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Results in Mathematics

Szász–Mirakyan Type Operators Which Fix Exponentials

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Abstract. In this paper, we construct a new general class of operators which have the classical Szász Mirakyan ones as a basis, and fix the functions e^{ax} and e^{2ax} with $a > 0$. The convergence of the corresponding sequences is discussed in exponential weighted spaces, and a Voronovskaya type result is given. Also we define a new weighted modulus of smoothness and determine the approximation order of the constructed operators. Finally, we study the goodness of the estimates of our new operators via saturation results.

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1. Introduction

Korovkin's approximation theorem is one of the most powerful criteria to check whether a given sequence $(L_n)_{n\geq 1}$ of positive linear operators tends to the identity operator with respect to the uniform norm of the space $C[a,b]$, that is to say, whether it represents or not an approximation process. This theorem given by P. P. Korovkin and independently by H. Bohman in the fifties says that the convergence towards f for the test functions $e_j(x) = x^j$, $j = 0, 1, 2$, $(L_n f)_{n\geq 1}$ assures that this approximation property holds as well for the funcions of the space $C[a,b]$. With this result, studies on linear positive operators have gained momentum and new constructions of approximation processes have been provided. One of the most known in this field is the King's operator. In [10], King presented modifications of the Bernstein operators which preserve e_0 and e_2

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in order to furnish a better error estimation. Inspiring by this fact Cárdenas-Morales et al. [6] introduced an operator of King type, which reproduced the function $e_2+\alpha e_1$ for $\alpha \geq 0$, having Bernstein basis functions. Also, for different Bernstein Durrmeyer type operators similar results were given in [7]. Agratini [2] applied a similar idea to more general discrete type operators, acting on either bounded or unbounded intervals, depending on a real parameter $\alpha \geq 0$ and preserving both the constants and the polynomial $e_2+\alpha e_1$. Also in [5] the author considered new operators of a general class preserving only two test functions, namely e_0 and e_1, e_0 and e_2, and e_1 and e_2, conditionally. For more results, we refer the readers to [3,4]. We emphasize that all the mentioned results were considered to obtain new sequences of operators which preserved test functions among the ones of the Korovkin set $\{e_0,\ e_1,\ e_2\}$.

Our aim with this paper is to introduce a family of linear positive operators having Szász Mirakyan basis function, that reproduce the functions e^{ax} and e^{2ax}, $a>0$, instead of the usual polynomial type ones. Such a first construction of Szász Mirakyan operators which preserve the functions 1 and e^{2ax}, $a>0$ has been recently investigated by Acar et al. [1]. In the present paper, we investigate the approximation behavior of the new operators for real valued functions belonging to certain exponential weighted space. To do this, we first present the moments and central moments of the operators. Then we present a result on the uniform approximation properties of the operators by means of a weighted Korovkin type theorem. Sections 4 and 5 are devoted to establishing the weighted approximation properties of the new class. We emphasize that our results are global ones. Generally, we work with exponential weighted spaces defined on $\mathbb{R}^+$. The last sections study the goodness of the new approximation processes, by stating an asymptotic formula and saturation results.

2. Construction of the Operators

We consider a sequence $(\mathcal{R}_n)_{n\geq 1}$ of linear positive operators based on the Szász–Mirakyan operators having the form

$$\mathcal{R}_n\left(f;x\right)=e^{-n\alpha_n(x)}\sum_{k=0}^{\infty}\frac{\left(n\beta_n\left(x\right)\right)^k}{k!}f\left(\frac{k}{n}\right),\quad n\in\mathbb{N},\, x\in\mathbb{R}^+, \tag{2.1}$$

where $\alpha_n,\beta_n:\mathbb{R}^+\rightarrow\mathbb{R}$ are positive functions to be considered in such a way that the operators hold fixed the functions e^{ax} and e^{2ax}. The operators act on an appropriate subspace of $C\left(\mathbb{R}^+\right)$ for which the above series is convergent.

Thus, to describe the sequences (α_n) and (β_n) explicitly we consider the following identities:

$$\mathcal{R}_n\left(e^{at};x\right)=e^{ax}, \tag{2.2}$$

$$\mathcal{R}_n\left(e^{2at};x\right)=e^{2ax}, \tag{2.3}$$

for each $n\in\mathbb{N}$ and $x\in\mathbb{R}^+$.

Directly, (2.1) permit us to write

$$e^{ax}=e^{n\left(\beta_n(x)e^{a/n}-\alpha_n(x)\right)}$$

and

$$e^{2ax}=e^{n\left(\beta_n(x)e^{2a/n}-\alpha_n(x)\right)}.$$

Using the above equalities, we have

$$\begin{aligned}2n\left(\beta_n\left(x\right)e^{a/n}-\alpha_n\left(x\right)\right)&=n\left(\beta_n\left(x\right)e^{2a/n}-\alpha_n\left(x\right)\right)\\ \beta_n\left(x\right)e^{a/n}\left(2-e^{a/n}\right)&=\alpha_n\left(x\right)\end{aligned}$$

and

$$\begin{aligned}ax&=n\left(\beta_n\left(x\right)e^{2a/n}-\alpha_n\left(x\right)\right)-n\left(\beta_n\left(x\right)e^{a/n}-\alpha_n\left(x\right)\right)\\ &=n\beta_n\left(x\right)e^{a/n}\left(e^{a/n}-1\right).\end{aligned}$$

Simple calculations give us the equalities

$$\beta_n\left(x\right)=\frac{ax}{ne^{a/n}\left(e^{a/n}-1\right)} \tag{2.4}$$

and

$$\begin{aligned}\alpha_n\left(x\right)&=\beta_n\left(x\right)e^{a/n}\left(2-e^{a/n}\right)\\ &=\frac{ax\left(2-e^{a/n}\right)}{n\left(e^{a/n}-1\right)}.\end{aligned} \tag{2.5}$$

Taking into account the above equalities we have

$$\begin{aligned}\mathcal{R}_n\left(e_0,x\right)&=e^{n\left(\beta_n(x)-\alpha_n(x)\right)}\\ &=e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}}.\end{aligned} \tag{2.6}$$

We proceed now to introduce the announced class of operators. Starting from (2.4), (2.5) and (2.1), we focus our attention on obtaining modified processes: for each $n\in\mathbb{N}$, we define the operators

$$\begin{aligned}\mathcal{R}_n\left(f;x\right)&=e^{-n\alpha_n(x)}\sum_{k=0}^{\infty}\frac{\left(n\beta_n\left(x\right)\right)^k}{k!}f\left(\frac{k}{n}\right)\\ &=e^{x\frac{a\left(2-e^{a/n}\right)}{\left(e^{a/n}-1\right)}}\sum_{k=0}^{\infty}\frac{\left(ax\right)^k}{k!e^{ak/n}\left(e^{a/n}-1\right)^k}f\left(\frac{k}{n}\right)\\ &=:\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)P_{n,k}\left(x\right).\end{aligned} \tag{2.7}$$

where $f\in\mathcal{F}\left(\mathbb{R}^+\right)$, which is the space of all functions $f:\mathbb{R}^+\to\mathbb{R}$ which are continuous on each compact interval of $\mathbb{R}^+$and for which the relation $\mathcal{R}_n\left(|f|\right)<$

∞ occurs. Examining (2.7) we deduce that $(\mathcal{R}_n)_{n\geq 1}$ is a sequence of positive linear operators.

Here we note that similar construction of Szász–Mirakyan operators which preserve the constant functions and e^{2ax} have been recently introduced in [1].

3. Auxiliary Results

In this section, we shall present the moments and the central moments of the operators (2.7) which will be necessary to prove our main results.

Lemma 1. *For the operator* $(\mathcal{R}_n)_{n\geq 1}$ *we have*

1. $\mathcal{R}_n(e_0;x) = e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}}$,
2. $\mathcal{R}_n(e_1;x) = \frac{ax}{ne^{a/n}\left(e^{a/n}-1\right)} e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}}$,
3. $\mathcal{R}_n(e_2;x) = \left\{\left(\frac{ax}{ne^{a/n}\left(e^{a/n}-1\right)}\right)^2 + \frac{ax}{n^2e^{a/n}\left(e^{a/n}-1\right)}\right\} e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}}$.

From the above lemma we can see that our operators $(\mathcal{R}_n)_{n\geq 1}$ do not reproduce the Korovkin test functions. Based on the Bohman-Korovkin theorem, the values of the limits of the above moments guarantee that $(\mathcal{R}_n)_{n\geq 1}$ is an approximation process on any compact $K \subset \mathbb{R}^+$.

Let us consider the central moment operator of order s by $\mu_n^s(x) = \mathcal{R}_n\left((t-x)^s;x\right),\ s = 0, 1, 2 \ldots$.

Using the equalities (2.4), (2.5) and Lemma 1 we immediately have the following lemma.

Lemma 2. *For the operator* $\mu_n^s(x)$ *we have*

1. $\mu_n^0(x) = e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}}$,
2. $\mu_n^1(x) = \left(\frac{ax}{ne^{a/n}\left(e^{a/n}-1\right)} - x\right) e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}}$,
3. $\mu_n^2(x) = \left\{\left(\frac{ax}{ne^{a/n}\left(e^{a/n}-1\right)} - x\right)^2 + \frac{ax}{n^2e^{a/n}\left(e^{a/n}-1\right)}\right\} e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}}$.

4. Weighted Approximation

We also analyze the behavior of the operators on some weighted spaces. Set $\varphi(x) = 1 + e^{2ax}$, $x \in \mathbb{R}^+$, and consider the following weighted spaces:

$$B_\varphi\left(\mathbb{R}^+\right) = \{f : \mathbb{R}^+ \to \mathbb{R} \ : \ |f(x)| \leq M_f\varphi(x),\ x \geq 0\},$$

$$C_\varphi\left(\mathbb{R}^+\right) = C\left(\mathbb{R}^+\right) \cap B_\varphi\left(\mathbb{R}^+\right),$$

$$C_\varphi^k\left(\mathbb{R}^+\right) = \left\{f \in C_\varphi\left(\mathbb{R}^+\right) : \lim_{x\to\infty}\frac{f(x)}{\varphi(x)} = k_f \text{ exists and it is finite}\right\},$$

where M_f is a constant depending on f and k_f is a constant depending on f. All three spaces are normed spaces with the norm

$$\|f\|_{\varphi} = \sup_{x\in\mathbb{R}^+} \frac{|f(x)|}{\varphi(x)}.$$

It is obvious that for any $f \in C_{\varphi}(\mathbb{R}^+)$ the inequality

$$\|\mathcal{R}_n(f)\|_{\varphi} \leq \|f\|_{\varphi}$$

holds and we conclude that $\mathcal{R}_n(f)$ maps $C_{\varphi}(\mathbb{R}^+)$ to $C_{\varphi}(\mathbb{R}^+)$.

Note that, the properties of linear positive operators acting on more general weighted spaces and related Korovkin type theorems have been studied in [8].

For the generalized Szász–Mirakyan operators (2.7) we have:

Theorem 1. *For each function* $f \in C_{\varphi}^{k}(\mathbb{R}^+)$

$$\lim_{n\to\infty} \|\mathcal{R}_n(f) - f\|_{\varphi} = 0.$$

Proof. Using the general result established in [8] it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \|\mathcal{R}_n(e^{va\cdot}) - e^{va\cdot}\|_{\varphi} = 0, \ v = 0, 1, 2. \tag{4.1}$$

Also using the property (2.6) we have

$$\|\mathcal{R}_n(e_0) - 1\|_{\varphi} = \sup_{x\in\mathbb{R}^+} \frac{e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}} - 1}{1+e^{2ax}}.$$

By means of the classical inequality $e^x - 1 \leq xe^x$ for $x \geq 0$, we get

$$\begin{aligned} e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}} - 1 &\leq ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}} e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}} \\ &\leq \frac{a^2x}{n} e^{\frac{a^2x}{n}} \end{aligned}$$

and moreover, since $\max_{x>0} xe^{-ax} = \frac{1}{ae}$, we obtain

$$\begin{aligned} \frac{e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}} - 1}{1+e^{2ax}} &\leq \frac{\frac{a^2x}{n}e^{\frac{a^2x}{n}}}{1+e^{2ax}} \\ &\leq \frac{a^2}{n}\frac{x}{e^{ax}}\frac{e^{\frac{a^2x}{n}}}{e^{ax}} \\ &= \frac{a^2}{n}\frac{x}{e^{ax}}\frac{1}{e^{ax\left(1-\frac{a}{n}\right)}} \\ &\leq \frac{a^2}{n}\frac{1}{ae} \end{aligned} \tag{4.2}$$

for sufficiently large n. This means that the conditions (4.1) are fulfilled for $v=0$. Since $\mathcal{R}_n\left(e^{at};x\right)=e^{ax}$ and $\mathcal{R}_n\left(e^{2at};x\right)=e^{2ax}$, the conditions (4.1) are fulfilled for $v=1$ and $v=2$. □

5. Rate of Convergence

Here we explore the rate of convergence of $\left(\mathcal{R}_n\right)_{n\geq 1}$ to the identity operator in terms of certain weighted modulus of continuity. In order to consider the approximation to unbounded functions, we consider exponential weighed space $C_a\left(\mathbb{R}^+\right)$ with a fixed $a>0$, which is the set of all real valued functions f continuous on $\mathbb{R}^+$ satisfying the condition $\left|f(x)\right|\leq Me^{ax}$, where M is a positive constant. This space is a normed space with the norm

$$\left\|f\right\|_a=\sup_{x\in\mathbb{R}^+}\frac{\left|f(x)\right|}{e^{ax}}.$$

Let $C_a^k\left(\mathbb{R}^+\right)$ be subspace of all the functions $f\in C_a\left(\mathbb{R}^+\right)$ such that $\lim_{x\to\infty}\frac{|f(x)|}{e^{ax}}=k$, where k is a positive constant. Our purpose is to extend the well known technique by Shisha and Mond [11] for functions belonging to the space $C_a^k\left(\mathbb{R}^+\right)$. They obtained the approximation error in terms of the classical modulus of continuity. Here we consider a weighted modulus of continuity with similar properties to that of the compact case, namely,

$$\widetilde{\omega}\left(f;\delta\right)=\sup_{|t-x|\leq\delta,\ x\geq 0}\frac{\left|f(t)-f(x)\right|}{e^{at}+e^{ax}},$$

for $f\in C_a^k\left(\mathbb{R}^+\right)$.

Firstly we note that our modulus of continuity is well behaved for functions $f\in C_a^k\left(\mathbb{R}^+\right)$, that is, from the property (2.2), the operator $\left(\mathcal{R}_n\right)_{n\geq 1}$ is a linear positive operator from $C_a\left(\mathbb{R}^+\right)$ to $C_a\left(\mathbb{R}^+\right)$.

Now we give some basic properties of $\widetilde{\omega}\left(f;\cdot\right)$.

Lemma 3. *Given* $f\in C_a^k\left(\mathbb{R}^+\right)$, $\lim_{\delta\to 0}\widetilde{\omega}\left(f;\delta\right)=0$.

Proof. Since $f\in C_a^k\left(\mathbb{R}^+\right)$, $\lim_{x\to\infty}\frac{|f(x)|}{e^{ax}}=k$, there exists x_0 such that

$$\begin{aligned}\widetilde{\omega}\left(f;\delta\right)&=\sup_{|t-x|\leq\delta,\ x\geq 0}\frac{\left|f(t)-f(x)\right|}{e^{at}+e^{ax}}\\&\leq\sup_{|t-x|\leq\delta,\ 0\leq x\leq x_0}\left|f(t)-f(x)\right|+\sup_{|t-x|\leq\delta,\ x>x_0}\frac{\left|f(t)-f(x)\right|}{e^{at}+e^{ax}}\\&\leq\omega\left(f;\delta\right)+\sup_{|t-x|\leq\delta,\ x>x_0}\left|\frac{f(t)}{e^{at}}-k\right|+\sup_{|t-x|\leq\delta,\ x>x_0}\left|\frac{f(x)}{e^{ax}}-k\right|\\&\leq\omega\left(f;\delta\right)+2\varepsilon,\end{aligned}$$

where $\omega(f;\delta)$ is the classical modulus of continuity of f on the interval $[0,x_0]$ such that $\omega(f;\delta)\to 0$ as $\delta\to 0$. As ε has been chosen arbitrarily, we have the desired result. □

The inequality that follows allows us to use with $\widetilde{\omega}(f;\cdot)$ the technique by Shisha and Mond.

Lemma 4. *For any integer m and $f\in C_a^k(\mathbb{R}^+)$ we have*

$$\widetilde{\omega}(f;m\delta)\le 2m\widetilde{\omega}(f;\delta).$$

Proof. Let $m\in\mathbb{N}$ and $f\in C_a^k(\mathbb{R}^+)$ be fixed. For each $x\in\mathbb{R}^+$ and $\delta>0$, based on the definition of $\widetilde{\omega}(f;\delta)$ we can write

$$\begin{aligned}|f(x+mh)-f(x)| &= \left|\sum_{j=1}^{m} f(x+jh)-f(x+(j-1)h)\right|\\ &\le \widetilde{\omega}(f;\delta)\sum_{j=1}^{m} e^{a(x+jh)}+e^{a(x+(j-1)h)}.\end{aligned}$$

The supremum being the least upper bound, we obtain

$$\begin{aligned}\widetilde{\omega}(f;m\delta) &= \sup_{|h|\le\delta,\ x\ge 0}\frac{|f(x+mh)-f(x)|}{e^{a(x+mh)}+e^{ax}}\\ &\le \widetilde{\omega}(f;\delta)\sup_{|h|\le\delta,\ x\ge 0}\sum_{j=1}^{m}\frac{e^{a(x+jh)}+e^{a(x+(j-1)h)}}{e^{a(x+mh)}+e^{ax}}\\ &\le 2m\widetilde{\omega}(f;\delta),\end{aligned}$$

for a positive integer m. □

Theorem 2. *For $f\in C_a^k(\mathbb{R}^+)$*

$$\|\mathcal{R}_n(f)-f\|_{5a/2}\le\frac{a}{e}\frac{1}{n}\|f\|_a+C\widetilde{\omega}\left(f;\frac{1}{\sqrt{n}}\right),$$

where C is positive constant.

Proof. From Lemma 4 we can write

$$\widetilde{\omega}(f;\lambda\delta)\le 2(1+\lambda)\widetilde{\omega}(f;\delta)$$

for $\lambda>0$. Also, from the definition of the weighted modulus of smoothness $\widetilde{\omega}(f;\delta)$ for $f\in C_a^k(\mathbb{R}^+)$ and $x,t\in\mathbb{R}^+$ and last inequality with $\lambda=\frac{|t-x|}{\delta}$, $\delta>0$, we have

$$\begin{aligned}|f(t)-f(x)| &\le \left(e^{at}+e^{ax}\right)\widetilde{\omega}(f;|t-x|)\\ &\le 2\left(e^{at}+e^{ax}\right)\left(1+\frac{|t-x|}{\delta}\right)\widetilde{\omega}(f;\delta).\end{aligned}\tag{5.1}$$

On the other hand, using Lemma 1, for each $x>0$, we have

$$\left|\mathcal{R}_n(f;x)-f(x)\right| \leq f(x)\left|1-e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}}\right| + \sum_{k=0}^{\infty}\left|f\left(\frac{k}{n}\right)-f(x)\right| P_{n,k}(x).$$

Further on, (5.1) imply that

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left|f\left(\frac{k}{n}\right)-f(x)\right| P_{n,k}(x) \\
&\leq 2\widetilde{\omega}(f;\delta)\sum_{k=0}^{\infty}\left(e^{a\frac{k}{n}}+e^{ax}\right)\left(1+\frac{1}{\delta}\left|\frac{k}{n}-x\right|\right)P_{n,k}(x) \\
&\leq 2\widetilde{\omega}(f;\delta)\left\{\sum_{k=0}^{\infty}e^{a\frac{k}{n}}P_{n,k}(x)+\frac{1}{\delta}\sum_{k=0}^{\infty}e^{a\frac{k}{n}}\left|\frac{k}{n}-x\right|P_{n,k}(x)\right. \\
&\quad\left. + e^{ax}e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}}+\frac{e^{ax}}{\delta}\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|P_{n,k}(x)\right\} \\
&\leq 2\widetilde{\omega}(f;\delta)e^{ax}\left\{1+\frac{1}{\delta}\left(\mu_n^2(x)\right)^{1/2}\right. \\
&\quad\left. + e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}}+\frac{1}{\delta}\left(e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}}\right)^{1/2}\left(\mu_n^2(x)\right)^{1/2}\right\}.
\end{aligned}$$

Besides, from Lemma 2 it follows that

$$\sup_{x>0}\frac{\mu_n^2(x)}{e^{2ax}}<\infty.$$

If we choose $\delta=\frac{1}{\sqrt{n}}$, we have the desired results. □

6. Voronovskaya Type Theorem

In this section, we obtain some asymptotic estimates of the pointwise convergence in the case of functions with exponential growth.

Theorem 3. *Let $f\in C_{\varphi}(\mathbb{R}^+)$. If f is twice differentiable in $x\in\mathbb{R}^+$ and f'' is continuous at x, then the following limit holds*:

$$\lim_{n\to\infty} n\left[\mathcal{R}_n(f,x)-f(x)\right]=a^2xf(x)-\frac{3}{2}axf'(x)+\frac{x}{2}f''(x).$$

Proof. By the Taylor's formula there exists η lying between x and t such that

$$f(t)=f(x)+f'(x)(t-x)+\frac{f''(x)}{2}(t-x)^2+h(t,x)(t-x)^2,$$

where

$$h(t,x) := \frac{f''(\eta) - f''(x)}{2}$$

and h is a continuous function which vanishes as $t \to x$. Applying the operator $(\mathcal{R}_n)_{n\geq 1}$ to the above equality, we get

$$\mathcal{R}_n(f,x) - f(x) + f(x) - e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}} f(x) = f'(x)\,\mu_n^1(x) + \frac{f''(x)}{2}\mu_n^2(x) + \mathcal{R}_n\left(h(t,x)(t-x)^2, x\right).$$

Also we can write that

$$n\left[\mathcal{R}_n(f,x) - f(x)\right] = f(x)\,n\left(e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}} - 1\right) + f'(x)\,n\mu_n^1(x) + \frac{f''(x)}{2}n\mu_n^2(x) + n\mathcal{R}_n\left(h(t,x)(t-x)^2, x\right). \tag{6.1}$$

After a straightforward computation, we have

$$\lim_{n\to\infty} n\left(e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}} - 1\right) = a^2 x$$

and

$$\begin{aligned}\lim_{n\to\infty} n\mu_n^1(x) &= \lim_{n\to\infty}\left(\frac{ax}{e^{a/n}\left(e^{a/n}-1\right)} - nx\right) e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}} \\ &= \lim_{n\to\infty} e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}} \lim_{n\to\infty} n\left(\frac{ax}{ne^{a/n}\left(e^{a/n}-1\right)} - x\right) \\ &= -\frac{3}{2}ax.\end{aligned}$$

Since

$$\begin{aligned}&\lim_{n\to\infty} n\left(\frac{ax}{ne^{a/n}\left(e^{a/n}-1\right)} - x\right)^2 \\ &= \lim_{n\to\infty} n\left(\frac{ax}{ne^{a/n}\left(e^{a/n}-1\right)} - x\right)\lim_{n\to\infty}\left(\frac{ax}{ne^{a/n}\left(e^{a/n}-1\right)} - x\right) \\ &= 0\end{aligned}$$

we have

$$\begin{aligned}\lim_{n\to\infty} n\mu_n^2(x) &= \left\{n\left(\frac{ax}{ne^{a/n}\left(e^{a/n}-1\right)} - x\right)^2 + \frac{ax}{ne^{a/n}\left(e^{a/n}-1\right)}\right\} e^{ax\frac{\left(e^{a/n}-1\right)}{e^{a/n}}} \\ &= x.\end{aligned}$$

In order to estimate the last term in (5.1), for every $\varepsilon > 0$ choose $\delta > 0$ such that $|h(t,x)| < \varepsilon$ for $|t-x| < \delta$. Therefore if $|t-x| < \delta$ then $\left|h(t,x)(t-x)^2\right| < \varepsilon (t-x)^2$while if $|t-x| \geq \delta$, then since $|h(t,x)| \leq M$ we have $\left|h(t,x)(t-x)^2\right| \leq \frac{M}{\delta^2}(t-x)^4$. So we can write

$$\mathcal{R}_n\left(h(t,x)(t-x)^2, x\right) < \varepsilon \mu_n^2(x) + \frac{M}{\delta^2}\mu_n^4(x).$$

Direct calculations show that

$$\mu_n^4(x) = O\left(\frac{1}{n^2}\right),$$

and we conclude

$$\lim_{n\to\infty} n\mathcal{R}_n\left(h(t,x)(t-x)^2, x\right) = 0.$$

This proves the theorem □

7. Saturation Results

Beyond the establishment of an asymptotic formula, the natural way to study to goodness of the estimates in approximation theory by sequences of linear operators is via saturation results. That is what we develop in this section.

First of all we observe that the right-hand side of the asymptotic formula can be written in terms of three weight functions as follows:

$$a^2 x f(x) - \frac{3}{2}axf'(x) + \frac{x}{2}f''(x) = \frac{1}{w_2(x)}\left(\frac{1}{w_1(x)}\left(\frac{f(x)}{w_0(x)}\right)'\right)',$$

where

$$w_0(x) = e^{ax}, \quad w_1(x) = ae^{ax}, \quad w_2(x) = \frac{2}{ax}e^{-2ax}.$$

Secondly, we observe that with easy modifications the results in [9] apply to the sequence $\mathcal{R}_n$ (in the particular case in which the operators are assumed to be only positive) despite the right-hand side of the asymptotic formula contains $f(x)$, in addition to $f'(x)$ and $f''(x)$.

Consequently, the following results concerning the trivial class and the saturation class of the sequence R_n remain valid.

Theorem 4. *Let $f \in C_\varphi(\mathbb{R}^+)$ and let $J \subset \mathbb{R}^+$ be a bounded open interval. Then, for each $x \in J$*

$$n(\mathcal{R}_n(f;x) - f(x)) = o(1) \text{ if and only if } f \in \langle e^{ax}, e^{2ax}\rangle.$$

Theorem 5. *Let $f \in C_\varphi(\mathbb{R}^+)$, let $J \subset \mathbb{R}^+$ be a bounded open interval and let $M \geq 0$. Then, for each $x \in J$, one has that*

$$n|\mathcal{R}_n(f;x) - f(x)| \leq M + o(1)$$

if and only if,

$$\left|a^2xf(x)-\frac{3}{2}atf'(x)+\frac{x}{2}f''(x)\right|\leq M,\quad \textit{for almost every } x\in J.$$

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