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Results in Mathematics

On a New Type of Hyperstability for Radical Cubic Functional Equation in Non-Archimedean Metric Spaces

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Abstract. Let $\mathbb{R}$ be the set of real numbers, $(G, +)$ be a commutative group and d be a complete ultrametric on G that is invariant (i.e., $d(x+z, y+z) = d(x, y)$ for $x, y, z \in G$). Under some weak natural assumptions on the function $\gamma : \mathbb{R}^2 \to [0, \infty)$, we study the generalised hyperstability results when $f : \mathbb{R} \to G$ satisfy the following radical cubic inequality

$$d\big(f(\sqrt[3]{x^3 + y^3}), f(x) + f(y)\big) \leq \gamma(x, y), \quad x, y \in \mathbb{R}\backslash\{0\},$$

with $x \neq -y$. The method is based on a quite recent fixed point theorem (cf. Brzdęk and Ciepliński in Nonlinear Anal 74:6861–6867, 2011, Theorem 1) in some functions spaces.

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1. Introduction

In this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the sets of all positive integers, and real numbers, respectively; we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$ and $\mathbb{R}_+ = [0, \infty)$. Moreover, E_1 and E_2 always stand for normed spaces.

Let us recall that a metric d on a nonempty set M is said to be non-Archimedean (or an ultrametric) provided $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for $x, y, z \in M$. One of important examples of the ultrametric spaces is a non-Archimedean normed space (see, for instance, [24]).

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Definition 1.1. By a non-Archimedean field, we mean a field $\mathbb{K}$ equipped with a function(valuation) $|.| : \mathbb{K} \to [0, \infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:

(1) $|r| = 0$ if and only if $r = 0$;
(2) $|rs| = |r||s|$;
(3) $|r + s| \leq \max(|r|, |s|)$ for all $r, s \in \mathbb{K}$.

Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. The function $|.|$ is called the trivial valuation if $|r| = 1$, $\forall r \in \mathbb{K}$, $r \neq 0$, and $|0| = 0$.

Definition 1.2. Let E be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|.|$. A function $\|.\| : E \to \mathbb{R}_+$ is non-Archimedean norm (valuation) if it satisfies the following conditions:

(1) $\|x\| = 0$ if and only if $x = 0$;
(2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in E$;
(3) $\|x + y\| \leq \max(\|x\|, \|y\|)$ for all $x, y \in E$.

Then, $(E, \|.\|)$ is called a non-Archimedean space or an ultrametric normed space. Due to the fact that

$$\|x_m - x_n\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\},$$

in which $n > m$, the sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. In a complete non-Archimedean space, every Cauchy sequence is convergent.

Clearly, if E is a non-Archimedean normed space, then the formula $d(x, y) := \|x - y\|$ defines an ultrametric d in E, that is invariant (i.e., $d(x+z, y+z) = d(x, y)$ for every $x, y, z \in E$).

The first example of a non-Archimedean field was provided by Hensel in [21], where he gave a description of the p-adic numbers [for each fixed prime number p and any nonzero rational number x, there exists a unique integer n_x such that $x = (a/b)p^{n_x}$, where a and b are integers not divisible by p; then $|x|_p := p^{-n_x}$ defines a non-Archimedean valuation on $\mathbb{Q}$ (the set of rational numbers)]. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, and it is called the p-adic number field.

The next definition describes the notion of hyperstability that we apply here (A^B denotes the family of all functions mapping a set $B \neq \emptyset$ into a set $A \neq \emptyset$).

Definition 1.3. Let $A \neq \emptyset$ be a nonempty set, (Z, d) be a metric space, $\gamma : A^n \to \mathbb{R}_+$, $B \subset A^n$ be nonempty, and $\mathcal{F}_1, \mathcal{F}_2$ map a nonempty $\mathcal{D} \subset Z^A$ into Z^{A^n}. We say that the conditional equation

$$\mathcal{F}_1\varphi(x_1, \ldots, x_n) = \mathcal{F}_2\varphi(x_1, \ldots, x_n), \qquad (x_1, \ldots, x_n) \in B, \tag{1.1}$$

is γ-hyperstable provided every $\varphi_0 \in \mathcal{D}$, satisfying

$$d\big(\mathcal{F}_1\varphi_0(x_1, \ldots, x_n), \mathcal{F}_2\varphi_0(x_1, \ldots, x_n)\big) \leq \gamma(x_1, \ldots, x_n), \qquad (x_1, \ldots, x_n) \in B, \tag{1.2}$$

is a solution to (1.1).

That notion is strictly connected with the well know issue of Ulam's stability for various (e.g., difference, differential, functional, integral, operator) equations. Let us recall that the study of such problems was motivated by the following question of Ulam (cf. [22,34]) asked in 1940.

Ulam's question. *Let* $(G_1, *)$, $(G_2, \star)$ *be two groups and* $\rho : G_2 \times G_2 \to [0, \infty)$ *be a metric. Given* $\delta > 0$, *does there exist* $\epsilon > 0$ *such that if a function* $g : G_1 \to G_2$ *satisfies the inequality*

$$\rho(g(x * y), g(x) \star g(y)) \leq \delta$$

for all $x, y \in G_1$, *then there is a homomorphism* $h : G_1 \to G_2$ *with*

$$\rho(g(x), h(x)) \leq \epsilon$$

for all $x \in G$ *?*

In 1941, Hyers [22] published the first answer to it, in the case of Banach space. The following theorem is the most classical result concerning Ulam's stability of the Cauchy equation

$$f(x + y) = f(x) + f(y), \qquad x, y \in E_1. \tag{1.3}$$

Theorem 1.1. *Let* $f : E_1 \to E_2$ *satisfy the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{1.4}$$

for all $x, y \in E_1 \backslash \{0\}$, *where* θ *and* p *are real constants with* $\theta > 0$ *and* $p \neq 1$. *Then the following two statements are valid.*

(a) *If* $p \geq 0$ *and* E_2 *is complete, then there exists a unique solution* $T : E_1 \to E_2$ *of* (1.3) *such that*

$$\|f(x) - T(x)\| \leq \frac{\theta}{|1 - 2^{p-1}|} \|x\|^p, \qquad x \in E_1 \backslash \{0\}. \tag{1.5}$$

(b) *If* $p < 0$, *then* f *is additive, i.e.,* (1.3) *holds.*

Note that Theorem 1.1 reduces to the first result of stability due to Hyers [22] if $p = 0$, Aoki [2] for $0 < p < 1$ (see also [31]). Afterward, Gajda [19] obtained this result for $p > 1$ and gave an example to show that Theorem 1.1 fails whenever $p = 1$. Further, Rassias [32] has noticed that a similar result is valid also for $p < 0$ (see [33, page 326] and [6]). Now, it is well-known that the statement (b) is valid, i.e., f must be additive in that case, which has been proved for the first time in [25] and next in [10] on the restricted domain. Also, the stability of some functional equations in the framework of non-Archimedean normed spaces and other spaces has been established (see e.g., [1,11,14,17,18,23,27–29]).

Recently, interesting results concerning radical cubic functional equation

$$f(\sqrt[3]{x^3 + y^3}) = f(x) + f(y), \qquad x, y \in \mathbb{R} \tag{1.6}$$

have been obtained in [1].

The hyperstability term was used for the first time probably in [26]; however, it seems that the first hyperstability result was published in [5] and concerned the ring homomorphisms. For further information concerning the notion of hyperstability we refer to the survey paper [9] (for recent related results see, e.g., [3,4,8,12,15,16,20]). For somewhat different approach to such terminology we refer to [30].

Let $(G,+)$ be a commutative group and d be an ultrametric on G that is invariant. We say that a function $f:\mathbb{R}\to G$ fulfils the radical cubic equation (1.6) on $\mathbb{R}_0$ (or is a solution to (1.6) on $\mathbb{R}_0$) provided

$$f(\sqrt[3]{x^3+y^3}) = f(x)+f(y), \qquad x,y\in\mathbb{R}_0,\ \ x\neq -y. \tag{1.7}$$

We consider functions $f:\mathbb{R}\to G$ fulfilling (1.7) approximately, i.e., satisfying the inequality

$$d\big(f(\sqrt[3]{x^3+y^3}), f(x)+f(y)\big)\le \gamma(x,y), \quad x,y\in\mathbb{R}\backslash\{0\}, \tag{1.8}$$

with $x\neq -y$ and $\gamma:\mathbb{R}^2\to\mathbb{R}_+$ is a given mapping. Under some additional assumptions on γ, we prove that the conditional functional equation (1.7) is γ-hyperstable in the class of functions $f:\mathbb{R}\to G$, i.e., each $f:\mathbb{R}\to G$ satisfying inequality (1.8) with such γ must fulfil equation (1.7). The method based on a fixed point result that can be easily derived from [7] and patterned on the ideas provided in [11].

2. Auxiliary Result

The main tool in the proofs of the main theorems of this paper is a fixed point result that can be derived from [7, Theorem 1]. We will use the following three hypotheses:

(H1) W is a nonempty set and (M,d) stands for a complete non-Archimedean metric space.

(H2) $\Lambda:\mathbb{R}_+^W\to\mathbb{R}_+^W$ is a non-decreasing operator satisfying

$$\lim_{n\to\infty}\Lambda\delta_n(x)=0 \quad \text{for every sequence} \quad (\delta_n)_{n\in\mathbb{N}}\in\mathbb{R}_+^W \quad \text{and} \quad x\in W$$
$$\text{with} \quad \lim_{n\to\infty}\delta_n(x)=0 \quad \text{for} \quad x\in W.$$

(We say that an operator $\Lambda:\mathbb{R}_+^W\to\mathbb{R}_+^W$ is non-decreasing if it satisfies the condition $\Lambda\delta_1(x)\le\Lambda\delta_2(x)$ for all $\delta_1,\delta_2\in\mathbb{R}_+^W$ and $x\in W$ with $\delta_1(x)\le\delta_2(x)$ for all $x\in W$).

(H3) $\mathcal{T}:M^W\to M^W$ is an operator satisfying the inequality

$$d\big(\mathcal{T}\xi(x),\mathcal{T}\mu(x)\big)\le\Lambda(\Delta(\xi,\mu))(x), \quad \text{for all} \ \ \xi,\mu\in M^W,\ \ x\in W,$$

where $\Delta(\xi,\mu)(x){:=}d(\xi(x),\mu(x))$ for all $x\in W$.

Now we are in a position to present the mentioned fixed point result.

Theorem 2.1. *Assume that hypotheses* **(H1)–(H3)** are satisfied. Suppose that there are functions $\varepsilon : W \to \mathbb{R}_+$ and $\varphi : W \to M$ such that

$$d\big(\mathcal{T}\varphi(x), \varphi(x)\big) \le \varepsilon(x), \quad x \in W,$$

and

$$\lim_{n\to\infty} \Lambda^n \varepsilon(x) = 0, \quad x \in W,$$

then for every $x \in W$ the limit

$$\psi(x) = \lim_{n\to\infty} \mathcal{T}^n \varphi(x)$$

exists and the function $\psi \in M^W$, defined in this way, is a fixed point of $\mathcal{T}$ with

$$d(\varphi(x), \psi(x)) \le \sup_{n\in\mathbb{N}_0} (\Lambda^n \varepsilon)(x), \quad x \in W. \tag{2.1}$$

Moreover, if

$$\Lambda\big(\sup_{n\in\mathbb{N}_0} (\Lambda^n \varepsilon)\big)(x) \le \sup_{n\in\mathbb{N}_0} (\Lambda^{n+1} \varepsilon)(x), \quad x \in W,$$

then ψ is the unique fixed point of $\mathcal{T}$ satisfying (2.1).

3. New Hyperstability Results for Eq. (1.7)

The following two theorems are the main results in this paper and concern the γ–hyperstability of (1.7). Namely, for

$$\gamma(x, y) = h_1(x^3)h_2(y^3),$$

with $h_1, h_2 : \mathbb{R} \to \mathbb{R}_+$ are two functions and

$$\gamma(x, y) = h(x^3) + h(y^3),$$

with $h : \mathbb{R} \to \mathbb{R}_+$ is a function, under some additional assumptions on the functions h, h_1, h_2, we show that the conditional functional equation (1.7) is γ-hyperstable in the class of functions $f : \mathbb{R} \to G$. The method of proof that we present here can be easily adjusted for various other examples of function γ.

In the remaining part of the paper, $(G, +)$ is a commutative group, d is a complete ultrametric in G which is invariant (i.e., $d(x + z, y + z) = d(x, y)$ for $x, y, z \in G$), and $\mathbb{N}_{m_0}$ denotes the set of all positive integers greater than or equal to a given $m_0 \in \mathbb{N}$.

Theorem 3.1. *Let $h_1, h_2 : \mathbb{R} \to \mathbb{R}_+$ be two functions such that*

$$\mathcal{M}_0 := \big\{n \in \mathbb{N}_2 : a_n := \max\{s_1(n^3)s_2(n^3), s_1(1-n^3)s_2(1-n^3)\} < 1\big\} \ne \emptyset, \tag{3.1}$$

where $s_i(\pm n) := \inf\{t \in \mathbb{R}_+ : h_i(\pm n x^3) \le t h_i(x^3) \ \text{ for all } \ x \in \mathbb{R}\}$ for $i = 1, 2$ and $n \in \mathbb{N}_2$, such that

$$\lim_{n\to\infty} s_1(n)s_2(-n) = 0.$$

Suppose that $f : \mathbb{R} \to G$ satisfies the inequality

$$d(f(\sqrt[3]{x^3+y^3}), f(x)+f(y)) \le h_1(x^3)h_2(y^3), \qquad x, y \in \mathbb{R}_0, \tag{3.2}$$

where $x \neq -y$, then (1.7) holds.

Proof. Replacing (x, y) by $(mx, -\sqrt[3]{m^3-1}x)$, where $m \in \mathbb{N}_2$, in (3.2), we get

$$d\big(f(x), f(mx) + f(-\sqrt[3]{m^3-1}x)\big) \le h_1(m^3x^3)h_2((1-m^3)x^3) \tag{3.3}$$

for all $x \in \mathbb{R}_0$. For each $m \in \mathbb{N}_2$, we will define the operators $\mathcal{T}_m : G^{\mathbb{R}_0} \to G^{\mathbb{R}_0}$ and $\Lambda_m : \mathbb{R}_+^{\mathbb{R}_0} \to \mathbb{R}_+^{\mathbb{R}_0}$ by

$$\begin{aligned}\mathcal{T}_m\xi(x) &:= \xi(mx) + \xi(-\sqrt[3]{m^3-1}x), \quad \xi \in G^{\mathbb{R}_0}, \;\; x \in \mathbb{R}_0\\ \Lambda_m\delta(x) \;&=\; \max\Big\{\delta(mx), \delta(-\sqrt[3]{m^3-1}x)\Big\}, \quad \delta \in \mathbb{R}_+^{\mathbb{R}_0}, \;\; x \in \mathbb{R}_0.\end{aligned}$$

Then $\Lambda_m{:=}\Lambda$ satisfies hypothesis (**H2**). Further, observe that

$$\begin{aligned}\varepsilon_m(x) : &= h_1(m^3x^3)h_2((1-m^3)x^3)\\ &\le s_1(m^3)s_2(1-m^3)h_1(x^3)h_2(x^3)\end{aligned} \tag{3.4}$$

for all $x \in \mathbb{R}_0$ and $m \in \mathbb{N}_2$.Then the inequality (3.3) takes the form

$$d(\mathcal{T}_mf(x), f(x)) \le \varepsilon_m(x), \qquad x \in \mathbb{R}_0, \;\; m \in \mathbb{N}_2.$$

Moreover, for every $\xi, \mu \in G^{\mathbb{R}_0}$, $m \in \mathbb{N}_2$ and $x \in \mathbb{R}_0$, we obtain

$$\begin{aligned} d\big(\mathcal{T}_m\xi(x), \mathcal{T}_m\mu(x)\big) &= d\big(\xi(mx) + \xi(-\sqrt[3]{m^3-1}x), \mu(mx) + \mu(-\sqrt[3]{m^3-1}x)\big)\\ &\le \max\big\{d\big(\xi(f_1(x)), \mu(f_1(x))\big), d\big(\xi(f_2(x)), \mu(f_2(x))\big)\big\}\\ &= \max_{1\le i\le 2} d\big(\xi(f_i(x)), \mu(f_i(x))\big)\\ &= \max_{1\le i\le 2} \Delta(\xi,\mu)\big(f_i(x)\big)\\ &= \Lambda_m\Delta(\xi,\mu)(x),\end{aligned}$$

where

$$f_1(x) = mx, \;\text{ and }\; f_2(x) = -\sqrt[3]{m^3-1}x, \quad x \in \mathbb{R}_0.$$

So, **(H3)** is valid for $\mathcal{T}_m$ with $m \in \mathbb{N}_2$.

By using mathematical induction, we will show that for each $x \in \mathbb{R}_0$ we have

$$\Lambda_m^n\varepsilon_m(x) \le s_1(m^3)s_2(1-m^3)a_m^nh_1(x^3)h_2(x^3), \tag{3.5}$$

for all $n \in \mathbb{N}_0$ and $m \in \mathcal{M}_0$. From (3.4), we obtain that the inequality (3.5) holds for $n = 0$. Next, we will assume that (3.5) holds for $n = l$, where $l \in \mathbb{N}_0$. Then we have

$$\begin{aligned}\Lambda_m^{l+1}\varepsilon_m(x) =&\Lambda_m(\Lambda_m^l\varepsilon_m(x))\\ =&\ \max\left\{\Lambda_m^l\varepsilon_m(mx),\Lambda_m^l\varepsilon_m(-\sqrt[3]{m^3-1}x)\right\}\\ \leq&\max\left\{s_1(m^3)s_2(1-m^3)a_m^l h_1(m^3x^3)h_2(m^3x^3),\right.\\ &\qquad\left. s_1(m^3)s_2(1-m^3)a_m^l h_1((1-m^3)x^3)h_2((1-m^3)x^3)\right\}\\ =&s_1(m^3)s_2(1-m^3)a_m^l\max\left\{h_1(m^3x^3)h_2(m^3x^3),\right.\\ &\left. h_1((1-m^3)x^3)h_2((1-m^3)x^3)\right\}\\ \leq&s_1(m^3)s_2(1-m^3)a_m^l h_1(x^3)h_2(x^3)\\ &\max\left\{s_1(m^3)s_2(m^3),s_1(1-m^3)s_2(1-m^3)\right\}\\ =&s_1(m^3)s_2(1-m^3)a_m^{l+1}h_1(x^3)h_2(x^3).\end{aligned}$$

This shows that (3.5) holds for $n = l + 1$. Now we can conclude that the inequality (3.5) holds for all $n \in \mathbb{N}_0$. Therefore, by (3.5), we obtain that

$$\lim_{n\to\infty}\Lambda_m^n\varepsilon_m(x) = 0,$$

for all $x \in \mathbb{R}_0$ and $m \in \mathcal{M}_0$. Further, for each $n \in \mathbb{N}_0$, $m \in \mathcal{M}_0$ and $x \in \mathbb{R}_0$ we have

$$\begin{aligned}\sup_{n\in\mathbb{N}_0}\Lambda_m^n\varepsilon_m(x) &= \Lambda_m^0\varepsilon_m(x) = \varepsilon_m(x),\\ \sup_{n\in\mathbb{N}_0}\Lambda_m^{n+1}\varepsilon_m(x) &= \Lambda_m\varepsilon_m(x) = \Lambda_m(\sup_{n\in\mathbb{N}_0}\Lambda_m^n\varepsilon_m)(x).\end{aligned}$$

Thus, according to Theorem 2.1, for each $m \in \mathcal{M}_0$ the mapping $C_m : \mathbb{R}_0 \to G$, given by $C_m(x) = \lim_{n\to\infty}\mathcal{T}_m^n f(x)$ for $x \in \mathbb{R}_0$, is a unique fixed point of $\mathcal{T}_m$, i.e,

$$C_m(x) = C_m(mx) + C_m(-\sqrt[3]{m^3-1}x)$$

for all $x \in \mathbb{R}_0$; moreover

$$d\big(f(x), C_m(x)\big) \leq \sup_{n\in\mathbb{N}_0}\Lambda_m^n\varepsilon_m(x),\ \ x \in \mathbb{R}_0,\ \ m \in \mathcal{M}_0.$$

Now we show that

$$d\big(\mathcal{T}_m^n f(\sqrt[3]{x^3+y^3}), \mathcal{T}_m^n f(x) + \mathcal{T}_m^n f(y)\big) \leq a_m^n h_1(x^3)h_2(y^3), \tag{3.6}$$

for every $n \in \mathbb{N}_0$, $m \in \mathcal{M}_0$ and $x, y \in \mathbb{R}_0$ with $x \neq -y$.

Clearly, if $n = 0$, then (3.6) is simply (3.2). So, fix $n \in \mathbb{N}_0$ and suppose that (3.6) holds for n and every $x, y \in \mathbb{R}_0$ with $x \neq -y$. Then, for every $x, y \in \mathbb{R}_0$ with $x \neq -y$,

$$\begin{aligned}
&d\big(\mathcal{T}_m^{n+1} f(\sqrt[3]{x^3+y^3}), \mathcal{T}_m^{n+1} f(x) + \mathcal{T}_m^{n+1} f(y)\big)\\
&= d\big(\mathcal{T}_m^{n} f(m\sqrt[3]{x^3+y^3}) + \mathcal{T}_m^{n} f(-\sqrt[3]{m^3-1}\sqrt[3]{x^3+y^3}), \mathcal{T}_m^{n} f(mx)\\
&\qquad + \mathcal{T}_m^{n} f(-\sqrt[3]{m^3-1}x) + \mathcal{T}_m^{n} f(my) + \mathcal{T}_m^{n} f(-\sqrt[3]{m^3-1}y)\big)\\
&\le \max\Big\{ d\big(\mathcal{T}_m^{n} f(m\sqrt[3]{x^3+y^3}), \mathcal{T}_m^{n} f(mx) + \mathcal{T}_m^{n} f(my)\big),\\
&\qquad d\big(\mathcal{T}_m^{n} f(-\sqrt[3]{m^3-1}\sqrt[3]{x^3+y^3}), \mathcal{T}_m^{n} f(-\sqrt[3]{m^3-1}x)\\
&\quad + \mathcal{T}_m^{n} f(-\sqrt[3]{m^3-1}y)\big)\Big\}\\
&\le \max\Big\{ a_m^n h_1(m^3x^3)h_2(m^3y^3), a_m^n h_1((1-m^3)x^3)h_2((1-m^3)y^3)\Big\}\\
&\le a_m^n h_1(x^3)h_2(y^3)\max\big\{s_1(m^3)s_2(m^3), s_1(1-m^3)s_2(1-m^3)\big\}\\
&= a_m^{n+1} h_1(x^3)h_2(y^3).
\end{aligned}$$

Thus, by induction, we have shown that (3.6) holds for all $x, y \in \mathbb{R}_0$ such that $x \neq -y$ and for all $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (3.6), we obtain that

$$C_m(\sqrt[3]{x^3+y^3}) = C_m(x) + C_m(y) \tag{3.7}$$

for every $x, y \in \mathbb{R}_0$ with $x \neq -y$.

In this way, for each $m \in \mathcal{M}_0$, we obtain a function C_m such that (3.7) holds for $x, y \in \mathbb{R}_0$ with $x \neq -y$ and

$$d\big(f(x), C_m(x)\big) \le \sup_{n\in\mathbb{N}_0} \Lambda_m^n \varepsilon_m(x) \le s_1(m^3)s_2(1-m^3)h_1(x^3)h_2(x^3)$$

for all $x \in \mathbb{R}_0$ and $m \in \mathcal{M}_0$.

Since

$$\lim_{m\to\infty} s_1(m)s_2(-m) = 0,$$

it follows, with $m \to \infty$, that f fulfils (1.7). □

Remark 1. The Theorem 3.1 remains true if we replace (3.1) by

$$\mathcal{N}_0 := \big\{n \in \mathbb{N} : \max\{s_1(n^3)s_2(n^3), s_1(n^3+1)s_2(n^3+1)\} < 1\big\} \neq \emptyset,$$

where $s_i(n) := \inf\{t \in \mathbb{R}_+ : h_i(nx^3) \le t h_i(x^3) \ \text{ for all } \ x \in \mathbb{R}\}$ for $n \in \mathbb{N}$ and $i = 1, 2$, such that

$$\lim_{n\to\infty} s_1(n) = 0 \quad \text{or} \quad \lim_{n\to\infty} s_2(n) = 0.$$

Theorem 3.2. *Let $h : \mathbb{R} \to \mathbb{R}_+$ be a function such that*

$$\mathcal{M}_0 := \big\{n \in \mathbb{N}_2 : b_n := \max\{s(n^3), s(1-n^3)\} < 1\big\} \neq \emptyset, \tag{3.8}$$

where $s(\pm n) := \inf\{t \in \mathbb{R}_+ : h(\pm nx^3) \le t h(x^3) \ \text{ for all } \ x \in \mathbb{R}\}$ for $n \in \mathbb{N}_2$, such that

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} s(-n) = 0.$$

Suppose that $f : \mathbb{R} \to G$ *satisfies the inequality*

$$d(f(\sqrt[3]{x^3+y^3}), f(x)+f(y)) \leq h(x^3)+h(y^3), \qquad x, y \in \mathbb{R}_0, \tag{3.9}$$

where $x \neq -y$*, then* (1.7) *holds.*

Proof. Replacing (x, y) by $(mx, -\sqrt[3]{m^3-1}x)$, where $m \in \mathbb{N}_2$, in (3.9), we get

$$d\big(f(x), f(mx)+f(-\sqrt[3]{m^3-1}x)\big) \leq h(m^3x^3)+h((1-m^3)x^3) \tag{3.10}$$

for all $x \in \mathbb{R}_0$. For each $m \in \mathbb{N}_2$, we will define operator $\mathcal{T}_m : G^{\mathbb{R}_0} \to G^{\mathbb{R}_0}$ by

$$\mathcal{T}_m\xi(x) := \xi(mx) + \xi(-\sqrt[3]{m^3-1}x),$$

for every $\xi \in G^{\mathbb{R}_0}$ and $x \in \mathbb{R}_0$. Further put

$$\begin{aligned} \varepsilon_m(x) :&= h(m^3x^3)+h((1-m^3)x^3) \\ &\leq \big(s(m^3)+s(1-m^3)\big)h(x^3) \end{aligned} \tag{3.11}$$

for all $m \in \mathbb{N}_2$ and $x \in \mathbb{R}_0$. Then the inequality (3.10) takes the form

$$d(\mathcal{T}_m f(x), f(x)) \leq \varepsilon_m(x), \qquad x \in \mathbb{R}_0, \ \ m \in \mathbb{N}_2.$$

Let $\Lambda_m : \mathbb{R}_+^{\mathbb{R}_0} \to \mathbb{R}_+^{\mathbb{R}_0}$ be an operator which is defined by

$$\Lambda_m\eta(x) = \max\Big\{\eta(mx), \eta(-\sqrt[3]{m^3-1}x)\Big\}$$

for all $x \in \mathbb{R}_0$, $m \in \mathbb{N}_2$ and $\eta \in \mathbb{R}_+^{\mathbb{R}_0}$. Then it is easily seen that $\Lambda_m := \Lambda$ satisfies hypothesis (**H2**).

Moreover, for every $\xi, \mu \in G^{\mathbb{R}_0}$, $m \in \mathbb{N}_2$ and $x \in \mathbb{R}_0$, we obtain

$$\begin{aligned} d\big(\mathcal{T}_m\xi(x), \mathcal{T}_m\mu(x)\big) &= d\big(\xi(mx)+\xi(-\sqrt[3]{m^3-1}x), \mu(mx)+\mu(-\sqrt[3]{m^3-1}x)\big) \\ &\leq \max\big\{d\big(\xi(f_1(x)), \mu(f_1(x))\big), d\big(\xi(f_2(x)), \mu(f_2(x))\big)\big\} \\ &= \max_{1\leq i\leq 2} \Delta(\xi,\mu)\big(f_i(x)\big) \\ &= \Lambda_m\Delta(\xi,\mu)(x), \end{aligned}$$

where

$$f_1(x) = mx, \ \text{ and } \ \ f_2(x) = -\sqrt[3]{m^3-1}x, \quad x \in \mathbb{R}_0.$$

Consequently, for each $m \in \mathbb{N}_2$, also **(H3)** is valid for $\mathcal{T}_m$.

Next, it easily seen that, by induction on n, from (3.11) we obtain

$$\Lambda_m^n\varepsilon_m(x) \leq \big(s(m^3)+s(1-m^3)\big)b_m^n h(x^3), \tag{3.12}$$

for all $n \in \mathbb{N}_0$ and $m \in \mathcal{M}_0$. Therefore, by (3.12), we obtain that

$$\lim_{n\to\infty} \Lambda_m^n\varepsilon_m(x) = 0,$$

for all $x \in \mathbb{R}_0$ and $m \in \mathcal{M}_0$. Further, for each $n \in \mathbb{N}_0$, $m \in \mathcal{M}_0$ and $x \in \mathbb{R}_0$ we have

$$\sup_{n\in\mathbb{N}_0} \Lambda_m^n \varepsilon_m(x) = \Lambda_m^0 \varepsilon_m(x) = \varepsilon_m(x),$$
$$\sup_{n\in\mathbb{N}_0} \Lambda_m^{n+1} \varepsilon_m(x) = \Lambda_m \varepsilon_m(x) = \Lambda_m(\sup_{n\in\mathbb{N}_0} \Lambda_m^n \varepsilon_m)(x).$$

Thus, according to Theorem 2.1, for each $m \in \mathcal{M}_0$ the mapping $C_m : \mathbb{R}_0 \to G$, given by $C_m(x) = \lim_{n\to\infty} \mathcal{T}_m^n f(x)$ for $x \in \mathbb{R}_0$, is a unique fixed point of $\mathcal{T}_m$, i.e,

$$C_m(x) = C_m(mx) + C_m(-\sqrt[3]{m^3-1}x)$$

for all $x \in \mathbb{R}_0$; moreover

$$d\big(f(x), C_m(x)\big) \le \sup_{n\in\mathbb{N}_0} \Lambda_m^n \varepsilon_m(x), \quad x \in \mathbb{R}_0, \quad m \in \mathcal{M}_0.$$

Now we show that

$$d\big(\mathcal{T}_m^n f(\sqrt[3]{x^3+y^3}), \mathcal{T}_m^n f(x) + \mathcal{T}_m^n f(y)\big) \le b_m^n(h(x^3) + h(y^3)), \tag{3.13}$$

for every $n \in \mathbb{N}_0$, $m \in \mathcal{M}_0$ and $x, y \in \mathbb{R}_0$ with $x \ne -y$.

Clearly, if $n = 0$, then (3.13) is simply (3.9). So, fix $n \in \mathbb{N}_0$ and suppose that (3.13) holds for n and every $x, y \in \mathbb{R}_0$ with $x \ne -y$. Then, for every $x, y \in \mathbb{R}_0$ with $x \ne -y$,

$$\begin{aligned}
&d\big(\mathcal{T}_m^{n+1} f(\sqrt[3]{x^3+y^3}), \mathcal{T}_m^{n+1} f(x) + \mathcal{T}_m^{n+1} f(y)\big)\\
&= d\big(\mathcal{T}_m^n f(m\sqrt[3]{x^3+y^3}) + \mathcal{T}_m^n f(-\sqrt[3]{m^3-1}\sqrt[3]{x^3+y^3}), \mathcal{T}_m^n f(mx)\\
&\qquad + \mathcal{T}_m^n f(-\sqrt[3]{m^3-1}x) + \mathcal{T}_m^n f(my) + \mathcal{T}_m^n f(-\sqrt[3]{m^3-1}y)\big)\\
&\le \max\Big\{d\big(\mathcal{T}_m^n f(m\sqrt[3]{x^3+y^3}), \mathcal{T}_m^n f(mx) + \mathcal{T}_m^n f(my)\big),\\
&\qquad d\big(\mathcal{T}_m^n f(-\sqrt[3]{m^3-1}\sqrt[3]{x^3+y^3}), \mathcal{T}_m^n f(-\sqrt[3]{m^3-1}x) + \mathcal{T}_m^n f(-\sqrt[3]{m^3-1}y)\big)\Big\}\\
&\le \max\Big\{b_m^n\big(h(m^3x^3) + h(m^3y^3)\big), b_m^n\big(h((1-m^3)x^3) + h((1-m^3)y^3)\big)\Big\}\\
&\le b_m^n\big(h(x^3) + h(y^3)\big) \max\big\{s(m^3), s(1-m^3)\big\}\\
&= b_m^{n+1}\big(h(x^3) + h(y^3)\big).
\end{aligned}$$

Thus, by induction, we have shown that (3.13) holds for all $x, y \in \mathbb{R}_0$ such that $x \ne -y$ and for all $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (3.13), we obtain that

$$C_m(\sqrt[3]{x^3+y^3}) = C_m(x) + C_m(y) \tag{3.14}$$

for every $x, y \in \mathbb{R}_0$ with $x \ne -y$.

In this way, for each $m \in \mathcal{M}_0$, we obtain a function C_m such that (3.14) holds for $x, y \in \mathbb{R}_0$ with $x \ne -y$ and

$$d\big(f(x), C_m(x)\big) \le \sup_{n\in\mathbb{N}_0} \Lambda_m^n \varepsilon_m(x) \le \big(s(m^3) + s(1-m^3)\big)h(x^3)$$

for all $x \in \mathbb{R}_0$ and $m \in \mathcal{M}_0$.

Since

$$\lim_{m\to\infty} s(m) = \lim_{m\to\infty} s(-m) = 0,$$

it follows, with $m \to \infty$, that f fulfils (1.7). □

Remark 2. The Theorems 3.1 and 3.2 also provide γ-hyperstability results in the case where the control function is $\gamma(x, y) = h(x^3)$ (or $\gamma(x, y) = h(y^3)$).

By using Theorems 3.1 and 3.2 and the same technique we get the following hyperstability results for the inhomogeneous radical cubic functional equation.

Corollary 3.1. *Let $h_1, h_2 : \mathbb{R} \to \mathbb{R}_+$ be two functions such that (3.1) is an infinite set, where $s_i(\pm n){:=}\inf\{t \in \mathbb{R}_+ : h_i(\pm nx^3) \le th_i(x^3)$ for all $x \in \mathbb{R}\}$ for $i = 1, 2$ and $n \in \mathbb{N}_2$, with*

$$\lim_{n\to\infty} s_1(n)s_2(-n) = 0.$$

Let $f : \mathbb{R} \to G$ and $F : \mathbb{R}^2 \to G$ be two functions such that

$$d\big(f(\sqrt[3]{x^3{+}y^3}), f(x) + f(y) + F(x, y)\big) \le h_1(x^3)h_2(y^3), \quad x, y \in \mathbb{R}_0, \quad x \ne -y.$$

Assume that the functional equation

$$f(\sqrt[3]{x^3 + y^3}) = f(x) + f(y) + F(x, y), \quad x, y \in \mathbb{R}_0, \quad x \ne -y \tag{3.15}$$

admits a solution $f_0 : \mathbb{R}_0 \to G$. Then f is a solution of (3.15).

Proof. Let $f_1(x){:=}f(x) - f_0(x)$ for $x \in \mathbb{R}_0$. Then

$$\begin{aligned}
&d\big(f_1(\sqrt[3]{x^3 + y^3}), f_1(x) + f_1(y)\big)\\
&\quad = d\big(f(\sqrt[3]{x^3 + y^3}) - f_0(\sqrt[3]{x^3 + y^3}), f(x) - f_0(x) + f(y)\\
&\qquad + F(x, y) - f_0(y) - F(x, y)\big)\\
&\quad \le \max\Big\{d\big(f(\sqrt[3]{x^3 + y^3}), f(x) + f(y) + F(x, y)\big),\\
&\qquad d\big(f_0(\sqrt[3]{x^3 + y^3}), f_0(x) + f_0(y) + F(x, y)\big)\Big\}\\
&\quad = d\big(f(\sqrt[3]{x^3 + y^3}), f(x) + f(y) + F(x, y)\big)\\
&\quad \le h(x^3)h(y^3), \qquad x, y \in \mathbb{R}_0, \quad x \ne -y.
\end{aligned}$$

It follows from Theorem 3.1 with f replaced by f_1 that f_1 satisfies the radical cubic functional equation (1.7). Therefore,

$$\begin{aligned}
f(\sqrt[3]{x^3 + y^3}) - f(x) - f(y) - F(x, y) &= f_1(\sqrt[3]{x^3 + y^3}) - f_1(x) - f_1(y)\\
&\quad + f_0(\sqrt[3]{x^3 + y^3}) - f_0(x) - f_0(y) - F(x, y) = 0
\end{aligned}$$

for all $x, y \in \mathbb{R}_0$ with $x \ne -y$. □

Analogously we prove the following.

Corollary 3.2. *Let $h : \mathbb{R} \to \mathbb{R}_+$ be a function such that (3.8) is an infinite set, where $s(\pm n) := \inf\{t \in \mathbb{R}_+ : h(\pm n x^3) \leq t h(x^3) \text{ for all } x \in \mathbb{R}\}$ for $n \in \mathbb{N}_2$, with*

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} s(-n) = 0.$$

Let $f : \mathbb{R} \to G$ and $F : \mathbb{R}^2 \to G$ be two functions such that

$$d\big(f(\sqrt[3]{x^3+y^3}), f(x) + f(y) + F(x,y)\big) \leq h(x^3) + h(y^3),$$
$$x, y \in \mathbb{R}_0, \quad x \neq -y.$$

Assume that the functional equation

$$f(\sqrt[3]{x^3+y^3}) = f(x) + f(y) + F(x,y), \quad x, y \in \mathbb{R}_0, \quad x \neq -y \tag{3.16}$$

admits a solution $f_0 : \mathbb{R}_0 \to G$. Then f is a solution of (3.16).

4. Some Particular Cases

According to Theorems 3.1, 3.2 and Corollaries 3.1, 3.2 with $G = X$ is a non-Archimedean Banach space over a field $\mathbb{K}$ with respect to the ultrametric $d(x,y) := \|x - y\|_d$ for all $x, y \in X$, $h(x) := c|x|^p$, $h_i(x) := c_i|x|^{p_i}$ for all $x \in \mathbb{R}_0$ where $c, p, c_i, p_i \in \mathbb{R}$ for $i = 1, 2$, we derive some particular cases from our main results.

Corollary 4.1. *Let X be an ultrametric Banach space and $c, p, q \in \mathbb{R}$ with $c \geq 0$ and $p + q < 0$. If $f : \mathbb{R} \to X$ satisfies the inequality*

$$\|f(\sqrt[3]{x^3+y^3}) - f(x) - f(y)\|_d \leq c|x|^p|y|^q, \qquad x, y \in \mathbb{R}_0,\ x \neq -y,$$

then (1.7) holds.

Corollary 4.2. *Let X be an ultrametric Banach space, $F : \mathbb{R}^2 \to X$ be a function and $c, p, q \in \mathbb{R}$ with $c \geq 0$ and $p + q < 0$. Let $f : \mathbb{R} \to X$ be a function such that*

$$\|f(\sqrt[3]{x^3+y^3}) - f(x) - f(y) - F(x,y)\|_d \leq c|x|^p|y|^q, \qquad x, y \in \mathbb{R}_0,\ x \neq -y.$$

Assume that the functional equation

$$f(\sqrt[3]{x^3+y^3}) = f(x) + f(y) + F(x,y), \quad x, y \in \mathbb{R}_0, \quad x \neq -y \tag{4.1}$$

admits a solution $f_0 : \mathbb{R}_0 \to X$. Then f is a solution of (4.1).

Corollary 4.3. *Let X be an ultrametric Banach space and $c, p \in \mathbb{R}$ with $c \geq 0$ and $p < 0$. If $f : \mathbb{R} \to X$ satisfies the inequality*

$$\|f(\sqrt[3]{x^3+y^3}) - f(x) - f(y)\|_d \leq c(|x|^p + |y|^p), \qquad x, y \in \mathbb{R}_0,\ x \neq -y,$$

then (1.7) holds.

Corollary 4.4. *Let X be an ultrametric Banach space, $F : \mathbb{R}^2 \to X$ be a function and $c, p \in \mathbb{R}$ with $c \geq 0$ and $p < 0$. Let $f : \mathbb{R} \to X$ be a function such that*

$$\|f(\sqrt[3]{x^3 + y^3}) - f(x) - f(y) - F(x,y)\|_d \leq c(|x|^p + |y|^p),$$
$$x, y \in \mathbb{R}_0,\ x \neq -y.$$

Assume that the functional equation

$$f(\sqrt[3]{x^3 + y^3}) = f(x) + f(y) + F(x,y), \quad x, y \in \mathbb{R}_0, \quad x \neq -y \tag{4.2}$$

admits a solution $f_0 : \mathbb{R}_0 \to X$. Then f is a solution of (4.2).

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