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Results in Mathematics

New Fixed Point Tools in Non-metrizable Spaces

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Abstract. The aim of this paper is to provide some sufficient conditions under which a self-mapping T defined on a non-empty set X endowed with some convergence property is a Picard operator. A relevant example showing that such a mapping T on a non-metrizable space is a Picard operator is given. Our results can be used to obtain some known fixed point theorems on generalized metric spaces.

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1. Introduction and Preliminaries

Metric fixed point theory, as it is well-known, plays an incontestable role in many branches of mathematics and in many other sciences. Also, more and more interesting and quaint applications are covered with the help of fixed point tools. The inspiration taken from the starting point of this theory, i.e. the famous Banach contraction principle, led to the great development in many directions, including the research on very general conditions on the mappings and on the spaces where they are defined on. The examples of recent investigations in this context one can find e.g. in [26–28], where the authors considered the contractive mappings of various type in uniform spaces equipped with the so called generalized pseudodistances. In turn, in [3] there was taken into consideration the well-known contraction of Suzuki type [21] and the appropriate conditions were delivered which imply that certain Banach spaces have fixed point property. Obviously there are many other significant contributions to

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the fixed point theory. A thorough research on the equivalence of some known nonlinear contractive conditions one can find e.g. in [7,8].

In the present work, for a nonempty set X, we will introduce the concept of certain mapping defined on the subset of $X \times X$ which will successfully substitute many known distance function studied so far. Next, we will define a new type of contractive mapping and prove some fixed point results which cover many known theorems in the literature. The novelty of the proposed project is guaranteed by the nontrivial example concerning non-metrizable topological space.

If X is a non-empty set on which a convergence property is defined, we say that a mapping $T : X \to X$ is a *Picard Operator* (abbreviated P.O.) if it has a unique fixed point ξ and $\xi = \lim_n T^n x$, for all $x \in X$, where T^n denotes the n-th composition of T.

1.1. *F*-contractions

In 2012, D. Wardowski in his work [24] proposed a new kind of contractive self-mapping T on a metric space (X, d), namely F-contraction. He considered the functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F1) F is increasing,
(F2) $F(t_n) \to -\infty$ if and only if $t_n \searrow 0$,
(F3) $\lim_{t\to 0} t^\lambda F(t) = 0$ for some $\lambda \in (0, 1)$.

A self mapping T on a metric space (X, d) is said to be an *F-contraction* if there exists $\tau > 0$ such that

$$F\big(\mathrm{d}(Tx, Ty)\big) + \tau \le F(\mathrm{d}(x, y)) \tag{1}$$

for all $x, y \in X$ with $Tx \ne Ty$.

Theorem 1.1. [24, Th. 2.1] *If (X, d) is a complete metric space and $T : X \to X$ an F-contraction, then T is a P.O.*

Putting concrete forms of F, Wardowski obtained other known types of contractions, including Banach contraction principle, and proved that F-contractions are really their generalizations. In the literature one can find many papers devoted to F-contractions. The articles [14,16,20,25] are some of them. In [23] M. Turinici showed that some class of F-contractions are contractions of Matkowski type [13]. In the recent article [18] Secelean and Wardowski extended the family of F-contractions by introducing so called ψF-contractions which include even the Picard operators without nonexpansiveness condition.

1.2. Some Generalized Metric Spaces

In 2000, Branciari [4] introduced the concept of *generalized metric space* or *rectangular metric space* (abbreviated RMS), where the sum on the right hand side of the triangular inequality in the definition of a standard metric is replaced by a three-term expression. This concept is very interesting because a RMS does not necessarily have a compatible topology.

Definition 1.1. Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a mapping such that, for all $x, y \in X$ and all distinct points $u, v \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0 \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral or rectangular inequality).

Then (X, d) is called *generalized* or *rectangular metric space* (RMS).

Definition 1.2. Let (X, d) be a RMS, (x_n) be a sequence in X and $x \in X$. We say that (x_n) *d-converges* to x, or simply *converges*, if $d(x_n, x) \to 0$. We denote this by $x_n \to x$.

We also say that (x_n) is a Cauchy sequence if $d(x_n, x_m) \underset{n,m}{\longrightarrow} 0$.

(X, d) is called *complete* if every Cauchy sequence converges to some $x \in X$.

Remark 1.1. 1. Clearly any metric space is a RMS.

2. A RMS (X, d) may not have a topology compatible with d, *i.e.* a topology τ on X such that a sequence $(x_n) \subset X$ τ-converges to some $x \in X$ if and only if it d-converges to x, as it follows from [22, Ex. 7].

In a complete RMS every Cauchy sequence of distinct points has a unique limit. More precisely, one has the following result established in [10, Lemma 1.10].

Lemma 1.1. *Let (X, d) be a RMS, $(x_n) \subset X$ be a Cauchy sequence. Assume that there is a positive integer N such that:*

(i) *$x_n \neq x_m$, for all $n, m > N$, $n \neq m$;*
(ii) *x_n and x are distinct points in X, for all $n > N$;*
(iii) *x_n and y are distinct points in X, for all $n > N$;*
(iv) *$\lim_n d(x_n, x) = \lim_n d(x_n, y) = 0$. Then $x = y$.*

Various fixed point results were established in such spaces. We describe such a result given by Jleli and Samet in [11].

Let Θ be the family of functions $\theta : (0, \infty) \to (1, \infty)$ that satisfy the following conditions:

(Θ_1) θ is non-decreasing;
(Θ_2) for each sequence $(t_n) \subset (0, \infty)$, $\lim_n \theta(t_n) = 1$ if and only if $t_n \to 0^+$;
(Θ_3) there exist $r \in (0, 1)$ and $l > 0$ such that $\lim_{t \to 0^+} \frac{\theta(t)-1}{t^r} = l$.

Theorem 1.2. [11, Th. 2.1] *Let (X, d) be a complete RMS and $T : X \to X$ be a given map. Suppose that there exist $\theta \in \Theta$ and $\lambda \in (0, 1)$ such that*

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \quad \Rightarrow \quad \theta\big(d(Tx, Ty)\big) \leq \big[\theta\big(d(x, y)\big)\big]^{\lambda}. \tag{2}$$

Then T has a unique fixed point.

Note that the Banach contraction principle follows from the previous theorem taking $\theta(t) = e^t$. On the other hand note that in metric spaces Theorem 1.2 can be derived from Theorem 1.1 by putting $F(t) = \ln(\ln(\theta(t)))$ and $\tau = -\ln \lambda$.

In the following we describe another generalization of the standard notion of metric given by Jleli and Samet in [9] which extends some other generalized metric structures such as: b-metric spaces introduced by Bakhtin [2] and dislocated metric defined by Hitzler and Seda in [6]. In these spaces several fixed point results are improved.

Let X be a nonempty set.

Definition 1.3. A function $\mathcal{D} : X \times X \to [0, \infty]$ is said to be a generalized metric if it satisfies the following conditions:

($\mathcal{D}_1$) for every $x, y \in X$, $\mathcal{D}(x, y) = 0 \;\Rightarrow\; x = y$;
($\mathcal{D}_2$) $\mathcal{D}(x, y) = \mathcal{D}(y, x)$, for every $x, y \in X$;
($\mathcal{D}_3$) there exists $C > 0$ such that

$$x, y \in X,\; (x_n) \subset X,\; \lim_n \mathcal{D}(x_n, x) = 0 \;\Rightarrow\; \mathcal{D}(x, y) \le C \limsup_n \mathcal{D}(x_n, y).$$

For no confusion we will call this function a $\mathcal{D}$-metric.

Definition 1.4. Let $(X, \mathcal{D})$ be a $\mathcal{D}$-metric space and $(x_n) \subset X$. We say that (x_n) *converges* to some $x \in X$ if $\mathcal{D}(x_n, x) \to 0$. Also, (x_n) is a *Cauchy sequence* if $\mathcal{D}(x_{n+p}, x_n) \underset{n,p}{\longrightarrow} 0$.

$(X, \mathcal{D})$ is *complete* if every Cauchy sequence is convergent.

In a $\mathcal{D}$-metric space every convergent sequence has a unique limit [9, Prop. 2.4].

Proposition 1.1. [9, Prop. 2.8, 2.10] *Every b-metric and every dislocated metric is a $\mathcal{D}$-metric.*

Theorem 1.3. [9, Th. 3.3] *Let $(X, \mathcal{D})$ be a complete $\mathcal{D}$-metric space and $T : X \to X$ be a mapping for which there is $k \in (0, 1)$ such that*

$$\mathcal{D}(Tx, Ty) \le k\mathcal{D}(x, y),\; \forall\, x, y \in X.$$

If there is $x_0 \in X$ such that $\sup_{i,j\in\mathbb{N}} \mathcal{D}(T^i x_0, T^j x_0) < \infty$, then $(T^n x_0)$ converges to a fixed point ω of T. Moreover, if ω' is another fixed point of T such that $\mathcal{D}(\omega, \omega') < \infty$, then $\omega = \omega'$.

Senapati et al. [19] generalized the notion of F-contraction to a $\mathcal{D}$-metric space and proved a fixed point theorem.

Definition 1.5. A self-mapping T on a $\mathcal{D}$-metric space is said to be an F-*contraction* if there exists $\tau > 0$ such that, for all $x, y \in X$,

$$\mathcal{D}(x, y) \in (0, \infty) \text{ and } \mathcal{D}(Tx, Ty) \in (0, \infty) \;\Rightarrow\; \tau + F\big(\mathcal{D}(Tx, Ty)\big) \le F\big(\mathcal{D}(x, y)\big),$$

where $F : (0, \infty) \to \mathbb{R}$ satisfy (F1) and (F2).

Theorem 1.4. [19, Th. 4.2] *Let $(X,\mathcal{D})$ be a $\mathcal{D}$-complete metric space and $T : X \to X$ an F-contraction. If there is $x_0 \in X$ such that $\sup_{i,j\in\mathbb{N}} \mathcal{D}(T^i x_0, T^j x_0) < \infty$, then $(T^n x_0)$ converges to a fixed point ω of T. Moreover, if ω' is another fixed point of T such that $\mathcal{D}(\omega,\omega') < \infty$, then $\omega = \omega'$.*

In [1] and [12] one can find more detailed and recent information about fixed points in generalized spaces.

2. Main Results

Let X be any non-empty set and $\Delta = \big\{(x,x);\ x \in X\big\}$.

We will denote by $\mathcal{F}$ the class of all functions $F : (0,\infty) \to \mathbb{R}$ that satisfy $(F2)$.

Definition 2.1. Consider a function $\rho : X \times X \setminus \Delta \to \mathbb{R}$. We say that a sequence $(x_n) \subset X$ *forward ρ-converges* (respectively *backward ρ-converges*) to some $x \in X$ whenever, for each $M > 0$, there exists $N_M \in \mathbb{N}$ such that, for every $n \geq N_M$, one has

$$x_n \neq x \ \Rightarrow\ \rho(x_n,x) < -M \text{ (resp. } x_n \neq x \ \Rightarrow\ \rho(x,x_n) < -M). \tag{3}$$

If each forward ρ-convergent sequence has a unique limit, then ρ is called a *ρ-metric* and the pair (X,ρ) is said to be a *ρ-space.*

Throughout the article we work only with the notion of forward ρ-convergence, therefore for simplicity we will omit "forward".

Definition 2.2. If (X,ρ) is a ρ-space and τ is a topology on X such that a sequence (x_n) converges to $x \in X$ in the topology τ if and only if it ρ-converges to x, we say that (X,τ,ρ) is a *topological ρ-space.*

Remark 2.1. 1. If (X,ρ) is a ρ-space and $(x_n) \subset X$ is stationary (*i.e.* there are $x \in X$ and $N \in \mathbb{N}$ such that $x_n = x$, for all $n \geq N$), then it ρ-converges to x. Also, if (x_n) is such that $x_n \neq x$ for all n greater that some $N \in \mathbb{N}$, then

$$x_n \to x \ \Leftrightarrow\ \rho(x_n,x) \to -\infty.$$

2 . In the above settings, if the set $A = \{n_k :\ x_{n_k} \neq x,\ k = 1,2,\ldots\}$ is infinite and $\rho(x_{n_k},x) \xrightarrow[k]{} -\infty$, then, by Definition 2.1, $x_n \to x$.

Example 2.1. If (X,d) is a metric space, then one can observe that $\rho(x,y) = -1/\mathrm{d}(x,y)$ is a ρ-metric. More generally, if $F \in \mathcal{F}$, then $\rho(x,y) := F\big(\mathrm{d}(x,y)\big)$ is a ρ-metric and X is a topological ρ-space.

If (X,d) is a RMS in which all convergent sequences have a unique limit and $F \in \mathcal{F}$, then $\rho(x,y) = F\big(d(x,y)\big)$ is a ρ-metric which may not be topological (see Remark 1.1, 2.)

Also, if $(X,\mathcal{D})$ is a $\mathcal{D}$-metric space and $F \in \mathcal{F}$, then, taking $\rho(x,y) = F\big(\mathcal{D}(x,y)\big)$, we obtain a ρ-space.

When the underlying topological space (X, τ) is non-metrizable, ρ-metric finds its real application and can be used to measure the "distance" between the elements in the topological ρ-space (X, τ, ρ). Such a situation is illustrated by the following example.

Example 2.2. Consider the Sorgenfrey line (lower limit topology), *i.e.* the topology τ_l on the set $\mathbb{R}$ generated by the basis of all half-open intervals

$$\mathcal{B} = \{[a,b)\colon a, b \in \mathbb{R}, a < b\}.$$

It is known that $(\mathbb{R}, \tau_l)$ is a Hausdorff topological space and it is not metrizable (see e.g. [5]). It is easy to observe that, in this topology, a given sequence converges whenever it converges in the standard topology and at most a finite number of elements are less than the limit.

Consider now the mapping $\rho\colon \mathbb{R} \times \mathbb{R} \setminus \Delta \to \mathbb{R}$ given by

$$\rho(x, y) = \frac{1}{y - x}, \ \forall\, x \neq y \in \mathbb{R}.$$

Taking any sequence (x_n) convergent to x, $x_n \neq x$, with respect to the topology τ_l, we have $x_n \searrow x$ in the standard topology, and due to the fact that $x_n > x$ for almost all $n \in \mathbb{N}$, we get $\rho(x_n, x) \to -\infty$. On the other side, taking any $(x_n) \subset \mathbb{R}$, $x \in \mathbb{R}$, $x_n \neq x$, such that $\rho(x_n, x) \to -\infty$ we must have $x_n > x$ except for finitely many n and $\frac{1}{x - x_n} \to -\infty$, so $x_n \searrow x$ in the standard topology and hence $x_n \underset{\tau_l}{\to} x$.

Summarizing, $(\mathbb{R}, \tau_l, \rho)$ is a topological ρ-space. The analogous conclusion can be also obtained for the mapping

$$\rho(x, y) = \begin{cases} c, & \text{for } y > x; \\ \frac{1}{y-x}, & \text{for } y < x, \end{cases}$$

where $c \in \mathbb{R}$. □

Definition 2.3. Let (X, ρ) be a ρ-space. A sequence $(x_n) \subset X$ is said to be *ρ-backward-Cauchy* (respectively *ρ-forward-Cauchy*) whenever, for every $M > 0$, there is $N \in \mathbb{N}$ such that, for all $n \geq N$ and $p \geq 1$, one has

$$x_{n+p} \neq x_n \ \Rightarrow \ \rho(x_{n+p}, x_n) < -M \ \ (\text{resp. } \rho(x_n, x_{n+p}) < -M).$$

A sequence $(x_n) \subset X$ is said to be *ρ-Cauchy* if for every $M > 0$, there is $N \in \mathbb{N}$ such that, for all $m, n \geq N$ one has

$$x_m \neq x_n \ \Rightarrow \ \rho(x_m, x_n) < -M.$$

We say that ρ is *backward complete* (resp. *forward complete*, *complete*) if every ρ-backward-Cauchy (resp. ρ-forward-Cauchy, ρ-Cauchy) sequence converges.

Remark 2.2. 1. If ρ is symmetric, then ρ-backward-Cauchy and ρ-forward-Cauchy properties coincide.

2. A ρ-space is backward complete (respectively forward complete, complete) if and only if every sequence of distinct elements $(x_n) \subset X$ such that
$$\rho(x_{n+p}, x_n) \underset{n,p}{\longrightarrow} -\infty, \text{ (resp. } \rho(x_n, x_{n+p}) \underset{n,p}{\longrightarrow} -\infty, \ \rho(x_m, x_n) \underset{m\neq n}{\longrightarrow} -\infty)$$
is ρ-convergent.
3. Let (x_n) be a sequence of different elements in a ρ-space. Then
$$\rho(x_n, x_{n+p}) \underset{n,p}{\longrightarrow} -\infty \text{ and } \rho(x_{n+p}, x_n) \underset{n,p}{\longrightarrow} -\infty \tag{4}$$
if and only if $\rho(x_m, x_n) \underset{m\neq n}{\longrightarrow} -\infty$. Therefore, (x_n) is ρ-Cauchy if and only if it is simultaneously ρ-backward-Cauchy and ρ-forward-Cauchy.

Proof. 1. and 2. are obvious. 3. If (x_n) is ρ-Cauchy, one has clearly (4). Conversely, the properties from (4) imply
$$\forall\, M > 0, \ \exists \ N \in \mathbb{N} \text{ such that } \rho(x_{n+p}, x_n) < -M \text{ and } \rho(x_n, x_{n+p}) < -M, \ \forall\, n \geq N, \ p \geq 1.$$
Choose $m, n \geq N$, $m \neq n$. If $m > n$, then $\rho(x_m, x_n) = \rho(x_{n+p}, x_n) < -M$, where $p = m - n$. Analogously, if $m < n$ one obtains $\rho(x_m, x_n) < -M$.

Therefore (x_n) is ρ-Cauchy. □

Example 2.3. Any of the ρ-metrics defined in Example 2.2 is forward complete, while it is not backward complete. Indeed, if $(x_n) \subset \mathbb{R}$ is a sequence of distinct elements such that $\rho(x_n, x_{n+p}) \underset{n,p}{\longrightarrow} -\infty$, then there is $n_0 \in \mathbb{N}$ such that $(x_n)_{n\geq n_0}$ is decreasing and $|x_n - x_{n+p}| \underset{n,p}{\longrightarrow} 0$. Hence (x_n) is Cauchy with respect to the Euclidean metric, so it is convergent. Therefore (x_n) converges in the Sorgenfrey topology.

Next, if $\rho(x_{n+p}, x_n) \underset{n,p}{\longrightarrow} -\infty$, then $(x_n)_{n\geq n_0}$ is increasing for some $n_0 \in \mathbb{N}$, so it does not converge in the Sorgenfrey line.

Remark 2.3. Let us consider a metric space (X, d), a function $F \in \mathcal{F}$ and a ρ-metric $\rho = F \circ \mathrm{d}$ (see Example 2.1). Then d is complete if and only if ρ is complete. The same assertion holds if we consider an RMS or a $\mathcal{D}$-metric space instead of a metric space.

Proof. Suppose that d is complete and $(x_n) \subset X$ is ρ-Cauchy. Assume that (x_n) is not d-Cauchy. Then there exist $\varepsilon > 0$ and the subsequences (x_{n_k}) and (x_{m_k}) of (x_n) such that
$$t_k = \mathrm{d}(x_{m_k}, x_{n_k}) > \varepsilon \text{ for all } k \in \mathbb{N}.$$
By (F2) $\rho(x_{m_k}, x_{n_k}) = F(t_k) \not\to -\infty$ which contradicts the fact that (x_n) is ρ-Cauchy (see Remark 2.2). Therefore (x_n) is d-Cauchy and hence convergent, i.e. $\mathrm{d}(x_n, x) \to 0$ for some $x \in X$. Now, using again (F2) one can easily see that (x_n) is ρ-convergent.

The proof of the second assertion is very similar. □

Remark 2.4. A ρ-convergent sequence need not be ρ-Cauchy as we can see in the following example. Set $X = [0, \infty)$ and let $\rho : X \times X \setminus \Delta \to \mathbb{R}$ be given by

$$\rho(x, y) = \begin{cases} \ln(x + y), & \text{if } x = 0 \text{ or } y = 0; \\ \ln(x + y + 1), & \text{otherwise.} \end{cases}$$

Then ρ is a ρ-metric. If $x_n = \frac{1}{n}$, $n = 1, 2, \ldots$, then $\rho(x_n, 0) = -\ln n \to -\infty$ so (x_n) is ρ-convergent. However, $\rho(x_{n+p}, x_n) = \rho(x_n, x_{n+p}) = \ln\left(\frac{1}{n+p} + \frac{1}{n} + 1\right) \underset{n,p}{\longrightarrow} 0$ hence (x_n) is neither ρ-backward-Cauchy nor ρ-forward-Cauchy.

A nonempty subset $B \subset X$ is said to be *ρ-bounded* whenever there exists $M > 0$ such that $\rho(u, v) \le M$, for all $u, v \in B$, $u \neq v$.

Let $\psi : (-\infty, \mu) \to (-\infty, \mu)$, where $\mu > M := \sup_{x \neq y \in X} \rho(x, y)$.

Definition 2.4. A mapping $T : X \to X$ is called $\rho\psi$-contraction if

$$\rho(Tx, Ty) \le \psi\big(\rho(x, y)\big), \ \forall\, x, y \in X, \ x \neq y, \ Tx \neq Ty. \tag{5}$$

For a mapping $T \colon X \to X$ and $x_0 \in X$ the *orbit of T starting at the point* x_0, denoted by $O(T, x_0)$, is the set

$$O(T, x_0) = \{x_0, Tx_0, T^2x_0, \ldots\}.$$

Theorem 2.1. *Let (X, ρ) be a ρ-space and assume that ρ is backward complete or forward complete and that ψ is a nondecreasing map with $\psi^n(t) \to -\infty$, for all $t \in (-\infty, \mu)$. If T is a $\rho\psi$-contraction with a ρ-bounded orbit $O(T, x_0)$ for some $x_0 \in X$, then it is a P.O.*

Proof. First note that $\psi(t) < t$, for all $t \in (-\infty, \mu)$. Indeed, if there is $t_0 \in (-\infty, \mu)$ such that $\psi(t_0) \ge t_0$, then $\psi^2(t_0) \ge \psi(t_0) \ge t_0$ and, inductively, $\psi^n(t_0) \ge t_0$, for all $n \in \mathbb{N}$. This contradicts $\psi^n(t_0) \to -\infty$.

Since

$$\rho(Tx, Ty) \le \psi\big(\rho(x, y)\big) < \rho(x, y), \ \forall\, x, y \in X, \ x \neq y, \ Tx \neq Ty, \tag{6}$$

it follows that T has at most one fixed point.

In order to establish the existence of fixed point of T and also its successive approximation, we have to investigate two cases.

Case I. If there exist $n, p \ge 1$ such that $T^{n+p}x_0 = T^p x_0$, then $T^p x_0$ is a fixed point for T^n. Next $T^{n+p+1}x_0 = T^{p+1}x_0$, hence $T^{p+1}x_0$ is, also, a fixed point of T^n. From the inequalities

$$\begin{aligned} \rho(T^n x, T^n y) &\le \psi\big(\rho(T^{n-1}x, T^{n-1}y)\big) \\ &< \rho(T^{n-1}x, T^{n-1}y) < \cdots < \rho(x, y), \ \forall\, x \neq y \in X, \ T^n x \neq T^n y, \end{aligned}$$

we deduce that T^n has only one fixed point. Therefore $T^{p+1}x_0 = T^p x_0$, so $\xi = T^p x_0$ is a fixed point of T.

Case II. Assume that, for every $n, p \geq 1$, one has $T^{n+p}x_0 \neq T^n x_0$. Then, according to (6), for every $n, p \geq 1$ we have

$$\begin{aligned}\rho(T^{n+p}x_0, T^n x_0) &\leq \psi\big(\rho(T^{n+p-1}x_0, T^{n-1}x_0)\big) \leq \cdots \leq \psi^n\big(\rho(T^p x_0, x_0)\big)\\ &\leq \psi^n(M) \xrightarrow[n]{} -\infty,\end{aligned}$$

where $M = \sup_{x\neq y} \rho(x, y)$, which means that $(T^n x_0)$ is ρ-backward-Cauchy.

Analogously, we obtain

$$\rho(T^n x_0, T^{n+p}x_0) \leq \cdots \leq \psi^n\big(\rho(x_0, T^p x_0)\big) \leq \psi^n\big(M\big) \xrightarrow[n]{} -\infty$$

and so $(T^n x_0)$ is ρ-forward-Cauchy.

By hypothesis, there exists $\xi \in X$ such that $T^n x_0 \xrightarrow[n]{} \xi$.

Set $A = \{n \in \mathbb{N};\ T^{n+1}x_0 \neq T\xi\}$. If A is finite, then $T^{n+1}x_0 \to T\xi$. Assume that A is infinite. Then $A = (n_k)_{k\in\mathbb{N}}$ and $T^{n_k}x_0 \neq \xi$, for all $k \geq 1$. Hence $\rho(T^{n_k}x_0, \xi) \xrightarrow[k]{} -\infty$.

Using, again (6), we get

$$\rho(T^{n_k+1}x_0, T\xi) < \rho(T^{n_k}x_0, \xi),\ \forall\, k \in N,$$

that is $T^{n_k+1}x_0 \xrightarrow[k]{} T\xi$.

Therefore, by Remark 2.1, $T^{n+1}x_0 \xrightarrow[n]{} T\xi$.

Since all convergent sequences in X have a unique limit, one obtains $T\xi = \xi$, so ξ is a fixed point of T.

In order to show the successive approximations of ξ, choose $x \in X$. If there is $n_0 \in \mathbb{N}$ such that $T^{n_0}x = \xi$, then the conclusion is obvious. Suppose that $T^n x \neq \xi$, for all $n \geq 1$. Then

$$\rho(T^n x, \xi) = \rho(T^n x, T^n \xi) \leq \psi\big(\rho(T^{n-1}x, T^{n-1}\xi)\big) \leq \cdots \leq \psi^n\big(\rho(x, \xi)\big) \xrightarrow[n]{} -\infty,$$

so $T^n x \xrightarrow[n]{} \xi$.

The proof is complete. □

In the following we provide an example of non-metrizable topological space in which the previous theorem can be applied.

Example 2.4. Take any $\lambda \in (0, 1)$ and set

$$X = \bigcup_{n=1}^{\infty} [\lambda^{2n-1}, \lambda^{2n-2}] \cup \{0\}$$

with the topology τ_l induced from the Sorgenfrey line. Let us consider a mapping $T\colon X \to X$ given by

$$Tx = \begin{cases} 0, & \text{for } x \in \bigcup\limits_{n=1}^{\infty} [\lambda^{2n-1}, \lambda^{2n-2}) \cup \{0\};\\ \lambda^{2n}, & \text{for } x = \lambda^{2n-2},\ n \in \mathbb{N} \end{cases}$$

and the function $\psi : (-\infty, \mu) \to (-\infty, \mu)$, $\mu > 0$, $\psi(t) = t + 2\ln\lambda$. Denote $\Lambda = \{\lambda^{2n-2} \colon n \in \mathbb{N}\}$ and define $\rho \colon X \times X \setminus \Delta \to \mathbb{R}$ as follows

$$\rho(x,y) = \begin{cases} \ln|x-y|, \text{ if } \ [x,y \in \Lambda \cup \{0\}] \text{ or } [x,y \notin \Lambda \text{ and } y < x]; \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

Then

a) (X, τ_l) is a non-metrizable Hausdorff topological space;
b) ρ is a forward complete topological ρ-metric;
c) T is continuous in (X, τ_l) while it is discontinuous with respect to the standard topology τ_{d} generated by the Euclidean metric d on $\mathbb{R}$;
d) T is a $\rho\psi$-contraction and P.O.

Proof. *a*) The Sorgenfrey line is Hausdorff, and so is its subspace (X, τ_l).

We will prove the non-metrizability of X using a direct way. Suppose that (X, τ_l) is metrizable, *i.e.* $\tau_l = \tau_\delta$, where τ_δ denotes the topology induced by some metric δ. Fix $x \in [\lambda, 1) \subset X$. Since $[x, 1) \in \tau_l = \tau_\delta$, there exists $n \in \mathbb{N}$ such that $B(x, \frac{1}{n}) \subset [x, 1)$. $B(x, \frac{1}{n}) \in \tau_l$, therefore there is also $m \in \mathbb{N}$ such that $[x, x + \frac{1}{m}) \subset B(x, \frac{1}{n})$. Summarizing, for each $a \in [\lambda, 1)$ one can find $s, t > 0$ such that $a \in V_{m,n}$, where

$$V_{m,n} = \Big\{x \in \big[\lambda, 1\big) \colon \Big[x, x + \frac{1}{m}\Big) \subset B\Big(x, \frac{1}{n}\Big) \subset [x, 1)\Big\}.$$

In consequence, we get $[\lambda, 1) \subset \bigcup_{m,n \in \mathbb{N}} V_{m,n}$. The set $[\lambda, 1)$ is uncountable, therefore there must exist $m_0, n_0 \in \mathbb{N}$ such that V_{m_0,n_0} is uncountable. Let (x_k) be any sequence of elements in $[\lambda, 1)$ such that $\lambda < x_1 < x_2 < \ldots < 1$, $x_k \to 1$, $x_1 - \lambda < \frac{1}{m_0}$ and $x_{k+1} - x_k < \frac{1}{m_0}$ for all $k \in \mathbb{N}$. In one of intervals $[\lambda, x_1]$ or $[x_k, x_{k+1}]$, $k \in \mathbb{N}$, there are uncountable many elements of V_{m_0,n_0}. Thus we can choose $u, v \in V_{m_0,n_0}$ such that $u > v$ and $u - v < \frac{1}{m_0}$. Hence, we have

$$u \in \Big[v, v + \frac{1}{m_0}\Big) \subset B\Big(v, \frac{1}{n_0}\Big),$$

and, in consequence

$$v \in B\Big(u, \frac{1}{n_0}\Big) \subset [u, 1).$$

From the above we obtain $v \geq u$ contradicting the choice of u, v. (X, τ_l) is therefore non-metrizable.

b) In order to show that ρ is a ρ-metric, consider a sequence of different elements $(x_n) \subset X$ which converges to $x \in X$ with respect to τ_l. Then we can assume that (x_n) is decreasing and $|x_n - x| \to 0$, that is $x_n \to x$ with respect to the Euclidean metric. Clearly, $x \notin \Lambda$. We claim that $\rho(x_n, x) \to -\infty$. Indeed, if there is $N \in \mathbb{N}$ such that $x_n \notin \Lambda$ for every $n \geq N$, then $\rho(x_n, x) = \ln|x_n - x|$. On the contrary, one can find a subsequence $(x_{n_k})_k \subset \Lambda$. In this case $x = 0$ and $\rho(x_n, 0) = \ln x_n$, for all $n \geq 1$. In both cases $\rho(x_n, x) \to -\infty$.

Conversely, let suppose that $(x_n) \subset X$ has different elements and satisfies $\rho(x_n, x) \to -\infty$, where $x \in X$. It follows that $x_n > x \geq 0$ except for a finite set of $n \in \mathbb{N}$. We will prove that $\rho(x_n, x) = \ln|x_n - x|$ hence $|x_n - x| \to 0$, that is $x_n \to x$ with respect to the topology τ_l.

If there exists $N \in \mathbb{N}$ such that $x_n, x \notin \Lambda$ for all $n \geq N$, then $\rho(x_n, x) = \ln|x_n - x|$. Assume that there is a subsequence $(x_{n_k}) \subset \Lambda$ of (x_n). Then $x_{n_k} \xrightarrow[k]{} 0$, so $x = 0$ and $\rho(x_n, x) = \ln x_n$.

It remains to prove the forward completeness of ρ. For this purpose, let $(x_n) \subset X$ be a sequence of different numbers such that $\rho(x_n, x_{n+p}) \xrightarrow[n,p]{} -\infty$. Then one can find $N \in \mathbb{N}$ such that $\rho(x_n, x_{n+p}) = \ln|x_n - x_{n+p}|$, for all $n \geq N$, $p \geq 1$. Moreover $(x_n)_{n \geq N}$ is decreasing because, if this is not so, one can find a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} < x_{n_{k+1}}$, for every $k \geq 1$. Hence $\rho(x_{n_k}, x_{n_{k+1}}) = 0$, for all $k \in \mathbb{N}$, which is a contradiction.

Therefore $|x_n - x_{n+p}| \xrightarrow[n,p]{} 0$, hence (x_n) is a Cauchy sequence in $(\mathbb{R}, \mathrm{d})$ so it converges to some $x \in \mathbb{R}$. Since $\complement X \in \tau_{\mathrm{d}}$, we deduce that X is closed in $(\mathbb{R}, \tau_{\mathrm{d}})$ so $x \in X$. Next, $(x_n)_{n \geq N}$ being decreasing, it follows that $x_n \to x$ with respect to τ_l.

Consequently ρ is forward complete.

c) Obviously, for every $n \in \mathbb{N}$, T is discontinuous in λ^{2n-2} with respect to the topology τ_{d}. However it is continuous with respect to τ_l. Indeed, for every $n \in \mathbb{N}$, it is enough to consider the open neighborhood U_n of λ^{2n-2} of the form

$$U_n = \{\lambda^{2n-2}\}.$$

Then, taking any $V_n \in \tau_l$ such that $\lambda^{2n} = T(\lambda^{2n-2}) \in V_n$ we have $T(U_n) \subset V_n$.

The continuity of T with respect to τ_l on the set $X \setminus \Lambda$ is easy to be verified.

d) In order to show that T is a $\rho\psi$-contraction consider first $x = \lambda^{2m-2}$, $y = \lambda^{2n-2}$, $m, n \in \mathbb{N}$, $x \neq y$. We have

$$\begin{aligned}\rho(x,y) - \rho(Tx,Ty) &= \rho(\lambda^{2m-2}, \lambda^{2n-2}) - \rho(\lambda^{2m}, \lambda^{2n}) \\ &= \ln \frac{|\lambda^{2m-2} - \lambda^{2n-2}|}{|\lambda^{2m} - \lambda^{2n}|} = \ln \frac{\lambda^{2m-2}|1 - \lambda^{2n-2m}|}{\lambda^{2m}|1 - \lambda^{2n-2m}|} = -2\ln\lambda.\end{aligned}$$

For $x = 0$ and $y = \lambda^{2n-2}$ we get

$$\rho(x,y) - \rho(Tx,Ty) = \rho(0, \lambda^{2n-2}) - \rho(0, \lambda^{2n}) = \ln \frac{\lambda^{2n-2}}{\lambda^{2n}} = -2\ln\lambda.$$

Next, taking $x \notin \Lambda$, $x \neq 0$, and $y = \lambda^{2n-2}$ for some $n \geq 1$, we obtain

$$\rho(x,y) - \rho(Tx,Ty) = \rho(x, \lambda^{2n-2}) - \rho(0, \lambda^{2n}) = -\ln\lambda^{2n} \geq -2\ln\lambda.$$

In the other cases in which $x, y \notin \Lambda$, one has $Tx = Ty = 0$ and (5) implicitly holds.

Finally observe that every orbit of T is ρ-bounded.

Consequently, all the assumptions of Theorem 2.1 are satisfied, so T is a P.O. □

Remark 2.5. In the space X endowed with the standard metric d the operator T from the previous example is neither nonexpansive nor expansive. This emphasizes that Theorem 2.1 offers a new method to establish that a self-mapping is a P.O.

Proof. Take, for some $n \in \mathbb{N}$, $x_1 = 0$, $\max\left\{\lambda^{2n-1}, \lambda^{2n-2}(1-\lambda^2)\right\} < x_2 < \lambda^{2n-2}$, $y = \lambda^{2n-2}$. The conclusion follows from the following inequalities

$$\mathrm{d}(Tx_1, Ty) < \mathrm{d}(x_1, y) \Leftrightarrow \lambda^{2n} < \lambda^{2n-2} \text{ and}$$
$$\mathrm{d}(Tx_2, Ty) > \mathrm{d}(x_2, y) \Leftrightarrow \lambda^{2n} > \lambda^{2n-2} - x_2 \;\Leftrightarrow\; x_2 > \lambda^{2n-2}(1-\lambda^2).$$

□

Lemma 2.1. *If* $\psi : (-\infty, \mu) \to (-\infty, \mu)$, $\mu \in [0, \infty]$, *is an upper semi-continuous function (or continuous) with* $\psi(t) < t$, *for all* $t < \mu$, *then* $\lim_n \psi^n(t) = -\infty$, *for all* $t < \mu$.

Proof. Fix $t \in (-\infty, \mu)$. Then $\psi^{n+1}(t) < \psi^n(t)$, for every $n \in \mathbb{N}$, hence the sequence $\left(\psi^n(t)\right)$ is decreasing so it has a limit $l \in [-\infty, \mu)$. If $l \in \mathbb{R}$, then $l \le \limsup_{t\to l} \psi(t) \le \psi(l)$ which is a contradiction. So $l = -\infty$. □

Remark 2.6. Theorem 1.2 follows from Theorem 2.1 as a corollary, taking $\rho(x, y) = \frac{1}{1-\theta(d(x,y))}$ for all $x, y \in X$, $x \ne y$, and $\psi : (-\infty, 0) \to (-\infty, 0)$, $\psi(t) = \frac{(-t)^\lambda}{(-t)^\lambda - (1-t)^\lambda}$.

Proof. First note that, following the proof of Theorem 2.1, one can suppose for the function ρ only that every ρ-Cauchy sequence has a unique limit instead of the uniqueness of the limit of all ρ-convergent sequences. Next, since $F(t) = \frac{1}{1-\theta(t)} \in \mathcal{F}$, it follows that ρ satisfies (3). We also note that, d being complete, one has ρ is complete.

A trivial verification shows that ψ is non-decreasing and that $\psi(t) < t$, for all $t \in (-\infty, 0)$. So, by Lemma 2.1, ψ satisfies the conditions from Theorem 2.1.

Now, since $\theta\big(d(x, y)\big) = 1 - \frac{1}{\rho(x,y)}$, one has

$$\theta\big(d(Tx, Ty)\big) \le \left[\theta\big(d(x, y)\big)\right]^\lambda \Leftrightarrow 1 - \frac{1}{\rho(Tx, Ty)} \le \Big(1 - \frac{1}{\rho(x, y)}\Big)^\lambda$$
$$\Leftrightarrow \rho(Tx, Ty) \le \frac{1}{1 - \big(1 - \frac{1}{\rho(x,y)}\big)^\lambda} = \psi\big(\rho(x, y)\big),$$

for all $x, y \in X$ with $Tx \ne Ty$. Finally, note that $\rho < 0$ and hence every orbit of T is ρ-bounded. □

Remark 2.7. Theorem 1.3 and [6, Th. 2.7] are simple consequences of Theorem 2.1 if we take $\rho(x,y) = -1/\mathcal{D}(x,y)$, for all $x,y \in X$, $x \neq y$, and $\psi(t) = \frac{1}{k}t$, for $t \in (-\infty, 0)$.

Remark 2.8. Theorem 1.4 is a particular case of Theorem 2.1 by taking $\rho(x,y) = (F \circ \mathcal{D})(x,y)$, for all $x \neq y \in X$, and $\psi(t) = t - \tau$, $t < 0$.

The following proposition states that the class of $\rho\psi$-contraction mappings on a complete metric space includes that of φ-contractions.

Proposition 2.1. *1. If $\mu = 0$, then a mapping $\psi : (-\infty,\mu) \to (-\infty,\mu)$ is nondecreasing with $\psi^n(t) \to -\infty$ for all $t < \mu$ if and only if $\varphi : (0,\infty) \to (0,\infty)$, $\varphi(t) = -1/\psi(-t^{-1})$, is a comparison function.*

2. Given a metric space (X, d), a mapping $T : X \to X$ is a φ-contraction if and only if it is a $\rho\psi$-contraction, where $\rho(x,y) = -1/\mathrm{d}(x,y)$, for all $x \neq y \in X$, and $\psi(t) = -1/\varphi(-t^{-1})$.

Proof. 1. If ψ is nondecreasing, then clearly φ is nondecreasing too. Next, it is easy to see that $\varphi^n(t) = -1/\psi^n(-t^{-1})$ for every $t > 0$ and $n \geq 1$. So

$$\varphi^n(t) \to 0 \Leftrightarrow \psi^n(-t^{-1}) \to -\infty.$$

Since $\psi(s) = -1/\varphi(-s^{-1})$, for all $s < 0$, the converse implication is obvious.

2. According to Example 2.1, ρ is a ρ-metric. For every $x, y \in X$ with $x \neq y$ and $Tx \neq Ty$, one has

$$\rho(Tx,Ty) \leq \psi\big(\rho(x,y)\big) \Leftrightarrow \frac{-1}{\mathrm{d}(Tx,Ty)} \leq \psi\Big(\frac{-1}{\mathrm{d}(x,y)}\Big) = \frac{-1}{\varphi\big(\mathrm{d}(x,y)\big)}$$
$$\Leftrightarrow \mathrm{d}(Tx,Ty) \leq \varphi\big(\mathrm{d}(x,y)\big).$$

□

For a given ρ-space (X,ρ), in order to obtain the next result, we will need the existence of a function $\Gamma\colon (-\infty,\mu) \to (0,\infty)$, $\mu = \sup_{x,y\in X, x\neq y} \rho(x,y)$, such that:

(Γ1) Γ is increasing;
(Γ2) $(\Gamma\circ\rho)(x,y) \leq (\Gamma\circ\rho)(x,z) + (\Gamma\circ\rho)(z,y)$ for all $x,y,z \in X$, $x \neq y \neq z \neq x$;
(Γ3) $t_n \to -\infty$ implies $\Gamma(t_n) \to 0$.

If X is a metrizable space, then some simple example of functions ρ and Γ can be found in the following example.

Example 2.5. Consider a metric space (X, d) and two functions $\Gamma : (-\infty, 0) \to (0,\infty)$ which satisfies (Γ1), (Γ3) and its inverse $\Gamma^{-1} : \Gamma\big((-\infty,0)\big) \to (-\infty,0)$. If $\rho : X \times X \setminus \Delta \to \mathbb{R}$ is given by $\rho(x,y) = \Gamma^{-1}\big(\mathrm{d}(x,y)\big)$, then ρ is a ρ-metric and (Γ1)-(Γ3) hold. In particular we can consider $\Gamma(t) = -1/t$ and $\rho(x,y) = -1/\mathrm{d}(x,y)$ or $\Gamma(t) = e^t$ and $\rho(x,y) = \ln \mathrm{d}(x,y)$.

Moreover in both cases the mapping Γ is continuous.

Proposition 2.2. *Consider a ρ-space X, a function $\Gamma\colon (-\infty,\mu) \to (0,\infty)$ satisfying $(\Gamma1)-(\Gamma3)$ and $\Omega \subset (-\infty,\mu)$ such that $(-\infty,\mu)\setminus\Omega$ is dense in $(-\infty,\mu)$. For every sequence $(x_n) \subset X$ of different elements, if $\rho(x_n,x_{n+1}) \to -\infty$, $\rho(x_{n+1},x_n) \to -\infty$ and (x_n) is not ρ-Cauchy, then there exist $M \in (-\infty,\mu)\setminus\Omega$ and the sequences (m_k), (n_k) of natural numbers such that*

(a) $\Gamma\big(\rho(x_{m_k},x_{n_k})\big) \searrow \Gamma(M),\ k \to \infty$;

(b) $\Gamma\big(\rho(x_{m_k+1},x_{n_k+1})\big) \underset{k}{\to} \Gamma(M)$.

Proof. Since (x_n) is not ρ-Cauchy and $(-\infty,\mu)\setminus\Omega$ is dense, there exists $M \in (-\infty,\mu)\setminus\Omega$ such that, for each $k \in \mathbb{N}$ one can find $m,n \in \mathbb{N}$, $k \le m < n$ such that $\rho(x_m,x_n) > M$. Denote

$$m_k = \min\big\{m \in \mathbb{N} :\ \exists\, n \in \mathbb{N},\ k \le m < n,\ \rho(x_m,x_n) > M\big\}.$$
$$n_k = \min\big\{n \in \mathbb{N} :\ k \le m_k < n,\ \rho(x_{m_k},x_n) > M\big\}.$$

Let $n_0 \in \mathbb{N}$ be such that $\rho(x_n,x_{n+1}) < M$ and $\rho(x_{n+1},x_n) < M$ for all $n \ge n_0$. By the definition of m_k and n_k, for all $k \ge n_0$, one must have $n_k \ge m_k + 2$ and $\rho(x_{m_k},x_{n_k-1}) \le M$. Therefore, using $(\Gamma2)$ and $(\Gamma3)$, for all $k \ge n_0$, we get

$$\begin{aligned}\Gamma(M) &\le \Gamma\big(\rho(x_{m_k},x_{n_k})\big) \le \Gamma\big(\rho(x_{m_k},x_{n_k-1})\big) + \Gamma\big(\rho(x_{n_k-1},x_{n_k})\big)\\ &\le \Gamma(M) + \Gamma\big(\rho(x_{n_k-1},x_{n_k})\big).\end{aligned}$$

In consequence, since $\Gamma\big(\rho(x_{n_k-1},x_{n_k})\big) > 0$, $\rho(x_{n_k-1},x_{n_k}) \underset{k}{\to} -\infty$ and due to $(\Gamma3)$, we obtain

$$\Gamma\big(\rho(x_{m_k},x_{n_k})\big) \searrow \Gamma(M),\ k \to \infty.$$

Also observe that, for all $k \in \mathbb{N}$, using couple times $(\Gamma2)$, we have the inequalities

$$\begin{aligned}&\Gamma\big(\rho(x_{m_k},x_{n_k})\big) - \Gamma\big(\rho(x_{m_k},x_{m_k+1})\big) - \Gamma\big(\rho(x_{n_k+1},x_{n_k})\big) \le \Gamma\big(\rho(x_{m_k+1},x_{n_k+1})\big)\\ &\quad\le \Gamma\big(\rho(x_{m_k+1},x_{m_k})\big) + \Gamma\big(\rho(x_{m_k},x_{n_k})\big) + \Gamma\big(\rho(x_{n_k},x_{n_k+1})\big).\end{aligned}$$

Letting $k \to \infty$ and applying $(\Gamma3)$ finally we obtain

$$\Gamma(\rho(x_{m_k+1},x_{n_k+1})) \underset{k}{\longrightarrow} \Gamma(M).$$

□

Theorem 2.2. *Let $T\colon X \to X$ be a $\rho\psi$-contraction defined on a complete ρ-space, where $\psi\colon (-\infty,\mu) \to (-\infty,\mu)$, $\mu > \sup_{x,y\in X, x\ne y}\rho(x,y)$, is an upper semicontinuous function satisfying $\psi(t) < t$ for all $t < \mu$. Assume that there exists a map $\Gamma\colon (-\infty,\mu) \to (0,\infty)$ continuous on a dense subset A of $(-\infty,\mu)$ which satisfies $(\Gamma1)$-$(\Gamma3)$. Then T is a P.O.*

Proof. First observe that, for all $x,y \in X$ with $x \ne y$ and $Tx \ne Ty$, we have

$$\rho(Tx,Ty) \le \psi(\rho(x,y)) < \rho(x,y),$$

which proves that T has at most one fixed point.

Consider any $x_0 \in X$ and denote $x_n = T^n x_0$. If $x_{n_0} = x_{n_0-1}$ for some $n_0 \in \mathbb{N}$ then one can see that $T^{n_0-1}x_0$ is a fixed point of T.

Assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$ and denote $\delta_n = \rho(x_n, x_{n+1})$, $n \in \mathbb{N}$. We have

$$\delta_n \leq \psi(\delta_{n-1}) < \delta_{n-1} \text{ for all } n \in \mathbb{N}.$$

Set $\lambda = \lim_n \delta_n$. If $-\infty < \lambda$ then, by the above, we get

$$\lambda = \lim_n \psi(\delta_{n-1}) \leq \limsup_{t\to\lambda} \psi(t) \leq \psi(\lambda),$$

which is a contradiction and consequently $\delta_n \searrow -\infty$.

Now, suppose that (x_n) is not ρ-Cauchy. Taking $\Omega = (-\infty, \mu) \setminus A$ in Proposition 2.2, it follows that there exist $M \in A$ and the sequences (m_k), (n_k) such that

$$\Gamma\big(\rho(x_{m_k}, x_{n_k})\big) \searrow \Gamma(M),\ \Gamma\big(\rho(x_{m_k+1}, x_{n_k+1})\big) \underset{k}{\to} \Gamma(M).$$

The continuity of Γ in M and its monotonicity imply $\rho(x_{m_k}, x_{n_k}) \searrow M$ and $\rho(x_{m_k+1}, x_{n_k+1}) \underset{k}{\to} M$. Hence, we obtain

$$\rho(x_{m_k+1}, x_{n_k+1}) \leq \psi\big(\rho(x_{m_k}, x_{n_k})\big), \text{ for all } k \in \mathbb{N}.$$

Letting $k \to \infty$ and using the upper semicontinuity of ψ, we get

$$M \leq \limsup_{k\to\infty} \psi\big(\rho(x_{m_k}, x_{n_k})\big) \leq \limsup_{t\to M} \psi(t) \leq \psi(M),$$

which is impossible. Therefore (x_n) is ρ-Cauchy and hence convergent. The rest of the proof is analogous as in the proof of Theorem 2.1. □

As a particular case we obtain [15, Th. 2.1]:

Corollary 2.1. *Let T be a self mapping on a complete metric space (X, d). Suppose that $F : (0, \infty) \to \mathbb{R}$ is a continuous function which satisfies* (F1) *and* (F2). *If there exists $\tau > 0$ such that* (1) *holds, then T is a P.O.*

Proof. The function $F : (0, \infty) \to (-\infty, M)$, where $M = \sup_{t>0} F(t)$, is invertible and $\Gamma := F^{-1}$ is continuous and satisfies (Γ1)-(Γ3). Next, taking $\rho := F \circ \mathrm{d}$ and $\psi(t) = t - \tau$, the conclusion follows immediately from Theorem 2.2. □

Remark 2.9. Corollary 2.1 generalizes [11, Th. 2.1] and [17, Th. 3.2].

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