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Results in Mathematics

Distances from the Vertices of a Regular Simplex

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Dedicated to Professor R. G. Swan

Abstract. If S is a given regular d-simplex of edge length a in the d-dimensional Euclidean space $\mathcal{E}$, then the distances $t_1, \ldots, t_{d+1}$ of an arbitrary point in $\mathcal{E}$ to the vertices of S are related by the elegant relation

$$(d+1)\left(a^4+t_1^4+\cdots+t_{d+1}^4\right)=\left(a^2+t_1^2+\cdots+t_{d+1}^2\right)^2.$$

The purpose of this paper is to prove that this is essentially the only relation that exists among $t_1, \ldots, t_{d+1}$. The proof uses tools from analysis, algebra, and geometry.

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1. Introduction

Much has been written about the elegant relation

$$(d+1)\left(a^4+\sum_{j=1}^{d+1}t_j^4\right)=\left(a^2+\sum_{j=1}^{d+1}t_j^2\right)^2 \tag{1}$$

that exists among the edge length a of a regular d-dimensional simplex in the Euclidean space $\mathbb{R}^d$ and the distances $t_1, \ldots, t_{d+1}$ from the vertices of that

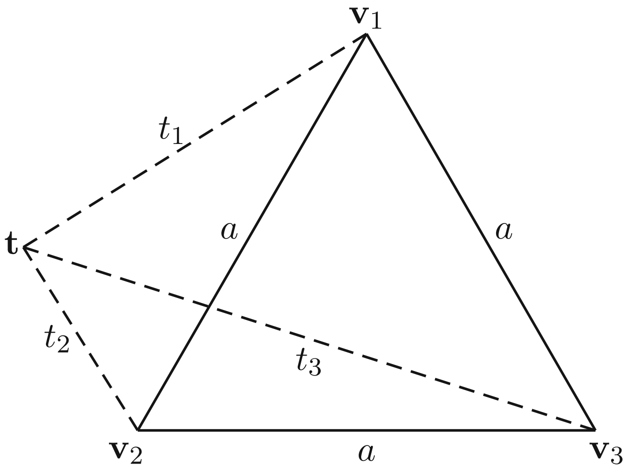

FIGURE 1. Illustrating Relation (2)

simplex to an arbitrary point in $\mathbb{R}^d$. The special case $d = 2$ is illustrated in Fig. 1 below, and the corresponding relation

$$3\left(a^4 + t_1^4 + t_2^4 + t_3^4\right) = \left(a^2 + t_1^2 + t_2^2 + t_3^2\right)^2 \tag{2}$$

was popularized by Martin Gardner in his article [5] that was reproduced in [6, Chapter 5, pp. 56–65]. Feedbacks on Gardner's article appear in [4,7], and [14], and in possibly other places. A proof of the general case can be found in [1], and another proof that uses the Cayley–Menger formula for the volume of a simplex is given in [9]. The relation (1) can also be derived using linear algebra.

One of the striking features of (1) is its symmetry, not only in $\{t_1, \ldots, t_{d+1}\}$ (which is only expected), but also in $\{a, t_1, \ldots, t_{d+1}\}$. The fact that a plays the same role in (1) as any other t_i does not seem to have a satisfactory explanation[1]. Due to this symmetry, it is customary to set $a = t_0$ in (1) and write it in the form

$$(d+1)\left(\sum_{j=0}^{d+1} t_j^4\right) = \left(\sum_{j=0}^{d+1} t_j^2\right)^2. \tag{3}$$

Another striking feature of (1) (or rather (3)) is its similarity with the relation

[1]Recently, Dr. Ismail Hammoudeh of Amman University has come up with a satisfactory conceptual explanation of this symmetry. He intends to write this in a separate paper.

$$d\left(\sum_{j=0}^{d+1}\left(\frac{1}{r_j}\right)^2\right)=\left(\sum_{j=0}^{d+1}\frac{1}{r_j}\right)^2 \tag{4}$$

that exists among the *oriented* radii $r_0,\ldots,r_{d+1}$ of $d+2$ spheres in $\mathbb{R}^d$ that are in mutual external touch; see [3,13].

It is now natural to ask whether there are other relations, beside the one in (1), among $t_1,\ldots,t_{d+1}$. We state this question precisely below, and we devote the rest of the paper to answering it.

Question 1.1. Let $S=[\mathbf{v}_1,\ldots,\mathbf{v}_{d+1}]$ be a regular d-simplex of side length t_0 in $\mathbb{R}^d$. For every $\mathbf{t}\in\mathbb{R}^d$, let t_j be the distance from $\mathbf{t}$ to $\mathbf{v}_j$, $1\le j\le d+1$. Let $R=\mathbb{R}[T_1,\ldots,T_{d+1}]$ be the ring of polynomials in the indeterminates T_1, …, T_{d+1} over the field $\mathbb{R}$ of real numbers. Let I be the ideal in R defined by

$$I=\{f\in R: f\left(t_1,\ldots,t_{d+1}\right)=0\ \forall\ \mathbf{t}\in\mathbb{R}^d\}, \tag{5}$$

and let

$$F=(d+1)\left(t_0^4+\sum_{j=1}^{d+1}T_j^4\right)-\left(t_0^2+\sum_{j=1}^{d+1}T_j^2\right)^2. \tag{6}$$

Is I generated by F?

We shall establish an affirmative answer to this question in Theorem 5.1 in Sect. 5. This essentially amounts to proving that I is a prime ideal of height 1 in the integral domain of functions that are real analytic on $\mathbb{R}^d\backslash\{\mathbf{v}_1,\ldots,\mathbf{v}_{d+1}\}$, that $F\in I$, and that F is irreducible. Rigorous proofs of these intuitively obvious facts turn out to involve tools from analysis, algebra, and geometry. As some of this material cannot be assumed to be known to the potential readers of this article, we have chosen to be self-contained, writing the definitions of the terms, and giving adequate references to, or proofs of, the theorems used. Readers are advised to read the main theorem, Theorem 5.1, and its proof first, and decide for themselves what sections of the paper they need to go back to and read.

The paper is organized as follows. Section 2 establishes the irreducibility of a certain polynomial. This is an essential step in the proof of the main theorem. Section 3 introduces the preliminary definitions and theorems from the theory of real analytic functions. The only reference that we have used is the book [11]. Section 4 puts together the necessary algebraic tools. These include the height of an ideal, algebraic independence over a field, transcendence bases and degree of an extension, the Krull dimension of a ring, and how these are related. The only reference that we referred to here is R. Y. Sharp's book [15]. We have also proved in this section that the distances from an arbitrary point in $\mathbb{R}^n$ to any n vertices of a regular n-simplex in $\mathbb{R}^n$ are algebraically independent (over $\mathbb{R}$). Section 5 contains the main theorem and its proof. The last section, Sect. 6, contains a list of problems that may generate further research. We expect that most, or all, of these problems are within reach of a young researcher, and we also expect the material in Sects. 2, 3, 4 and 5 to be useful to such a researcher and to others, and easier to refer to than to refer to various books.

2. Irreducibility of a Certain Class of Polynomials

The following theorem will be used later in this article. It follows from the results in [8], where the irreduciblity of the more general polynomial

$$t\left(a^4+\sum_{j=1}^{n} x_j^4\right)-\left(a^2+\sum_{j=1}^{n} x_j^2\right)^2 \in k[x_1,\ldots,x_n], \tag{7}$$

for all fields k and all $t, a \in k$ was fully investigated. However, we chose to provide an independent proof, thus removing dependence on [8]. In the proof, we freely use the fact that factors of a homogeneous polynomial are homogeneous; see [16, Theorem 10.5, p. 28].

Theorem 2.1. *Let $\mathbb{R}[x_1,\ldots,x_n]$ be the polynomial ring in the indeterminates $x_1,\ldots,x_n$ over the real number field $\mathbb{R}$, and let*

$$f=t\left(a^4+\sum_{j=1}^{n} x_j^4\right)-\left(a^2+\sum_{j=1}^{n} x_j^2\right)^2 \in \mathbb{R}[x_1,\ldots,x_n], \tag{8}$$

where $t, a \in \mathbb{R}$, $n \geq 3$, and $t \geq 3$. Then f is irreducible.

Proof. It is easy to see that a polynomial in $\mathbb{R}[x_1,\ldots,x_n]$ is irreducible if its leading homogeneous component is irreducible. Since the leading homogeneous component of f is given by

$$g=g(x_1,\ldots,x_n) \;=\; t\left(\sum_{j=1}^{n} x_j^4\right)-\left(\sum_{j=1}^{n} x_j^2\right)^2 \in \mathbb{R}[x_1,\ldots,x_n], \tag{9}$$

it is sufficient to show that g is irreducible. Letting F be the polynomial obtained from g by setting $x_4=\cdots=x_n=0$, and renaming x_1, x_2, and x_3 as x, y, and z, it is again easy to see that g is irreducible if F is. This follows from the fact that if A is a homogeneous polynomial in many variables of degree d, and if A^* is obtained from A by substituting 0 for some of these variables, then either $A^*=0$ or A^* is homogeneous of degree d. Thus we are left with showing that the polynomial

$$F=t\left(x^4+y^4+z^4\right)-\left(x^2+y^2+z^2\right)^2 \in \mathbb{R}[x,y,z] \tag{10}$$

is irreducible.

To prove that F is irreducible, we first show that F cannot have a non-constant symmetric factor. Letting

$$s = x + y + z,\ p = xy + yz + zx,\ q = xyz,$$

we write F as a polynomial $\phi = \phi(s,p,q)$ in the polynomial ring $\mathbb{R}[s,p,q]$, and we show that ϕ is irreducible. Direct calculations show that

$$\begin{aligned} F &= (t-1)\left(x^4+y^4+z^4\right) - 2\left(x^2y^2+y^2z^2+z^2x^2\right) \\ &= (t-1)\left(x^2+y^2+z^2\right)^2 - 2t\left(x^2y^2+y^2z^2+z^2x^2\right) \\ &= (t-1)(s^2-2p)^2 - 2t(p^2-2qs) \\ &= (t-1)s^4 + 2(t-2)p^2 - 4(t-1)s^2p + 4tqs. \end{aligned}$$

Thus $\phi = (t-1)s^4 + 2(t-2)p^2 - 4(t-1)s^2p + 4tqs$. Since ϕ is linear in q with leading term $4tsq$, it follows that the only possible factorizations of ϕ are of the form

$$\phi = (4tsq + \phi_1(s,p))\phi_2(s,p) \text{ and } \phi = (4tq + \phi_1(s,p))\phi_2(s,p).$$

Comparing coefficients of q, we see that either $\phi_2 = 1$, in which case the factorization is trivial, or $\phi_2 = s$, which is impossible since s does not divide ϕ. Thus ϕ is irreducible, and therefore F has no non-constant symmetric factors.

We next prove that F has no linear factor. If F has a factor A of degree 1, then the set $\mathcal{Z}(A)$ of zeros of A is a plane (in the (xyz)-space). However, we shall see now that $\mathcal{Z}(F)$ is contained in a set of four lines, contradicting the fact that $\mathcal{Z}(A) \subseteq \mathcal{Z}(F)$. We rewrite F as follows:

$$\begin{aligned} F &= t\left(x^4+y^4+z^4\right) - \left(x^2+y^2+z^2\right)^2 \\ &= (t-1)\left(\left(x^4+y^4+z^4\right) - \left(x^2y^2+y^2z^2+z^2x^2\right)\right) \\ &\quad +(t-3)\left(x^2y^2+y^2z^2+z^2x^2\right) \\ &= \frac{t-1}{2}\left((x^2-y^2)^2+(y^2-z^2)^2+(z^2-x^2)^2\right) \\ &\quad +(t-3)\left(x^2y^2+y^2z^2+z^2x^2\right). \end{aligned}$$

Since $t \geq 3$, it follows that $F \geq 0$ for all $x,y,z \in \mathbb{R}$, and that $F = 0$ if and only if ($t = 3$ and $x^2 = y^2 = z^2$) or ($t > 3$ and $x = y = z = 0$). Hence $\mathcal{Z}(F)$ is contained in a set of four lines. Thus $\deg(A)$ cannot be 1.

We next show that F cannot have a factor A of degree 2. If it does, then by the previous steps, A is irreducible and non-symmetric. Let

$$A = ax^2 + by^2 + cz^2 + \alpha yz + \beta zx + \gamma xy,$$

where $a,b,c,\alpha,\beta,\gamma$ are in $\mathbb{R}$. Letting σ be the permutation $(x \mapsto y \mapsto z \mapsto x)$, we see that F is divisible by A, $\sigma(A)$, and $\sigma^2(A)$. Since $\deg(A\sigma(A)\sigma^2(A)) = 6 > \deg F$, and since $A, \sigma(A), \sigma^2(A)$ are irreducible, it follows that two (and hence all) of the polynomials A, $\sigma(A)$, and $\sigma^2(A)$ are associates (i.e., constant multiples of each other). Thus $A = \lambda\sigma(A)$ for some $\lambda \in \mathbb{R}$. Since $\sigma^3(A) = A$, it follows that $\lambda^3 = 1$ and hence $\lambda = 1$. Thus $A = \sigma(A)$. Since

$$\sigma(A) = ay^2 + bz^2 + cx^2 + \alpha zx + \beta xy + \gamma yz,$$

it follows that $a = b = c$ and $\alpha = \beta = \gamma$, contradicting the assumption that A is not symmetric.

Thus F, and hence f, is irreducible, as desired. $\square$

3. Real Analytic Functions in Several Variables

In this section, we present the basic material on real analytic functions that will be needed in the proof of the main theorem, Theorem 5.1. The treatment is self contained, and all necessary definitions are given. We feel that this interesting subject (of real analytic functions) is not usually covered in standard required courses in graduate schools, and we also feel that there are not many textbooks on the subject. Our only reference is [11], and any differences between our presentation and that in [11] are very slight and trivial, and they are made with the permission of the first author of [11].

Definition 3.1. (**[11, Definition 2.1.4, p. 27]**) Let $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ be the set of non-negative integers, and let $(\mathbb{Z}^+)^m$, $m \in \mathbb{N}$, be denoted by $\Lambda(m)$. If $\mathbf{e} = (e_1, \dots, e_m) \in \Lambda(m)$, and if $\mathbf{r} = (r_1, \dots, r_m) \in G^m$, where G is any commutative ring with 1, then $\mathbf{r}^{\mathbf{e}}$ stands for the product

$$\prod_{j=1}^{m} r_j^{e_j}.$$

A power series in the m variables $\mathbf{x} = (x_1, \dots, x_m)$ with center at $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}^m$ is a formal expression

$$\sum_{\mathbf{e} \in \Lambda(m)} c_{\mathbf{e}} (\mathbf{x} - \mathbf{a})^{\mathbf{e}}, \text{ where } c_{\mathbf{e}} \in \mathbb{R}. \tag{11}$$

The power series (11) is said to converge at $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{R}^m$ if some rearrangement of it converges. More precisely, the power series (11) converges at $\mathbf{b} = (b_1, \dots, b_m)$ if there is a bijection $\phi : \mathbb{Z}^+ \to \Lambda(m)$ such that the sequence of partial sums of the series

$$\sum_{j=0}^{\infty} c_{\phi(j)} (\mathbf{b} - \mathbf{a})^{\phi(j)}$$

converges.

Theorem 3.2. (**[11, Proposition 2.1.7 (Abel's Lemma), p. 27]**) *If the power series* $\sum_{\mathbf{e} \in \Lambda(m)} c_{\mathbf{e}} \mathbf{x}^{\mathbf{e}}$ *converges at a point* $\mathbf{x} = \mathbf{b} = (b_1, \dots, b_m) \in \mathbb{R}^m$, *then it converges uniformly and absolutely on compact subsets of the silhouette of* $\mathbf{b}$, *i.e., the open box*

$$(-|b_1|, |b_1|) \times \cdots \times (-|b_m|, |b_m|).$$

Corollary 3.3. *If a power series converges at every point in an open set U, then it converges absolutely at every point in U. Consequently, it defines a function on U.*

Proof. Let $q \in U$. Then U contains a closed box B centered at q. Thus q belongs to the silhouette S of some vertex, say $\mathbf{v}$, of B. Since the given power series converges at $\mathbf{v}$, it follows from Abel's lemma that it converges absolutely on S, and hence at q. □

Definition 3.4. ([**11, Definition 2.2.1, p. 29**]) We say that the real-valued function f is *real analytic* at a point p in $\mathbb{R}^n$ if p has a neighborhood U on which f can be represented as a power series centered at p. We say that f is real analytic on a set U if f is real analytic at every point of U.

Theorem 3.5. *If f is real analytic at $p \in \mathbb{R}^n$, then f is real analytic on a neighborhood U of p. Consequently, the set where f is real analytic is open.*

Proof. Let $F_p(z)$ be a power series centered at p (i.e., in powers of $(z-p)$) that represents f on some neighborhood U of p. By Proposition 2.2.7 (p. 32) of [11], $F_p(z)$ is real analytic on U. But $F_p(z)$ coincides with f on U. Therefore f is real analytic on U. □

Theorem 3.6. **([11, Theorem 2.2.2, p. 29])** *If f, g are real analytic on the subsets U, V of $\mathbb{R}^n$, respectively, then $f+g$, fg are real analytic on $U \cap V$, and f/g is real analytic on $U \cap V \cap \{p : g(p) \neq 0\}$.*

Theorem 3.7. **([11, Corollary 1.2.7, p. 14])** *If f is real analytic on the open interval $U \subseteq \mathbb{R}$, and if the set of zeros of f has an accumulation point in U, then f is identically zero on U.*

Remark 3.8. Theorem 3.7 does not seem to have an analogue in higher dimensions. For example, the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = xy$ is real entire, and its zero set consists of the x- and y- axes. But f is not identically zero.

Lemma 3.9. *Suppose that $f = f(x_1, \ldots, x_n)$ is real analytic at $a = (a_1, \ldots, a_n)$, and let $1 \leq k \leq n$. Let*

$$f^* = f^*(x_{k+1}, \ldots, x_n) = f(a_1, \ldots, a_k, x_{k+1}, \ldots, x_n).$$

Then f^* is real analytic at $a^* = (a_{k+1}, \ldots, a_n)$. The same holds for the function f_* and the point a_* defined by

$$f_* = f_*(x_1, \ldots, x_k) = f(x_1, \ldots, x_k, a_{k+1}, \ldots, a_n), \; a_* = (a_1, \ldots, a_k).$$

Proof. Just substitute $a_1, \ldots, a_k$ for $x_1, \ldots, x_k$ in the power series. □

Theorem 3.10. *Let $U \subseteq \mathbb{R}^d$ be a connected open set and let f be an analytic function on U. If $f = 0$ on some non-empty open subset V of U, then $f = 0$ on U.*

Proof. If $d = 1$, then $U \subseteq \mathbb{R}$ is an open interval, and the result follows from Theorem 3.7. So we take $d > 1$.

If U is convex, choose a point p in V and consider lines L through p. Since $L \cap U$ is an open interval of L and $L \cap V$ is non-empty, it follows from

Lemma 3.9 and a change of coordinates that the restriction of f on $L \cap U$ is 0. Therefore $f = 0$ on U since the intervals $L \cap U$ cover U by convexity.

In the general case, let $p \in V$ and let q be any other point of U. We shall show that $f(q) = 0$. Since U is connected, we can find a finite number of open convex neighborhoods W_i, $1 \leq i \leq n$, such that

$$p \in W_1, \ q \in W_n, \ \text{and } W_i \cap W_{i+1} \text{ is non-empty for } i = 1, \ldots, n-1.$$

Letting f_i denote the restriction of f on W_i, we conclude by the previous case that if $f_i = 0$, then $f_{i+1} = 0$. Therefore $f_n = 0$, and hence $f(q) = 0$. □

Corollary 3.11. *The ring A of all real analytic functions on a connected open set $U \subseteq \mathbb{R}^d$ is an integral domain.*

Proof. Let $fg = 0$ in A, and let $V = \{x \in U : g(x) \neq 0\}$. If V is empty, we are done. Otherwise, the restriction of f on V is 0, and therefore $f = 0$ on U by Theorem 3.10. □

Remark 3.12. On the set of all functions that are real analytic at a point $p \in \mathbb{R}^m$, we define an equivalence relation $\equiv$ by

$$f \equiv g \Longleftrightarrow f = g \text{ on some open neighborhood } U = U_{f,g} \text{ of } p.$$

Each equivalence class is called a *germ* or a p-germ. It is not difficult to define addition and multiplication on the set G of p-germs so that G becomes a ring. Now the proof above can be mimicked to show that G is an integral domain. This is stronger than Corollary 3.11.

Theorem 3.13. **([11, Proposition 2.2.8, p. 33])** *If f_j, $1 \leq j \leq d$, are real analytic at $p \in \mathbb{R}^n$, and if g is real analytic at $(f_1(p), \ldots, f_d(p)) \in \mathbb{R}^d$, then the composition $g(f_1, \ldots, f_d)$ is real analytic at $p \in \mathbb{R}^n$.*

Corollary 3.14. *If f is real analytic on some open set $U \subseteq \mathbb{R}^n$, and if $f(p) > 0$ for some $p \in U$, then there exists a neighborhood $W \subseteq U$ of p such that $\sqrt{f}$ exists and is real analytic on W. In particular, if $p \in \mathbb{R}^n$, then the function $h : \mathbb{R}^n \to \mathbb{R}$ defined by $h(x) = \|x - p\|$ is real analytic at all points except at p.*

Proof. Observe that the square root function defined on $\{x \in \mathbb{R} : x > 0\}$ by $x \mapsto \sqrt{x}$ is real analytic because

$$\sqrt{a + x} = \sqrt{a} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \left(\frac{x}{a}\right)^n$$

for small x. The desired result now follows from Theorem 3.13. □

4. Algebraic Independence of Certain Functions

Let S be a regular d-simplex in $\mathbb{R}^d$, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, $n < d + 1$, be n vertices of S. Let the functions $\phi_j : \mathbb{R}^d \to \mathbb{R}$, $1 \leq j \leq d$, be defined by $\phi_j(\mathbf{x}) = \|\mathbf{x} - \mathbf{v}_j\|$. Let $\mathcal{U}$ be an open everywhere dense subset of $\mathbb{R}^d$, and let $\mathcal{A}$

be the ring of real-valued functions that are real analytic on $\mathcal{U}$. We have seen in Corollary 3.11 that $\mathcal{A}$ is an integral domain, and we have seen in Corollary 3.14 that the functions ϕ_j, $1 \leq j \leq d$, belong to $\mathcal{A}$. In this section, we shall show that these functions are algebraically independent over $\mathbb{R}$. Actually, the same holds if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is any set of affinely independent points in $\mathbb{R}^d$, but the proof is lengthier.

We start with the necessary preliminaries that we need from algebra.

4.1. Algebraic Preliminaries

Our reference is [15], and *all rings referred to are commutative with 1.* If B is any ring, then $B[T_1, \ldots, T_n]$ stands for the polynomial ring over B in the n indeterminates $T_1, \ldots, T_n$.

Let S be a ring (commutative with identity $1 \neq 0$).

If P is a prime ideal in S, then the *height* of P, written ht$P$, is the supremum of all non-negative integers k for which there exists a sequence $P_0, P_1, \ldots, P_k = P$ of prime ideals in S of the form

$$P_0 \subset P_1 \subset \cdots \subset P_k = P,$$

where $\subset$ stands for *strict* inclusion. If the supremum does not exist, we write ht$P = \infty$. The *dimension* of S, written $\dim S$, is the supremum of all non-negative integers k for which there exists a sequence $P_0, P_1, \ldots, P_k = P$ of prime ideals in S of the form

$$P_0 \subset P_1 \subset \cdots \subset P_k = P.$$

If the supremum does not exist, we write $\dim S = \infty$. Thus

$$\dim S = \sup\{\text{ht}P : P \text{ is a prime ideal in } S\},$$

where the supremum is defined for all subsets of $[-\infty, +\infty]$. It is easy to see, as done in Remark 14.18 (viii) (page 279) of [15], that

$$\text{ht } P + \dim S/P \leq \dim S, \tag{12}$$

where we adopt the convention that $\infty + \infty = \infty$, and $\infty + n = \infty$ for all integers n.

Now let R be a subring of S.

A finite subset $\{s_1, \ldots, s_n\}$ of S is said to be *algebraically independent* over R if there does not exist a non-zero polynomial $f = f(T_1, \ldots, T_n)$ in the polynomial ring $R[T_1, \ldots, T_n]$ such that $f(s_1, \ldots, s_n) = 0$. An arbitrary subset of S is said to be *algebraically independent* over R if every finite subset of S is algebraically independent over R. This is Definition 1.14 (page 8) of [15]. If both R and S are fields, then a subset $B = \{s_1, \ldots, s_n\}$ of S that is algebraically independent over R is called a (finite) *transcencdence basis* of S over R if every subset of S that properly contains B is algebraically dependent over R. This is Definition 12.54 (page 239) of [15]. By Theorem 12.53 (page 239) of [15], if S has a finite transcendence basis over R, then any two such bases have the same number of elements. This number is called the *transcendence degree* of S over R, and is denoted by $\text{tr.deg.}_R S$.

If S is an integral domain that contains a field R and that is finitely generated over R, i.e., S is an affine R-algebra, and if L is the field of quotients of S, then

$$\dim S = \text{tr.deg.}_R L. \tag{13}$$

This is Corollary 14.29 (page 282) in [15]. It follows that if R is a field, then

$$\dim R[T_1, \ldots, T_n] = n. \tag{14}$$

We also shall need the facts that if R is a field (or any unique factorization domain), then the polynomial ring $R[T_1, \ldots, T_n]$ is a unique factorization domain, and that an irreducible element in a unique factorization domain D generates a prime ideal of D; see [15, Theorem 1.42, p. 17] and [15, Exercise 3.42, p. 49]. Now the following theorem, that we will use later, follows immediately.

Theorem 4.1. *Let R be a field, and let P be a prime ideal of $R[T_1, \ldots, T_n]$ that contains an irreducible polynomial f. If ht $P = 1$, then P is generated by f.*

Proof. Let Q be the ideal generated by f. Then Q is a prime ideal. If $Q \neq P$, then the chain $P \supset Q \supset \{0\}$ would imply the contradiction that $\text{ht} P \geq 2$. Therefore $P = Q$, as claimed. □

4.2. Algebraic Independence of Distances to Vertices of a Regular Simplex

In this section we prove that the functions $\phi_j : \mathcal{U} \to \mathbb{R}$, $1 \leq j \leq d$, defined in the previous section are algebraically independent over $\mathbb{R}$.

Theorem 4.2. *Let $\mathbb{E} = \mathbb{R}^d$, and let $1 \leq n \leq d$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be n vertices of a regular d-simplex in $\mathbb{E}$, and let the functions $\phi_1, \ldots, \phi_n : \mathbb{E} \to \mathbb{R}$ be defined by*

$$\phi_j(p) = \|p - \mathbf{v}_j\|, \ 1 \leq j \leq n.$$

Then the functions $\phi_1, \ldots, \phi_n$, (thought of as elements in the ring of all real-valued functions on $\mathbb{E}$) are algebraically independent (over $\mathbb{R}$).

Proof. Identify $\mathbb{R}^n$ with the hyperplane H of $\mathbb{R}^{n+1}$ defined by

$$H = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + \cdots + x_{n+1} = 0\}.$$

We can assume that $\mathbf{v}_i = \mathbf{e}_i$ for $1 \leq i \leq n$, where $\mathbf{e}_i$ is the standard unit vector having 1 in the i-th place and 0 everywhere else. Let $\phi_i(\mathbf{x}) = \|\mathbf{x} - \mathbf{e}_i\|$. We think of these functions as elements in the ring A of real analytic functions on a connected dense open subset of $\mathbb{R}^n$, and we do our calculations in the quotient field of A.

To show that ϕ_i, $1 \leq i \leq n$, are algebraically independent is equivalent to showing that $\text{tr.deg.}_{\mathbb{R}} \mathbb{R}(\phi_1, \ldots, \phi_n) = n$. Let

$$f_i(\mathbf{x}) = (\phi_i(\mathbf{x}))^2 = \|\mathbf{x} - \mathbf{e}_i\|^2.$$

The field extension $\mathbb{R}(f_1, \ldots, f_n) \subset \mathbb{R}(\phi_1, \ldots, \phi_n)$ is algebraic, and so the two fields have the same transcendence degree over $\mathbb{R}$. For $\mathbf{x} = (x_1, \ldots, x_n)$,

$$f_i(\mathbf{x}) = \|\mathbf{x} - \mathbf{e}_i\|^2 = g(x) - 2x_i + 1,$$

where $g(\mathbf{x}) = x_1^2 + \cdots + x_n^2$. Therefore $2x_i = g + 1 - f_i$. Squaring and summing on i, we get

$$4g = 4\sum_{n=1}^{n} x_i^2 \;=\; n(g^2 + 2g + 1) - 2(g+1)\sum_{i=1}^{n} f_i + \sum_{i=1}^{n} f_i^2.$$

This is a quadratic equation for g over $\mathbb{R}(f_1, \ldots, f_n)$, and so $\mathbb{R}(f_1, \ldots, f_n, g)$ is algebraic over $\mathbb{R}(f_1, \ldots, f_n)$. But $\mathbb{R}(f_1, \ldots, f_n, g) = \mathbb{R}(x_1, \ldots, x_n)$, which has transcendence degree n over $\mathbb{R}$. The same is therefore true for $\mathbb{R}(f_1, \ldots, f_n)$ and $\mathbb{R}(\phi_1, \ldots, \phi_n)$. □

The next corollary is what we actually need in the proof of the main theorem, Theorem 5.1 below.

Corollary 4.3. *Let $n \geq 1$. Let $\mathbb{E}$ be a Euclidean space of dimension greater than or equal to n, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be n vertices of a regular n-simplex in $\mathbb{E}$. Let $\mathbb{E}_0$ be any everywhere dense subset of $\mathbb{E}$, and let the functions $\phi_1, \ldots, \phi_n : \mathbb{E}_0 \to \mathbb{R}$ be defined by*

$$\phi_j(p) = \|p - \mathbf{v}_j\|, \;\; 1 \leq j \leq n.$$

Then the functions ϕ_j, $1 \leq j \leq n$, thought of as elements in the ring of all real-valued functions on $\mathbb{E}_0$, are algebraically independent (over $\mathbb{R}$). This is true in particular if $\mathbb{E}\backslash\mathbb{E}_0$ is a finite set.

Proof. Let $g \in \mathbb{R}[T_1, \ldots, T_n]$ be a non-zero polynomial for which

$$g(\phi_1(p), \ldots, \phi_n(p)) = 0$$

for all $p \in \mathbb{E}_0$. Since $g(\phi_1, \ldots, \phi_n)$ is continuous, its set of zeros is closed, and hence contains the closure $\mathbb{E}$ of $\mathbb{E}_0$. Thus

$$g\left(\|p - \mathbf{v}_1\|, \cdots, \|p - \mathbf{v}_n\|\right) = 0$$

for all $p \in \mathbb{E}$. □

5. Answering Question 1.1: The Main Result

In this section, we establish, in Theorem 5.1 an affirmative answer to Question 1.1 posed in Sect. 1. Thus we prove that the ideal I of R is indeed the principal ideal generated by F.

Theorem 5.1. *Let $S = [\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}]$, $d \geq 2$, be a regular d-simplex in $\mathbb{R}^d$ having edge length a. Let $t_1, \ldots, t_{d+1}$ be the distances from an arbitrary point $\mathbf{t}$ in $\mathbb{R}^d$ to the vertices of S. Let $R = \mathbb{R}[T_1, \ldots, T_{d+1}]$ be the ring of polynomials* *over* $\mathbb{R}$ *in the indeterminates* $T_1, \dots, T_{d+1}$*, and let* I *be the ideal of* R *defined by*

$$I = \{f \in R : f\,(t_1, \dots, t_{d+1}) = 0 \ \forall\ \mathbf{t} \in \mathbb{R}^d\}. \tag{15}$$

Let

$$F = (d+1)\left(a^4 + \sum_{j=1}^{d+1} T_j^4\right) - \left(a^2 + \sum_{j=1}^{d+1} T_j^2\right)^2. \tag{16}$$

Then I *is the principal ideal generated by* F*.*

Proof. Let $U = \mathbb{R}^d \backslash \{\mathbf{v}_1, \dots, \mathbf{v}_{d+1}\}$. Let $\mathcal{A}$ be the set of (real-valued) functions that are real analytic on U. By Corollary 3.11, $\mathcal{A}$ is an integral domain. By Corollary 3.14, the functions $\phi_j : U \to \mathbb{R}$, $1 \le j \le d+1$, defined by $\phi_j(\mathbf{t}) = \|\mathbf{t} - \mathbf{v}_j\|$ belong to $\mathcal{A}$.

Let $\Phi : \mathbb{R}[T_1, \dots, T_{d+1}] \to \mathcal{A}$ be the ring $\mathbb{R}$-homomorphism defined by $\Phi(T_j) = \phi_j$ for $1 \le j \le d+1$. The kernel of Φ consists of all $f \in R$ such that $f(t_1, \dots, t_{d+1}) = 0$ for all $\mathbf{t}$ in U. By continuity, this is the set of all $f \in R$ such that $f(t_1, \dots, t_{d+1}) = 0$ for all $\mathbf{t}$ in $\mathbb{R}^d$, i.e., the ideal I. The image $\mathcal{A}_0$ of Φ, being a subring of the integral domain $\mathcal{A}$, is itself an integral domain. Since $R/I \cong \mathcal{A}_0$, by the first isomorphism theorem, it follows that I is a prime ideal of R.

Also, $\mathcal{A}_0$ contains the d elements $\phi_1, \dots, \phi_d$, and these are algebraically independent over $\mathbb{R}$, by Corollary 4.3. It follows from the definition that $\text{tr.deg.}_{\mathbb{R}}(QF(\mathcal{A}_0)) \ge d$, where $QF(.)$ denotes the quotient field. Since $\mathcal{A}_0$ and R/I are isomorphic as $\mathbb{R}$-algebras, it follows that $\text{tr.deg.}_{\mathbb{R}}(QF(R/I)) \ge d$. Also, the integral domain R/I is an affine $\mathbb{R}$-algebra. Therefore $\text{tr.deg.}_{\mathbb{R}}(QF(R/I)) = \dim(R/I)$. Therefore

$$\begin{aligned} d \le \text{tr.deg.}_{\mathbb{R}}(QF(\mathcal{A}_0)) \ &= \ \text{tr.deg.}_{\mathbb{R}}(QF(R/I)) \ = \ \dim(R/I) \\ \le \dim(R) - \text{ht}(I) \ &= \ d + 1 - \text{ht}(I). \end{aligned}$$

Hence $\text{ht}(I) \le 1$. Since I contains the polynomial F, which is irreducible by Theorem 2.1, it follows that I contains the prime ideal generated by F. Hence $\text{ht}(I) = 1$. By Theorem 4.1, I is generated by F, as claimed. □

Remark 5.2. When talking about a d-simplex, one usually assumes that $d \ge 2$, since a 1-simplex is a line segment with a poor geometry. However, it is legitimate to wonder whether Theorem 5.1 still holds when $d = 1$, and it may be interseting to know that it does not. In fact, if one takes $d = 1$ and if one defines I and F as in Theorem 5.1, then it turns out that F is not irreducible any more, as it factors into

$$\begin{aligned} F &= 2(a^4 + T_1^4 + T_2^4) - (a^2 + T_1^2 + T_2^2)^2 \\ &= (T_1 + T_2 + a)(T_1 + T_2 - a)(T_1 - T_2 + a)(T_1 - T_2 - a), \end{aligned}$$

and that I is the principal ideal generated by

$$F_0 = (T_1 + T_2 - a)(T_1 - T_2 + a)(T_1 - T_2 - a),$$

and not by F. To see this, let $\mathbf{v}_1, \mathbf{v}_2$ be two distinct points in $\mathbb{R}$, and let

$$I = \{f(T_1, T_2) \in \mathbb{R}[T_1, T_2] : f(t_1, t_2) = 0 \;\forall\; \mathbf{t} \in \mathbb{R}\}.$$

Assuming that $\mathbf{v}_1 < \mathbf{v}_2$, we let I_1, I_2, and I_3 be the ideals defined by

$$I_1 = \{f(T_1, T_2) \in \mathbb{R}[T_1, T_2] : f(t_1, t_2) = 0 \;\forall\; \mathbf{t} \leq \mathbf{v}_2\},$$
$$I_2 = \{f(T_1, T_2) \in \mathbb{R}[T_1, T_2] : f(t_1, t_2) = 0 \;\forall\; \mathbf{t} \geq \mathbf{v}_1\},$$
$$I_3 = \{f(T_1, T_2) \in \mathbb{R}[T_1, T_2] : f(t_1, t_2) = 0 \;\forall\; \mathbf{t} \in (\mathbf{v}_1, \mathbf{v}_2)\}.$$

It is easy to see that the polynomials

$$H_1 = T_1 - T_2 - a \in I_1,\; H_2 = T_1 - T_2 + a \in I_2,\; H_3 = T_1 + T_2 - a \in I_3.$$

Being of total degree 1, H_i, $1 \leq i \leq 3$, is irreducible, and hence generates I_i. Thus if $H \in I$, then $H \in I_1 \cap I_2 \cap I_3$, and therefore H_i divides H for $1 \leq i \leq 3$. Since H_1, H_2, and H_3 are pairwise relatively prime, and since $\mathbb{R}[T_1, T_2]$ is a unique factorization domain, it follows that $H_1H_2H_3$ divides H. This shows that I is generated by $H_1H_2H_3$, as claimed.

6. Questions for Further Research

Question 6.1. Let $S = [\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}]$ be a regular d-simplex of side length t_0 in $\mathbb{R}^d$. For every $\mathbf{t} \in \mathbb{R}^d$, let t_j be the distance from $\mathbf{t}$ to $\mathbf{v}_j$, $1 \leq j \leq d+1$. Let $R_0 = \mathbb{R}[T_0, T_1, \ldots, T_{d+1}]$ be the ring of polynomials in the indeterminates T_0, T_1, $\cdots$, T_{d+1} over the field $\mathbb{R}$ of real numbers. Allowing t_0 to vary, let I_0 be the ideal in R_0 defined by

$$I_0 = \{f \in R_0 : f\,(t_0, t_1, \ldots, t_{d+1}) = 0 \;\forall\; \mathbf{t} \in \mathbb{R}^d\}, \tag{17}$$

and let

$$F_0 = (d+1)\left(T_0^4 + \sum_{j=1}^{d+1} T_j^4\right) - \left(T_0^2 + \sum_{j=1}^{d+1} T_j^2\right)^2. \tag{18}$$

Is I_0 generated by F_0?

Question 6.2. Suppose that $S = [\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}]$ is a regular d-simplex of side length a in $\mathbb{R}^d$, and suppose that the positive numbers $t_1, \ldots, t_{d+1}$ satisfy (1). Does there exist a point $\mathbf{t} \in \mathbb{R}^d$ such that the distance from $\mathbf{t}$ to $\mathbf{v}_j$, $1 \leq j \leq d+1$, is t_j?

Notice the relation to a result of Klamkin in [10], which states that if $t_1, \ldots, t_n$ are positive and if

$$(t_1^2 + \cdots + t_n^2)^2 > (n-1)(t_1^4 + \cdots + t_n^4),$$

then there exists a regular $(n-1)$-simplex $[A_1, \ldots, A_n]$ (of edge length a) and a point P in its affine hull such that $|PA_j| = a_j$.

Question 6.3. Let $S = [\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}]$ be a regular d-simplex of side length t_0 in $\mathbb{R}^d$, and let Γ be the circumsphere of S. For every $\mathbf{t} \in \Gamma$, let t_j be the distance from $\mathbf{t}$ to $\mathbf{v}_j$, $1 \leq j \leq d+1$. Let $R = \mathbb{R}[T_1, \ldots, T_{d+1}]$ be the ring of polynomials in the indeterminates $T_1, \cdots, T_{d+1}$ over the field $\mathbb{R}$ of real numbers, and let J be the ideal in R defined by

$$J = \{f \in R : f(t_1, \ldots, t_{d+1}) = 0 \ \forall \ \mathbf{t} \in \Gamma\}. \tag{19}$$

Clearly J contains the polynomial F defined by

$$F = (d+1)\left(a^4 + \sum_{j=1}^{d+1} T_j^4\right) - \left(a^2 + \sum_{j=1}^{d+1} T_j^2\right)^2.$$

It is also known, and not difficult to prove, that J contains the polynomials G and H defined by

$$G = \left(\sum_{j=1}^{d+1} T_j^2\right) - da^2, \ H \ = \ \left(\sum_{j=1}^{d+1} T_j^4\right) - da^4.$$

Thus one may ask whether J is generated by F, G, and H. However, it is easy to check that

$$(d+1)H = F + ((d+1)a^2 + G)^2 - (d+1)^2 a^4, \tag{20}$$

and hence F is generated by G and H, and H is generated by F and G. Thus we ask

Is J generated by G and H? Is J generated by F and G?

Question 6.4. Is the Soddy relation (4) essentially the only one? Given positive numbers that satisfy (4), do there exist spheres having these numbers as radii and mutually touching each other?

Question 6.5. If, instead of taking a regular n-simplex, one takes a general n-simplex with given edge lengths, then the relation among the distances of an arbitrary point in its affine hull is expected to be complicated. In fact this problem is addressed for a triangle in [2], and the relation is found to be quite unmanageable. However, one may try to consider a tetrahedron which is not quite general. For example, a reasonable analog of the triangle in 3-space is, besides the regular tetrahedron, the tetrahedron having congruent faces. These tetrahedra are called *equifacial*, and have attracted much attention.

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