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Results in Mathematics

A Generalization of Nadler's Fixed Point Theorem

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Abstract. In the present paper, we generalize the well-known Nadler's fixed point theorem (Nadler in Pac J Math 30:475–488, 1969), and one of some Dhompongsa and Yingtaweesittikul type theorems for multi-valued operators, see (Dhompongsa and Yingtaweesittikul in Fixed Point Theory Appl, 2007). Also, we give an example showing that our result is a proper generalization of some previous theorems.

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Keywords. Complete metric space, Nadler's theorem, Multi-valued mappings.

1. Introduction and Preliminaries

Banach contraction principle plays an important role in many branches of mathematics. For instance, it has been used to study the existence of solutions for nonlinear Volterra integral equations, nonlinear integro-differential equations in Banach space and to prove the convergence of algorithms in computational mathematics. Because of its importance for mathematical theory, Banach contraction principle has been generalized by many authors in various directions, see [3,4,6,7,13,14,27,28]. One such generalization is due to Meir and Keeler [21]. In 1969, they proved the following very interesting fixed-point theorem.

Theorem 1.1. (Meir and Keeler [21]) *Let* $(X,\ d)$ *be a complete metric space and let* T *be a mapping on* X*. Assume that for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \text{ implies } d(Tx,Ty) < \varepsilon,$$

for $x, y \in X$. Then T has a unique fixed point.

In 2001, Lim [20] introduced the notion of an L-function and characterized Meir-Keeler contractions (MKC, for short).

Definition 1.1. (Lim [20]) A function φ from $[0, \infty)$ into itself is called an L-function if $\varphi(0) = 0$, $\varphi(s) > 0$ for $s \in (0, \infty)$, and for every $s \in (0, \infty)$ there exists $\delta > 0$ such that $\varphi(t) \leq s$ for all $t \in [s, s + \delta]$.

Theorem 1.2. (Lim [20]) *Let $(X,\ d)$ be a complete metric space and let T be a mapping on X. Then T is an MKC if and only if there exists an (nondecreasing, right continuous) L-functions φ such that*

$$d(Tx, Ty) < \varphi(d(x, y))$$

for all $x, y \in X$ with $x \neq y$.

Another valuable generalization is due to Geraghty [16].

Theorem 1.3. (Geraghty [16]) *Let $(X,\ d)$ be a complete metric space and let $T : X \to X$ be a mapping such that for each $x, y \in X$,*

$$d(Tx, Ty) \leq \alpha(d(x, y))\, d(x, y),$$

where $\alpha \in S$, and S is the class of functions from $[0, \infty)$ into $[0, 1)$ which satisfy the simple condition $\alpha(t_n) \to 1 \Longrightarrow t_n \to 0$. Then T has a unique fixed point $z \in X$, and $\{T^n(x)\}$ converges to z, for each $x \in X$ (T is a Picard operator).

In 1969, Nadler [24] proved a fundamental fixed point theorem for multi-valued maps. Given a metric space (X, d), by $P(X)$ and $CB(X)$ we will denote the family of nonempty subsets of X and the family of all nonempty closed and bounded subsets of X, respectively. It is obvious that, $CB(X) \subseteq P(X)$. For A, $B \in CB(X)$, let

$$H(A, B) = \max\left\{\sup_{x\in A} D(x, B), \quad \sup_{y \in B} D(y, A)\right\},$$

where $D(x, B) = \inf\{d(x, y) : y \in B\}$. Then H is a metric on $CB(X)$, which is called the Pompeiu-Hausdorff metric induced by d.

Theorem 1.4. (Nadler [24]) *Let $(X,\ d)$ be a complete metric space and let T be a multi-valued map on X such that Tx is nonempty closed bounded subset of X ($Tx \in CB(X)$) for any $x \in X$. If there exists $c \in (0, 1)$ such that*

$$H(Tx, Ty) \leq c \cdot d(x, y), \forall x, y \in X,$$

then T has a fixed point in X (there exists $z \in X$ such that $z \in Tz$).

Since then, a lot of generalizations of the result of Nadler have been given (see, for example, [1,2,8,9,11,12,15,17–19,22,23,26,29,30])

Gorgji et al. [17] introduced a notion called special multi-valued map and for this type of multi-valued map they generalized the Geraghty's fixed point theorem.

Definition 1.2. (Gorgji et al. [17]) Let $(X,\ d)$ be a complete metric space. A mapping $T : X \to CB(X)$ is called special multi-valued if

$$\inf_{y\in Tx}\{d(x,y)+d(y,z)\}=D(x,Tx)+D(z,Tx),$$

for all $x, z \in X$.

It is obvious that every single valued mapping is special multi-valued mapping.

Theorem 1.5. (Gorgji et al. [17]) *Let $(X,\ d)$ be a complete metric space and let T be special multi-valued mapping such that*

$$\begin{aligned} H\left(Tx,Ty\right) \le\ & \alpha\left(d\left(x,y\right)\right)d\left(x,y\right)+\beta\left(d\left(x,y\right)\right)\left[D(x,Tx)+D(y,Ty)\right] \\ & +\gamma\left(d\left(x,y\right)\right)\left[D(x,Ty)+D(y,Tx)\right] \end{aligned}$$

for all $x, y \in X$, where α, β, γ are mappings from $[0,\infty)$ into $[0,1)$ such that $\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} \in S$ and $\beta\left(t\right) \ge \gamma\left(t\right)$ for all $t \in [0,\infty)$. Then T has a fixed point.

Putting $\beta=\gamma=0$ in Theorem 1.5, we have the following result, which can be regarded as an extension of Geraghty's fixed point theorem.

Corollary 1.1. (Gorgji et al. [17]) *Let $(X,\ d)$ be a complete metric space and let T be special multi-valued mapping, $\alpha \in S$ such that*

$$H\left(Tx,Ty\right)\le\alpha\left(d\left(x,y\right)\right)d\left(x,y\right)$$

for all $x, y \in X$. Then T has a fixed point.

Proposition 1.1. *Let $(X,\ d)$ be a complete metric space and $T : X \to CB\left(X\right)$. Then T is a special multi-valued mapping if and only if $d\left(x,y\right) = D\left(x,Tx\right)$ for every $y \in Tx$ and $x \in X$.*

Proof. Assume that T is a special multi-valued map. Taking $z \in Tx$, we have

$$\inf_{y\in Tx}\{d(x,y)+d(y,z)\}=D(x,Tx).$$

But

$$D(x,Tx)\le d(x,y)\le d(x,y)+d(y,z) \text{ for all } \ y,z\in Tx,$$

hence $d(x,y) = D(x,Tx)$ for all $y \in Tx$. Now, we suppose that $d(x,y) = D(x,Tx)$ for all $y \in Tx$ and $x \in X$. Since

$$D\left(z,Tx\right)=\inf_{y\in Tx}d(y,z),$$

we have

$$\begin{aligned} \inf_{y\in Tx}\{d(x,y)+d(y,z)\} &= \inf_{y\in Tx}\{D(x,Tx)+d(y,z)\} \\ &= D(x,Tx)+\inf_{y\in Tx}d(y,z) \\ &= D(x,Tx)+D\left(z,Tx\right), \end{aligned}$$

so T is a special multi-valued map. □

A metric space (X, d) is hyperconvex if for any family of points $\{x_\alpha\}$ in X and any family of positive numbers $\{r_\alpha\}$ satisfying $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, we have $\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset$, where $B(x, r)$ is the closed ball with center at x and radius r. A subset E of X is said to be externally hyperconvex if for any of those families $\{x_\alpha\}$, $\{r_\alpha\}$ with $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ and $D(x_\alpha, E) \leq r_\alpha$, we have $\bigcap_\alpha B(x_\alpha, r_\alpha) \bigcap E \neq \emptyset$. The class of all externally hyperconvex subsets of X will be denoted by $E(X)$.

A selfmapping T on a metric space (X, d) is said to be asymptotically regular (cf. [5]) if

$$\lim_n d\left(T^n x, T^{n+1} x\right) = 0$$

for each $x \in X$.

Theorem 1.6. (Proinov [25]) *Let T be a continuous and asymptotically regular selfmapping on a complete metric space satisfying the following conditions:*

(i) there exists $\varphi \in \Phi_1$ (i.e., $\varphi : [0, \infty) \to [0, \infty)$ satisfying: for any $\varepsilon > 0$, there exists $\delta > \varepsilon$ such that $\varepsilon < t < \delta$ implies $\varphi(t) \leq \varepsilon$) such that $d(Tx, Ty) \leq \varphi(M(x, y))$ for all $x, y \in X$;

(ii) $d(Tx, Ty) < M(x, y)$ for all $x, y \in X$ with $x \neq y$.

Then T is a Picard operator, where $M(x, y) = d(x, y) + r[d(x, Tx) + d(y, Ty)]$, $r \geq 0$.

Dhompongsa and Yingtaweesittikul [10] proved a multivalued version of Theorem 1.6 on the class of hyperconvex metric space.

Theorem 1.7. ([10]) *Let (X, d) be a bounded hyperconvex metric space, and let $T : X \to E(X)$ be asymptotically regular satisfying the following conditions:*

(i) there exists $\varphi \in \Phi_1$ such that $\varphi(x) \leq x$, $\varphi(x + y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in [0, \infty)$, $\varphi(x) = 0$ if and only if $x = 0$, and $H(Tx, Ty) \leq \varphi(d(x, y))$ for all $x, y \in X$;

(ii) $H(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$.

Then, if $\delta(T^n x) \to 0$ for each $x \in X$, T has a contractive fixed point, that is, there exists a unique point ξ in X such that, for each $x \in X$, $T^n x \to \{\xi\} = FixT$.

(Here $\delta(A) := \sup\{d(x, y) : x, y \in A\}$ is the diameter of $A \subset X$.)

In this paper, we generalize Theorems 1.4 and 1.7.

2. Main Results

We start our work with two lemmas.

Lemma 2.1. *Let (X, d) be a complete metric space and $T : X \to CB(X)$. Then for every $x, y \in X$ we have*

$$d(x, y) \leq D(x, Tx) + H(Tx, Ty) + D(y, Ty) + \delta(Ty).$$

Proof. Let $\varepsilon > 0$ be fixed and $x, y \in X$. Then there exists $a \in Tx$, $b \in Ty$ such that $d(x,a) \leq D(x,Tx) + \varepsilon$ and $d(y,b) \leq D(y,Ty) + \varepsilon$. Also, for $a \in Tx$ there exists $c \in Ty$ such that $d(a,c) \leq H(Tx,Ty) + \varepsilon$. Therefore,

$$\begin{aligned} d(x,y) &\leq d(x,a) + d(a,c) + d(c,b) + d(b,y) \\ &\leq D(x,Tx) + \varepsilon + H(Tx,Ty) + \varepsilon + \delta(Ty) + D(y,Ty) + \varepsilon \\ &= D(x,Tx) + H(Tx,Ty) + D(y,Ty) + \delta(Ty) + 3\varepsilon. \end{aligned}$$

Since ε is arbitrar we have that

$$d(x,y) \leq D(x,Tx) + H(Tx,Ty) + D(y,Ty) + \delta(Ty).$$

Lemma 2.2. *Let $\varphi \in \Phi_1$ such that $\varphi(x) \leq x$, $\varphi(x+y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in [0, \infty)$, $\varphi(x) = 0$ if and only if $x = 0$. Then:*

(a) $\varphi(x) < x$ for every $x > 0$;
(b) for every sequence $\{d_n\}$ such that $d_n \to d$ as $n \to \infty$, $d_n \geq d$ we have $\limsup_{n\to\infty} \varphi(d_n) \leq \varphi(d)$.

Proof. (a) Suppose there exists $a > 0$ such that $\varphi(a) = a$. Let $x < a$ arbitrarly fixed. Then we have

$$a = \varphi(a) = \varphi(x + a - x) \leq \varphi(x) + \varphi(a - x) \leq x + a - x = a.$$

This implies $\varphi(x) = x$ for every $x \leq a$. Since $\varphi \in \Phi_1$, for $\varepsilon = \frac{a}{2}$ there exists $\delta > \frac{a}{2}$ such that $\varphi(t) \leq \frac{a}{2}$ for every $t \in \left(\frac{a}{2}, \delta\right)$. If $t \in \left(\frac{a}{2}, \delta\right) \cap \left(\frac{a}{2}, a\right)$ we have $\varphi(t) = t > \frac{a}{2}$, contradiction.

(b) Let the sequence $\{d_n\}$ such that $d_n \to d$ as $n \to \infty$, $d_n \geq d$. Then we have

$$\varphi(d_n) = \varphi(d + d_n - d) \leq \varphi(d) + \varphi(d_n - d) \leq \varphi(d) + d_n - d$$

Therefore $\limsup_{n\to\infty} \varphi(d_n) \leq \varphi(d)$.

Theorem 2.1. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ such that there exists $\varphi \in \Phi_1$ satisfying $\varphi(x) \leq x$, $\varphi(x+y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in [0, \infty)$, $\varphi(x) = 0$ if and only if $x = 0$, and $H(Tx,Ty) \leq \varphi(d(x,y))$ for all $x, y \in X$. Then, if $\delta(T^n x) \to 0$ for each $x \in X$, T has a unique contractive fixed point.*

Proof. Let $x_0 \in X$. If $x_0 \in Tx_0$ then x_0 is a fixed point of T. Suppose that $x_0 \notin Tx_0$. Then there exists $x_1 \in Tx_0$ such that

$$d(x_0, x_1) < D(x_0, Tx_0) + 1$$

If $x_1 \in Tx_1$ then x_1 is a fixed point of T. Suppose that $x_1 \notin Tx_1$. From Lemma 2.2 and hypothesis we have $D(x_1, Tx_1) \leq H(Tx_0, Tx_1) \leq \varphi(d(x_0, x_1)) < d(x_0, x_1)$, so there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) < D(x_1, Tx_1) + \frac{1}{2} \text{ and } d(x_1, x_2) < d(x_0, x_1).$$

Therefore inductively, there exists a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$ and

$$d(x_n, x_{n+1}) < D(x_n, Tx_n) + \frac{1}{n} \text{ and } d(x_n, x_{n+1}) < d(x_{n-1}, x_n).$$

Hence the sequence $\{d(x_n, x_{n+1})\}$ is strictly decreasing and there exists $d \geq 0$ such that

$$d_n = d(x_n, x_{n+1}) \to d.$$

Since $d_n < D(x_n, Tx_n) + \frac{1}{n} \leq d_n + \frac{1}{n}$, we have that $D_n = D(x_n, Tx_n) \to d$ as $n \to \infty$. But

$$\begin{aligned} D_{n+1} = D(x_{n+1}, Tx_{n+1}) &\leq H(Tx_n, Tx_{n+1}) \\ &\leq \varphi(d(x_n, x_{n+1})) \leq d(x_n, x_{n+1}) = d_n. \end{aligned}$$

Letting $n \to \infty$ we get $\lim_n \varphi(d_n) = d$.

From Lemma 2.2 we have $\lim_n \varphi(d_n) \leq \varphi(d)$,hence $d \leq \varphi(d) \leq d$. Thus $\varphi(d) = d$ and from Lemma 2.2 we get $d = 0$. This means that $H(Tx_n, Tx_{n+1}) \to 0$, $d_n \to 0$, $D_n \to 0$.

Now we prove that $\{x_n\}$ is a Cauchy sequence. Suppose the contrary, there exists $\varepsilon > 0$ and $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ two subsequences of $\{x_n\}$ such that $d(x_{n(k)}, x_{m(k)}) \geq \varepsilon$ for all k, where $n(k) > m(k) > k$. We can choose $n(k)$ such that $d(x_{n(k)-1}, x_{m(k)}) < \varepsilon$ for all k. Then

$$\varepsilon \leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)-1}, x_{m(k)}) + d(x_{n(k)-1}, x_{n(k)}) < \varepsilon + d_{n(k)-1}.$$

Letting $k \to \infty$ we have $\lim_k d(x_{n(k)}, x_{m(k)}) = \varepsilon$.

Since $Tx_{n(k)} \subset T^{n(k)+1}x_0$ and $\delta(T^n x_0) \to 0$ as $n \to \infty$, then $\delta(Tx_{n(k)}) \to 0$ as $k \to \infty$.

From Lemma 2.1 and hypothesis we have

$$\begin{aligned} &d(x_{n(k)}, x_{m(k)}) - D(x_{n(k)}, Tx_{n(k)}) - D(x_{m(k)}, Tx_{m(k)}) - \delta(Tx_{m(k)}) \\ &\quad \leq H(Tx_{n(k)}, Tx_{m(k)}) \leq \varphi(d(x_{n(k)}, x_{m(k)})) \leq d(x_{n(k)}, x_{m(k)}) \end{aligned}$$

for all k. Letting $k \to \infty$ we get

$$\lim_k \varphi(d(x_{n(k)}, x_{m(k)})) = \varepsilon.$$

But, from Lemma 2.2 $\lim_k \varphi(d(x_{n(k)}, x_{m(k)})) \leq \varphi(\varepsilon) < \varepsilon$, so we have a contradiction. Hence $\{x_n\}$ is a Cauchy sequence.

From the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$.

Since

$$\begin{aligned} D(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + D(x_{n+1}, Tx^*) \leq d(x^*, x_{n+1}) + H(Tx_n, Tx^*) \\ &\leq d(x^*, x_{n+1}) + \varphi(d(x_n, x^*)) \leq d(x^*, x_{n+1}) + d(x_n, x^*), \end{aligned}$$

letting $n \to \infty$, we have $D(x^*, Tx^*) = 0$. Therefore $x^* \in Tx^*$, by where $x^* \in T^n x^*$ for every $n \geq 1$.

Let $y^*, x^* \in Tx^*$. There exists sequence $\{y_n\}$, $y_1 \in Tx^*, y_n \in Ty_{n-1},\ n \geq 2$ such that $d(y_n, y^*) \leq H(Ty_{n-1}, Tx^*) + \frac{1}{n}$. We have $y_n \in Ty_{n-1} \subset T^2 y_{n-2} \subset \ldots \subset T^{n-1} y_1 \subset T^n x^*$.

Then

$$\begin{aligned} d(x^*, y^*) &\leq d(x^*, y_n) + d(y_n, y^*) \leq \delta(T^n x^*) + H(Ty_{n-1}, Tx^*) + \frac{1}{n} \\ &\leq \delta(T^n x^*) + \varphi(d(y_{n-1}, x^*)) + \frac{1}{n} \leq \delta(T^n x^*) + d(y_{n-1}, x^*) + \frac{1}{n} \\ &\leq \delta(T^n x^*) + \delta\left(T^{n-1} x^*\right) + \frac{1}{n}. \end{aligned}$$

Letting $n \to \infty$ we get $d(x^*, y^*) = 0$, so $Tx^* = \{x^*\}$.

Now suppose that there exists $y^* \neq x^*$ such that $y^* \in Ty^*$. Since

$$d(y^*, x^*) = D(y^*, Tx^*) \leq H(Ty^*, Tx^*) \leq \varphi(d(y^*, x^*)) < d(y^*, x^*)$$

we have a contradiction. □

Example 2.2. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$, then (X, d) is complete metric space. Define a mapping $T : X \to CB(X)$ as:

$$Tx = \begin{cases} \{0\} & , x = 0 \\ [0, \ln(1 + x)] & , x > 0 \end{cases}.$$

Taking $x = 0,\ y > 0$, we have $H(T0, Ty) = \ln(1 + y)$ and $d(0, y) = y$. Since $\lim_{y \to 0} \frac{\ln(1+y)}{y} = 1$, we get $\lim_{y \to 0} \frac{H(T0,Ty)}{d(0,y)} = 1$. This implies $(\nexists)\ c \in [0, 1)$ such that $H(Tx, Ty) \leq cd(x, y)\ \ (\forall)\ \ x, y \in X$, hence T does not verify hypothesis of Theorem 1.4.

Let $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(x) = \ln(1 + x)$.

For $\varepsilon > 0$, arbitrary, we consider $\delta = e^\varepsilon - 1 > \varepsilon$, obviously. For $t \in (\varepsilon, \delta)$ we have $\varphi(t) = \ln(1 + t) < \ln e^\varepsilon = \varepsilon$, so $\varphi \in \Phi_1$.

It is clear that

(i) $\varphi(x) = \ln(1 + x) \leq x$, for every $x \in X$
(ii) $\varphi(x) = 0 \ \Leftrightarrow\ x = 0$

For $(\forall)\ \ x, y \in X$ we have

$$\begin{aligned} \varphi(x) + \varphi(y) &= \ln(1 + x) + \ln(1 + y) = \ln(1 + x + y + xy) \\ &\geq \ln(1 + x + y) = \varphi(x + y). \end{aligned}$$

For $y > x \geq 0$ we have $Tx \subset Ty$, so

$$\begin{aligned} H(Tx, Ty) &= \sup\{D(b, Tx) : b \in Ty\} \\ &= \ln(1 + y) - \ln(1 + x) = \ln\frac{1 + y}{1 + x}. \end{aligned}$$

Since $0 \leq x \leq y$ we get $x^2 \leq xy$, and then

$$\begin{aligned} 1+y &\leq (1+x)(1+y-x) \Leftrightarrow \\ \frac{1+y}{1+x} &\leq 1+y-x \Leftrightarrow \\ \ln \frac{1+y}{1+x} &\leq \ln(1+y-x). \end{aligned}$$

This implies

$$H(Tx, Ty) \leq \varphi(d(x, y)) \tag{1}$$

Let $x_0 \in X$ arbitrarly fixed and the sequence $\{x_n\}_{n\geq 0}$ such that

$$x_n = \ln(1 + x_{n-1}), \ (\forall) n \geq 1. \tag{2}$$

It is clear that $0 \leq x_n \leq x_{n-1}$, hence $\{x_n\}$ converges to a limit $l \geq 0$. Taking the limit as $n \to \infty$ in (2), we get $l = \ln(1+l)$ which implies $l = 0$.

On the other hand we have

$$Tx_0 = \begin{cases} \{0\}, & x_0 = 0 \\ [0, \ln(1+x_0)] = [0, x_1], & x_0 > 0 \end{cases}$$

$$T^2 x_0 = \begin{cases} \{0\}, & x_0 = 0 \\ [0, x_2], & x_0 > 0 \end{cases}$$

Inductively, we get

$$T^n x_0 = \begin{cases} \{0\}, & x_0 = 0 \\ [0, x_n], & x_0 > 0 \end{cases}$$

Thus $\delta(T^n x_0) = x_n \to 0$. Therefore T satisfies the hypothesis of Theorem 2.1. $(T0 = \{0\})$

Let $\alpha : [0, \infty) \to [0, 1)$,

$$\alpha(x) = \begin{cases} \frac{\ln(1+x)}{x}, & x > 0 \\ 0, & x = 0 \end{cases}.$$

Since

$$\begin{aligned} &\lim_{x\to\infty} \alpha(x) = 0 \\ &\lim_{x\to 0} \alpha(x) = 1 \\ &\lim_{x\to a} \alpha(x) = \alpha(a) \in (0, 1), (\forall) \ a \in (0, \infty) \end{aligned}$$

then $\alpha(t_n) \to 1$ implies $t_n \to 0$, hence $\alpha \in S$.

Let $(\forall) \ x, y \in [0, \infty)$. If $x = y$ we have $H(Tx, Ty) = 0 \leq \alpha(d(x, y)) \cdot d(x, y) = 0$. If $x \neq y$ by (1) we have $H(Tx, Ty) \leq \varphi(d(x, y)) = \ln(1 + d(x, y)) = \alpha(d(x, y)) \cdot d(x, y)$. In conclusion $H(Tx, Ty) \leq \alpha(d(x, y)) \cdot d(x, y)$, $(\forall) \ x, y \in [0, \infty)$, where T satisfies the inequality of Corollary 1.1. For $x = e - 1$ we have $d(e-1, 0) = e - 1$, $T(e-1) = [0, 1]$ and $D(e-1, T(e-1)) = e - 2$, from where $d(e-1, 0) \neq D(e-1, T(e-1))$. Therefore, from Proposition 1.1, T is not a special multi-valued mapping.

In Theorem 1.7 (X, d) is a bounded hyperconvex metric space, but in our example (X, d) is an unbounded metric space.

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