ and ψ is surjective on $\mathcal{F}_1(X)$ so $\delta'(y \otimes g) = 0$.
- Finally, we show that $d' \equiv 0$ on $\mathcal{B}(\mathcal{X})$. For each $A \in \mathcal{B}(\mathcal{X})$ and $x \otimes f \in \mathcal{F}_1(X)$ we have

$$0 = d'(A(x \otimes f)) = d'(A)\phi(x \otimes f) + \psi(A)\delta'(x \otimes f).$$

  Since $\delta' |_{\mathcal{F}_1(X)} \equiv 0$ and ϕ is surjective on $\mathcal{F}_1(X)$, we obtain that $d'(A) = 0$. Hence we conclude that $d(A) = S\phi(A) - \psi(A)T$, for $A \in \mathcal{B}(\mathcal{X})$. □

In the next theorem we show that a linear mapping on $\mathcal{B}(\mathcal{X})$ which behave like a generalized (ϕ, ψ)-derivation on pairs of elements with zero product is a generalized (ϕ, ψ)-derivation. Also, we recal that it can be assumed $\phi^(1) = 1$ and $\delta(1) = 0$.

Theorem 4. *Let $\mathcal{X}$ be a Banach algebra and let $\phi, \psi : \mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ be homomorphisms such that ϕ be surjective on $\mathcal{F}_1(\mathcal{X})$. If d is a linear mapping on $\mathcal{B}(\mathcal{X})$ such that for every $A, B \in \mathcal{B}(\mathcal{X})$ with $AB = 0$, $d(AB) = d(A)\phi(B) + \psi(A)\delta(B)$ in which δ is a (ϕ, ψ)-derivation. Then d is a generalized (ϕ, ψ)-derivation related to δ.*

Proof. Let $p \in \mathcal{B}(\mathcal{X})$ be an idempotent. For every $A \in \mathcal{B}(\mathcal{X})$ we have $Ap(I - p) = 0 = A(I - p)p$. Since

$$\begin{aligned} d(Ap(I-p)) &= d(Ap)\phi(I-p) + \psi(Ap)\delta(I-p) \\ &= d(Ap) - d(Ap)\phi(p) - \psi(Ap)\delta(p) \end{aligned}$$

and

$$\begin{aligned} d(A(I-p)p) &= d(A - Ap)\phi(p) + \psi(A - Ap)\delta(p) \\ &= d(A)\phi(p) - d(Ap)\phi(p) + \psi(A)\delta(p) - \psi(Ap)\delta(p). \end{aligned}$$

By comparing these equalities we obtain that $d(Ap) = d(A)\phi(p) + \psi(A)\delta(p)$ for $A \in \mathcal{B}(\mathcal{X})$. For $x_0 \in \mathcal{X}$ and $f \in \mathcal{X}^*$, if $f(x_0) = 1$ then $x_0 \otimes f$ is an idempotent. Now we substitude p by $x_0 \otimes f$. So we have for all $A \in \mathcal{B}(\mathcal{X})$

$$d(A(x_0 \otimes f)) = d(A)\phi(x_0 \otimes f) + \psi(A)\delta(x_0 \otimes f). \tag{3.1}$$

For each $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, we choose $\lambda \in \mathbb{C}$ such that $\mu = f(\lambda x + x_0) \neq 0$. Put $y := \lambda x + x_0$, so

$$\begin{aligned} &\mu^{-1}\lambda d(A(x \otimes f)) + \mu^{-1}d(A(x_0 \otimes f)) = d(A(\mu^{-1}y \otimes f)) \\ &\quad = d(A)\phi(\mu^{-1}y \otimes f) + \psi(A)\delta(\mu^{-1}y \otimes f) \\ &\quad = d(A)\phi(\mu^{-1}\lambda x \otimes f) + d(A)\phi(\mu^{-1}x_0 \otimes f) \\ &\qquad + \psi(A)\delta(\mu^{-1}\lambda x \otimes f) + \psi(A)\delta(\mu^{-1}x_0 \otimes f). \end{aligned}$$

These equalities together with 3.1 implies

$$d(A(x \otimes f)) = d(A)\phi(x \otimes f) + \psi(A)\delta(x \otimes f)$$

for all $A \in \mathcal{B}(\mathcal{X})$, $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$. Replacing A by AB for $A, B \in \mathcal{B}(\mathcal{X})$, we get

$$d(AB(x \otimes f)) = d(AB)\phi(x \otimes f) + \psi(AB)\delta(x \otimes f). \tag{3.2}$$

On the other hand

$$\begin{aligned} d(A(Bx \otimes f)) &= d(A)\phi(Bx \otimes f) + \psi(A)\delta(Bx \otimes f) \\ &= d(A)\phi(B)\phi(x \otimes f) + \psi(A)\delta(B)\phi(x \otimes f) + \psi(A)\psi(B)\delta(x \otimes f). \end{aligned}$$

Comparing Eq. 3.2 and the last equation, we conclude

$$(d(AB) - d(A)\phi(B) - \psi(A)\delta(B))\phi(x \otimes f) = 0.$$

Hence by surjectivity of ϕ on $\mathcal{F}_1(\mathcal{X})$, for each $y \in \mathcal{X}$ we have

$$\begin{aligned}&(d(AB) - d(A)\phi(B) - \psi(A)\delta(B))y \otimes g \\ &\quad = (d(AB) - d(A)\phi(B) - \psi(A)\delta(B))\phi(x \otimes f) = 0.\end{aligned}$$

Therefore d is a generalized (ϕ, ψ)-derivation related to δ. □

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