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Classification of the Locally Strongly Convex Centroaffine Hypersurfaces with Parallel Cubic Form

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Abstract. In this paper, we study locally strongly convex centroaffine hypersurfaces with parallel cubic form with respect to the Levi–Civita connection of the centroaffine metric. As the main result, we obtain a complete classification of such centroaffine hypersurfaces. The result of this paper is a centroaffine version of the complete classification of locally strongly convex equiaffine hypersurfaces with parallel cubic form due to Hu et al. (J Differ Geom 87:239–307, 2011).

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1. Introduction

In centroaffine differential geometry, we study properties of hypersurfaces in the $(n+1)$-dimensional affine space $\mathbb{R}^{n+1}$ equipped with its standard affine flat connection D, that are invariant under the centroaffine transformation group G in $\mathbb{R}^{n+1}$. Here, by definition, G is the subgroup of affine transformation group in $\mathbb{R}^{n+1}$ which keeps the origin $O \in \mathbb{R}^{n+1}$ invariant. Let M^n be an n-dimensional smooth manifold. An immersion $x : M^n \to \mathbb{R}^{n+1}$ is said to be centroaffine hypersurface if the position vector x (from O) for each point $x \in M^n$ is transversal to the tangent plane of M at x. In this case, the position vector x defines the so-called *centroaffine normalization* modulo orientation.

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For any vector fields X and Y tangent to M, we have the centroaffine formula of Gauss:

$$D_X x_*(Y) = x_*(\nabla_X Y) + h(X,Y)(-\varepsilon x), \tag{1.1}$$

where $\varepsilon = 1$ or -1. In this paper, we always assume that $x : M^n \to \mathbb{R}^{n+1}$ is a non-degenerate centroaffine hypersurface, i.e., the bilinear 2-form h, defined by (1.1), remains non-degenerate. Moreover, associated with (1.1) we call $-\varepsilon x$, ∇ and h the centroaffine normal, the induced connection and the *centroaffine metric* induced by $-\varepsilon x$, respectively.

Let $N(h)$ denote the dimension of the maximal negative definite subspaces of the bilinear form h with respect to $\varepsilon = -1$. For a locally strongly convex centroaffine hypersurface, i.e., $N(h) = 0$ or $N(h) = n$, we can choose ε such that the centroaffine metric h is positive definite. In that situation, if $\varepsilon = 1$ we say that the hypersurface is elliptic and, if $\varepsilon = -1$ we call the hypersurface hyperbolic (cf. Section 2 of [14]). We refer to [7,18,23] for some interesting studies on centroaffine hypersurfaces.

Given a non-degenerate centroaffine hypersurface $x : M^n \to \mathbb{R}^{n+1}$, we denote by $\hat{\nabla}$ the Levi–Civita connection of h. Then the difference tensor K, defined by $K(X,Y) := K_X Y := \nabla_X Y - \hat{\nabla}_X Y$, and the cubic form $C := \nabla h$ are related by the equation

$$C(X,Y,Z) = -2h(K_X Y, Z) = -2h(K_X Z, Y). \tag{1.2}$$

It is well-known (cf. [14,16,21]) that a centroaffine hypersurface immersion is uniquely determined, up to a centroaffine transformation, by its centroaffine metric and its cubic form (this means that the cubic form plays the role of the affine second fundamental form). Hence, in centroaffine differential geometry the problem of classifying affine hypersurfaces with parallel cubic form (i.e., $\hat{\nabla} C = 0$) is quite natural and important. In [17], Li and Wang considered this problem the first time by studying the so-called *canonical* centroaffine hypersurface. Here, a centroaffine hypersurface is called *canonical* if its centroaffine metric h is flat and its cubic form C satisfies $\hat{\nabla} C = 0$.

We should recall that in equiaffine differential geometry, the problem of classifying locally strongly convex affine hypersurfaces with parallel cubic form has been studied intensively, from the earlier beginning paper by Bokan et al. [2], and then [5,6,9,10] by some others, to the very recent complete classification of Hu et al. [12]. We also refer to the latest development due to Hildebrand [8], however, from the geometric viewpoint the arguments in [8] is difficult to be followed for us.

In centroaffine differential geometry, compared with its counterpart in equiaffine differential geometry, the important *apolarity condition* does not exist. The lack of the apolarity condition brings serious difficulties to the solution of the problem of classifying centroaffine hypersurfaces with parallel cubic form. To our knowledge, besides Li and Wang [17], the only known results concentrating on this problem is by Liu and Wang [19], where 2-dimensional centroaffine surfaces were classified under the condition that the traceless cubic form $\tilde{C}$ is parallel, i.e. $\hat{\nabla}\tilde{C} = 0$. As $\hat{\nabla}C = 0$ implies that $\hat{\nabla}\tilde{C} = 0$, Liu and Wang's classification list should include all immersions satisfying $\hat{\nabla}C = 0$.

In this paper, restricting our attention to locally strongly convex centroaffine hypersurfaces in $\mathbb{R}^{n+1}$, we will solve the above problem by establishing a complete classification of all centroaffine hypersurfaces with parallel cubic form. Similar to the one in [10–12], our classification depends heavily on the characterization of the so-called (generalized) Calabi product construction of centroaffine hypersurfaces (cf. [14,17]). Indeed, such characterization tells how to decompose a complicated centroaffine hypersurface into lower dimensional ones that have been well known.

To state the main result of this paper, we first recall that if $\psi_i : M_i \to \mathbb{R}^{n_i+1}$, where $i = 1, 2$, are non-degenerate centroaffine hypersurfaces, then, following [14,17], for a constant $\lambda \neq 0, -1$, we can define the (generalized) Calabi product of M_1 and M_2 by

$$\psi(u, p, q) = (e^u \psi_1(p), e^{-\lambda u}\psi_2(q)), \quad p \in M_1,\ q \in M_2,\ u \in \mathbb{R}. \tag{1.3}$$

Similarly, the (generalized) Calabi product of M_1 and a point is defined by

$$\tilde{\psi}(u, p) = (e^u \psi_1(p), e^{-\lambda u}), \quad p \in M_1,\ u \in \mathbb{R}. \tag{1.4}$$

Note that a straightforward calculation shows that the Calabi product of two centroaffine hypersurfaces with parallel cubic form (resp. the Calabi product of a centroaffine hypersurface with parallel cubic form and a point) again has parallel cubic form. The decomposition theorems, which can be seen as the converse of the previous Calabi product constructions, were first obtained in terms of h and K in [17] (Theorem 4.5 therein) and will be modified more quantitatively in the present paper (cf. Theorems 3.2 and 3.4 below) for maintaining consistency with Theorems 3 and 4 of [11]. In this paper, we further develop the techniques, started in [10,12] when dealing with equiaffine hypersurfaces, in order to obtain the following complete classification.

Theorem 1.1. *Let M^n be an n-dimensional locally strongly convex centroaffine hypersurface in $\mathbb{R}^{n+1}$ with $\hat{\nabla}C = 0$. Then, we have either*

(i) *M^n is an open part of a locally strongly convex hyperquadric with $C = 0$; or*
(ii) *M^n is obtained as the Calabi product of a lower dimensional locally strongly convex centroaffine hypersurface with parallel cubic form and a point; or*
(iii) *M^n is obtained as the Calabi product of two lower dimensional locally strongly convex centroaffine hypersurfaces with parallel cubic form; or*
(iv) *$n = \frac{1}{2}m(m+1) - 1$, $m \geq 3$, M^n is centroaffinely equivalent to the standard embedding of $\mathrm{SL}(m, \mathbb{R})/\mathrm{SO}(m) \hookrightarrow \mathbb{R}^{n+1}$; or*

(v) $n = \frac{1}{4}(m+1)^2 - 1$, $m \geq 5$, M^n *is centroaffinely equivalent to the standard embedding* $\mathrm{SL}(\frac{m+1}{2}, \mathbb{C})/\mathrm{SU}(\frac{m+1}{2}) \hookrightarrow \mathbb{R}^{n+1}$*; or*

(vi) $n = \frac{1}{8}(m+1)(m+3) - 1$, $m \geq 9$, M^n *is centroaffinely equivalent to the standard embedding* $\mathrm{SU}^*(\frac{m+3}{2})/\mathrm{Sp}(\frac{m+3}{4}) \hookrightarrow \mathbb{R}^{n+1}$*; or*

(vii) $n = 26$, M^n *is centroaffinely equivalent to the standard embedding* $\mathrm{E}_{6(-26)}/\mathrm{F}_4 \hookrightarrow \mathbb{R}^{27}$*; or*

(viii) M^n *is locally centroaffinely equivalent to the canonical centroaffine hypersurface* $x_{n+1} = \frac{1}{2x_1}\sum_{k=2}^n x_k^2 + x_1 \ln x_1$.

Remark 1.1. Compared to its counterpart of the Classification Theorem in equiaffine situation [12], the case (viii) in Theorem 1.1 is exceptional and it is completely newly appeared.

Remark 1.2. Theorem 1.1 implies that all canonical centroaffine hypersurfaces but that in (viii) can be decomposed as the Calabi product.

Remark 1.3. Related to Theorem 1.1 we have established in [4] the classification of locally strongly convex *isotropic* centroaffine hypersurfaces. From the comparison of the main results in [1,4] one sees that the *isotropic* condition again have different implications in both equiaffine theory of hypersurfaces and centroaffine theory of hypersurfaces, just like Theorem 1.1 here and the Classification Theorem in [12].

As direct consequence of Theorem 1.1, and without paying attention to the Calabi product constructions, the classification of locally strongly convex canonical centroaffine hypersurfaces can be formulated as follows:

Corollary 1.1 (cf. [17]). *Let* $x : M^n \to \mathbb{R}^{n+1}$ *be a locally strongly convex canonical centroaffine hypersurface. Then it is locally centroaffinely equivalent to one of the following hypersurfaces:*

(i) $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1$, *where either* $\alpha_i > 0$ $(1 \leq i \leq n+1)$, *or*

$$\alpha_1 < 0 \quad \textit{and} \quad \alpha_i > 0 \ (2 \leq i \leq n+1) \ \textit{such that} \ \sum_{i=1}^{n+1} \alpha_i < 0.$$

(ii) $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} (x_n^2 + x_{n+1}^2)^{\alpha_n} \exp(\alpha_{n+1} \arctan \frac{x_n}{x_{n+1}}) = 1$, *where*

$$\alpha_i < 0 \ (1 \leq i \leq n-1) \quad \textit{such that} \quad 2\alpha_n + \sum_{i=1}^{n-1} \alpha_i > 0,$$

(iii) $x_{n+1} = \frac{1}{2x_1}(x_2^2 + \cdots + x_{v-1}^2) - x_1(-\ln x_1 + \alpha_v \ln x_v + \cdots + \alpha_n \ln x_n)$, *where* $2 \leq v \leq n+1$, α_i $(v \leq i \leq n)$ *are real numbers satisfying*

$$\alpha_i > 0 \ (v \leq i \leq n) \quad \textit{and} \quad \sum_{i=v}^{n} \alpha_i < 1.$$

Remark 1.4. More general canonical centroaffine non-degenerate hypersurfaces have been discussed by Li and Wang [17], where the classification of canonical centroaffine hypersurfaces in $\mathbb{R}^{n+1}$ with $N(h) \leq 1$ was established. According to [17], it is easily seen that if $N(h) = 0$ then such hypersurfaces are centroaffinely equivalent to the following hypersurfaces

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1,$$

where α_i $(1 \leq i \leq n+1)$ are positive real numbers.

This paper is organized in twelve sections. In Sect. 2, we fix notations and recall relevant material for centroaffine hypersurfaces in affine differential geometry. In Sect. 3, we study both the Calabi product of centroaffine hypersurfaces and their characterizations. In Sect. 4, properties of centroaffine hypersurfaces with parallel cubic form in terms of a typical basis are presented, so that the classification problem of such hypersurfaces is divided into $(n+1)$ cases, namely: $\{\mathfrak{C}_m\}_{1\leq m\leq n}$ and an exceptional case $\mathfrak{B}$, depending on the decomposition of the tangent space into three orthogonal distributions, i.e., $\mathcal{D}_1$ (of dimension one), $\mathcal{D}_2$ and $\mathcal{D}_3$. The two cases $\mathfrak{C}_1$ and $\mathfrak{C}_n$ will be settled in this section. In Sect. 5, we settle the exceptional Case $\mathfrak{B}$. In Sect. 6, we classify locally strongly convex centroaffine surfaces in $\mathbb{R}^3$ with parallel cubic form. The result of Sect. 6 is necessary not only because it is indispensable to the induction procedure of Theorem 1.1, but also because it fills in a gap in the result of Liu and Wang [19].

To consider the cases $\{\mathfrak{C}_m\}_{2\leq m\leq n-1}$, we follow closely the same procedure as in [12]: we introduce two extremely important operators, i.e., an isotropic bilinear map $L : \mathcal{D}_2 \times \mathcal{D}_2 \to \mathcal{D}_3$ in Sect. 4.3, and, for any unit vector $v \in \mathcal{D}_2$, the symmetric linear map $P_v : \mathcal{D}_2 \to \mathcal{D}_2$ in Sect. 4.4. With the help of L and P_v, we can give a remarkable decomposition of $\mathcal{D}_2$ in Sect. 4.5. Then in Sects. 7–11, according to the decomposition of $\mathcal{D}_2$ we analyze these cases in much detail in order to achieve the corresponding conclusion, respectively. Finally in Sect. 12 we complete the proof of Theorem 1.1.

2. Preliminaries

In this section, we recall basic facts about centroaffine hypersurfaces. For more details see also [20,21]. Given a centroaffine hypersurface, let ∇, $\hat{\nabla}$, K and C denote the induced connection, the Levi–Civita connection for the centroaffine metric h, the difference tensor and the cubic form, respectively, and let X, Y, Z denote the tangent vector fields. We define the Tchebychev form $\hat{T}$ and the Tchebychev vector field T, respectively, by

$$n\hat{T}(X) = \mathrm{Tr}\,(K_X), \quad h(T, X) = \hat{T}(X). \tag{2.1}$$

If $T = 0$, or equivalently, $\mathrm{Tr}\,K_X = 0$ for any tangent vector X, then M^n is reduced to be the so-called *proper* (equi-)*affine hypersphere* centered at the origin O (cf. also [16], p. 279, and for more details, in Sect. 1.15.2-3 therein). Using the cubic form C and the Tchebychev form $\hat{T}$ one can define a traceless symmetric cubic form $\widetilde{C}$ by

$$\widetilde{C}(X,Y,Z) := -\tfrac{1}{2}C(X,Y,Z) \\ -\tfrac{n}{n+2}\big[\hat{T}(X)h(Y,Z)+\hat{T}(Y)h(X,Z)+\hat{T}(Z)h(X,Y)\big]. \quad (2.2)$$

It is well-known that $\widetilde{C}$ vanishes if and only if $f : M^n \to \mathbb{R}^{n+1}$ is a hyperquadric (cf. Section 7.1 in [21]; Lemma 2.1 and Remark 2.2 in [15]).

Let $\hat{R}$ denote the curvature tensor of $\hat{\nabla}$. Then, according to the integrability conditions, we have

$$\hat{R}(X,Y)Z = \varepsilon(h(Y,Z)X - h(X,Z)Y) - [K_X, K_Y]Z, \quad (2.3)$$

$$\hat{\nabla}C(X,Y,Z,W) = \hat{\nabla}C(Y,X,Z,W), \quad (2.4)$$

where $\hat{\nabla}C(X,Y,Z,W) := (\hat{\nabla}_X C)(Y,Z,W)$.

We define the curvature tensor acting as derivation by

$$(\hat{R}(X,Y)K)(Z,U) = \hat{R}(X,Y)K(Z,U) - K(\hat{R}(X,Y)Z,U) - K(Z,\hat{R}(X,Y)U).$$

Notice that $\hat{\nabla}C = 0$ if and only if $\hat{\nabla}K = 0$. Thus if $\hat{\nabla}C = 0$, we have

$$\hat{R}(X,Y)K(Z,U) = K(\hat{R}(X,Y)Z,U) + K(Z,\hat{R}(X,Y)U). \quad (2.5)$$

3. Characterizations of the Generalized Calabi Product

To prove Theorem 1.1, we should study the (generalized) Calabi products of centroaffine hypersurfaces as defined in (1.3) and (1.4). In this section, we first state some elementary calculations on Calabi product, formulated as Propositions 3.1 and 3.2. Then, considering the converse of these propositions, we will prove Theorems 3.1, 3.2, 3.3 and 3.4, which demonstrate the characterizations of the Calabi product in terms of their centroaffine invariants.

Let $\psi_i : M_i \to \mathbb{R}^{n_i+1}$ be a locally strongly convex centroaffine hypersurface of dimension n_i $(i = 1, 2)$. Denote by h^i the centroaffine metric of ψ_i $(i = 1, 2)$, respectively. Given the Calabi product ψ and $\tilde{\psi}$ defined as in (1.3) and (1.4), i.e., for constant $\lambda \neq 0, -1$, we have

$$\psi(u,p,q) = (e^u\psi_1(p),\, e^{-\lambda u}\psi_2(q)), \quad p \in M_1, \quad q \in M_2, \quad u \in \mathbb{R}, \quad (3.1)$$

$$\tilde{\psi}(u,p) = (e^u\psi_1(p),\, e^{-\lambda u}), \quad p \in M_1, \quad u \in \mathbb{R}. \quad (3.2)$$

Let $\{u_1,\ldots,u_{n_1}\}$ and $\{u_{n_1+1},\ldots,u_{n_1+n_2}\}$ be local coordinates for M_1 and M_2, respectively. For simplicity, we use the following range of indices:

$$1 \le i,j,k \le n_1, \quad n_1+1 \le \alpha,\beta,\gamma \le n_1+n_2.$$

According to Section 4 of Li and Wang [17], we can state the following result.

Proposition 3.1 (cf. [17]). *The Calabi product of M_1 and M_2*

$$\psi : M^{n_1+n_2+1} := \mathbb{R} \times M_1 \times M_2 \to \mathbb{R}^{n_1+n_2+2},$$

defined by (3.1) *is a non-degenerate centroaffine hyersurface, the centroaffine metric h induced by ψ is given by*

$$h = \lambda du^2 \oplus \tfrac{\lambda}{1+\lambda} h^1 \oplus \tfrac{1}{1+\lambda} h^2, \tag{3.3}$$

with the property

$$N(h) = \begin{cases} N(h^1) + N(h^2), & \lambda > 0, \\ n_1 + 1 - N(h^1) + N(h^2), & -1 < \lambda < 0, \\ n_2 + 1 + N(h^1) - N(h^2), & \lambda < -1. \end{cases} \tag{3.4}$$

The difference tensor K of ψ takes the following form:

$$\begin{aligned} &K\left(\tfrac{\psi_u}{\sqrt{|\lambda|}}, \tfrac{\psi_u}{\sqrt{|\lambda|}}\right) = \lambda_1 \tfrac{\psi_u}{\sqrt{|\lambda|}}, \quad K\left(\tfrac{\psi_u}{\sqrt{|\lambda|}}, \psi_{u_i}\right) = \lambda_2 \psi_{u_i}, \\ &K\left(\tfrac{\psi_u}{\sqrt{|\lambda|}}, \psi_{u_\alpha}\right) = \lambda_3 \psi_{u_\alpha}, \qquad K\left(\psi_{u_i}, \psi_{u_\alpha}\right) = 0, \end{aligned} \tag{3.5}$$

where $\lambda_1, \lambda_2, \lambda_3$ are constants satisfying

$$\lambda_1 = \lambda_2 + \lambda_3, \ \ \lambda_2\lambda_3 = -\operatorname{sgn}\lambda, \ \ \lambda_2 \neq \lambda_3. \tag{3.6}$$

Moreover, ψ is flat (*resp. of parallel cubic form*) *if and only if both ψ_1 and ψ_2 are flat* (*resp. of parallel cubic form*).

Similarly, the following result can be verified easily:

Proposition 3.2. *The Calabi product of M_1 and a point*

$$\tilde{\psi} : M^{n_1+1} = \mathbb{R} \times M_1 \to \mathbb{R}^{n_1+2}$$

defined by (3.2) *is a non-degenerate centroaffine hyersurface, the centroaffine metric $\tilde{h}$ induced by $\tilde{\psi}$ is given by*

$$\tilde{h} = \lambda du^2 \oplus \tfrac{\lambda}{1+\lambda} h^1, \tag{3.7}$$

with the property

$$N(\tilde{h}) = \begin{cases} N(h^1), & \lambda > 0, \\ n_1 + 1 - N(h^1), & -1 < \lambda < 0, \\ N(h^1) + 1, & \lambda < -1. \end{cases} \tag{3.8}$$

The difference tensor $\tilde{K}$ of $\tilde{\psi}$ takes the following form:

$$\tilde{K}\left(\tfrac{\tilde{\psi}_u}{\sqrt{|\lambda|}}, \tfrac{\tilde{\psi}_u}{\sqrt{|\lambda|}}\right) = \lambda_1 \tfrac{\tilde{\psi}_u}{\sqrt{|\lambda|}}, \quad \tilde{K}\left(\tfrac{\tilde{\psi}_u}{\sqrt{|\lambda|}}, \tilde{\psi}_{u_i}\right) = \lambda_2 \tilde{\psi}_{u_i}, \tag{3.9}$$

where λ_1, λ_2 are constants satisfying

$$\lambda_1 \neq 2\lambda_2, \quad \lambda_1\lambda_2 - \lambda_2^2 = -\operatorname{sgn}\lambda. \tag{3.10}$$

Moreover, $\tilde{\psi}$ is flat (*resp. of parallel cubic form*) *if and only if ψ_1 is flat* (*resp. of parallel cubic form*).

Remark 3.1. From (3.4) and (3.8), it is easily seen that if the Calabi product ψ (resp. $\tilde{\psi}$) is locally strongly convex, then the centroaffine metric of ψ (resp. $\tilde{\psi}$) induced by $-\varepsilon'\psi$ (resp. $-\varepsilon'\tilde{\psi}$) is positive, where $\varepsilon' = -\operatorname{sgn}\lambda$.

Next, as the converse of Proposition 3.1, we can prove the following theorem.

Theorem 3.1. *Let $\psi : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface. Assume that there exist distributions $\mathcal{D}_1$ (of dimension 1, spanned by a unit vector field T), $\mathcal{D}_2$ (of dimension n_1) and $\mathcal{D}_3$ (of dimension n_2) such that*

(i) $1 + n_1 + n_2 = n$,
(ii) *the centroaffine metric h induced by $-\varepsilon\psi$ ($\varepsilon = \pm 1$) is positive definite,*
(iii) *$\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are mutually orthogonal with respect to the centroaffine metric h,*
(iv) *there exist constants λ_1, λ_2, and λ_3 such that*

$$\begin{gathered} K(T,T) = \lambda_1 T, \quad K(T,V) = \lambda_2 V, \quad K(T,W) = \lambda_3 W, \quad K(V,W) = 0, \\ \forall\, V \in \mathcal{D}_2, \quad W \in \mathcal{D}_3; \quad \lambda_1 = \lambda_2 + \lambda_3,\ \lambda_2\lambda_3 = \varepsilon,\ \lambda_2 \neq \lambda_3. \end{gathered} \tag{3.11}$$

Then $\psi : M^n \to \mathbb{R}^{n+1}$ can be locally decomposed as the Calabi product of two lower dimensional locally strongly convex centroaffine hypersurfaces $\psi_1 : M_1^{n_1} \to \mathbb{R}^{n_1+1}$ and $\psi_2 : M_2^{n_2} \to \mathbb{R}^{n_2+1}$.

Proof. First of all, we have the following lemma, whose proof can be given exactly by following the proof of Lemmas 1, 2, 3 and 4 of [11].

Lemma 3.1. *Under the assumptions of Theorem 3.1, for any vector $X \in TM$, $V \in \mathcal{D}_2$ and $W \in \mathcal{D}_3$, the following hold*

$$\hat{\nabla}_X T = 0,\ \hat{\nabla}_X V \in \mathcal{D}_2,\ \hat{\nabla}_X W \in \mathcal{D}_3.$$

Lemma 3.1, together with the de Rham decomposition theorem, implies that (M, h) is locally isometric to $\mathbb{R} \times M_1 \times M_2$, where T is tangent to $\mathbb{R}$, whereas $\mathcal{D}_2$ (resp. $\mathcal{D}_3$) is tangent to M_1 (resp. M_2).

The product structure of M implies the existence of local coordinates (u, p, q) for M based on an open subset containing the origin of $\mathbb{R}^{n_1+n_2+1}$, such that $\mathcal{D}_1$ is given by $dp = dq = 0$, $\mathcal{D}_2$ (resp. $\mathcal{D}_3$) is given by $du = dq = 0$ (resp. $du = dp = 0$). We may assume that $T = \lambda_2 \frac{\partial}{\partial u}$. Put

$$\psi_1 = f(T - \lambda_3\psi), \quad \psi_2 = g(\lambda_2\psi - T), \tag{3.12}$$

where f and g are assumed to be nonzero functions which depend only on the variable u, and are given by

$$f(u) = \tfrac{1}{\lambda_2 - \lambda_3} e^{-u}, \quad g(u) = \tfrac{1}{\lambda_2 - \lambda_3} e^{-\frac{\lambda_3}{\lambda_2} u}.$$

A straightforward computation, by (3.12) and (1.1), shows that

$$\begin{aligned} D_T\psi_1 &= -\lambda_2 f(T-\lambda_3\psi) + f D_T(T-\lambda_3\psi) \\ &= f(\lambda_3\lambda_2-\varepsilon)\psi + f(-\lambda_2+\lambda_1-\lambda_3)T \\ &= 0. \end{aligned}$$

Similarly

$$D_W\psi_1 = 0, \quad D_T\psi_2 = D_V\psi_2 = 0.$$

The above relations imply that ψ_1 (resp. ψ_2) reduces to a map of M_1 (resp. M_2) in $\mathbb{R}^{n+1}$. The facts

$$\begin{aligned} d\psi_1(V) &= D_V\psi_1 = f(\lambda_2-\lambda_3)V, \\ d\psi_2(W) &= D_W\psi_2 = g(\lambda_2-\lambda_3)W \end{aligned}$$

show that both maps ψ_1 and ψ_2 are actually immersions. Denoting by ∇^1 (resp. ∇^2) the $\mathcal{D}_2$ (resp. $\mathcal{D}_3$) component of ∇, we further find that

$$\begin{aligned} D_V d\psi_1(\tilde V) &= f(\lambda_2-\lambda_3)D_V\tilde V \\ &= f(\lambda_2-\lambda_3)\Big(\nabla^1_V\tilde V - \varepsilon h(V,\tilde V)\psi + \lambda_2 h(V,\tilde V)T\Big) \\ &= d\psi_1(\nabla^1_V\tilde V) + (\lambda_2-\lambda_3)\lambda_2 h(V,\tilde V)\psi_1. \end{aligned}$$

Hence ψ_1 can be interpreted as a centroaffine immersion contained in an (n_1+1)-dimensional vector subspace of $\mathbb{R}^{n+1}$ with induced connection ∇^1 and centroaffine metric

$$h^1 = \lambda_2(\lambda_2-\lambda_3)h. \tag{3.13}$$

Similarly, we obtain that ψ_2 can be interpreted as a centroaffine immersion contained in an (n_2+1)-dimensional vector subspace of $\mathbb{R}^{n+1}$ with induced connection ∇^2 and centroaffine metric

$$h^2 = \lambda_3(\lambda_3-\lambda_2)h. \tag{3.14}$$

As both subspaces are complementary, we may assume that, up to a linear transformation, the (n_1+1)-dimensional subspace is spanned by the first n_1+1 coordinates of $\mathbb{R}^{n+1}$, whereas the (n_2+1)-dimensional subspace is spanned by the last n_2+1 coordinates of $\mathbb{R}^{n+1}$.

Solving (3.12) for the immersion ψ, we have

$$\psi = \tfrac{1}{(\lambda_2-\lambda_3)f}\psi_1 + \tfrac{1}{(\lambda_2-\lambda_3)g}\psi_2 = (e^u\psi_1, e^{\frac{\lambda_3}{\lambda_2}u}\psi_2).$$

From Proposition 3.1 we see that ψ is given as the Calabi product of the immersions ψ_1 and ψ_2. Moreover, from (3.13) and (3.14), we know that both ψ_1 and ψ_2 are locally strongly convex.

We have completed the proof of Theorem 3.1. □

In Theorem 3.1, if additionally M has parallel cubic form, equivalently, $\hat\nabla K = 0$, then by the totally same proof as that of Theorem 3 in [11], we can prove the following theorem.

Theorem 3.2. *Let $\psi : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface. Assume that $\hat{\nabla}K = 0$ and there exist h-orthogonal distributions $\mathcal{D}_1$ (of dimension 1, spanned by a unit vector field T), $\mathcal{D}_2$ (of dimension n_1) and $\mathcal{D}_3$ (of dimension n_2) such that*

$$\begin{aligned} &K(T,T) = \lambda_1 T, \quad K(T,V) = \lambda_2 V, \quad K(T,W) = \lambda_3 W,\\ &\forall\, V \in \mathcal{D}_2, \quad W \in \mathcal{D}_3; \quad \lambda_1 \neq 2\lambda_2, \quad \lambda_1 \neq 2\lambda_3, \quad \lambda_2 \neq \lambda_3. \end{aligned} \tag{3.15}$$

Then $\psi : M^n \to \mathbb{R}^{n+1}$ can be locally decomposed as the Calabi product of two locally strongly convex centroaffine hypersurfaces $\psi_1 : M_1^{n_1} \to \mathbb{R}^{n_1+1}$ and $\psi_2 : M_2^{n_2} \to \mathbb{R}^{n_2+1}$ with parallel cubic form.

Similarly, as the converse of Proposition 3.2, we can prove the following theorem.

Theorem 3.3. *Let $\psi : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface. Assume that there exist two distributions $\mathcal{D}_1$ (of dimension 1, spanned by a unit vector field T), $\mathcal{D}_2$ (of dimension $n-1$) such that*

(i) *the centroaffine metric h induced by $-\varepsilon\psi$ ($\varepsilon = \pm 1$) is positive definite,*
(ii) *$\mathcal{D}_1$ and $\mathcal{D}_2$ are orthogonal with respect to the centroaffine metric h,*
(iii) *there exist constants λ_1 and λ_2 such that*

$$\begin{aligned} &K(T,T) = \lambda_1 T, \quad K(T,V) = \lambda_2 V, \quad \forall\, V \in \mathcal{D}_2;\\ &\lambda_1 \neq 2\lambda_2, \quad \lambda_1\lambda_2 - \lambda_2^2 = \varepsilon. \end{aligned} \tag{3.16}$$

Then $\psi : M^n \to \mathbb{R}^{n+1}$ can be locally decomposed as the Calabi product of a locally strongly convex centroaffine hypersurface $\psi_1 : M_1^{n-1} \to \mathbb{R}^n$ and a point.

Proof. First, it is easily seen from (3.16) that we have

$$\lambda_2 \neq 0.$$

Next, by a proof similar to those for Lemmas 5.6 and 5.7 in [9], we can prove the following lemma.

Lemma 3.2. *Under the assumptions of Theorem 3.3, for any vector $X \in TM$ and $V \in \mathcal{D}_2$, there hold*

$$\hat{\nabla}_X T = 0, \quad \hat{\nabla}_X V \in \mathcal{D}_2.$$

From Lemma 3.2, applying the de Rham decomposition theorem, we see that (M,h) is locally isometric with $\mathbb{R} \times M_1$ such that T is tangent to $\mathbb{R}$ and $\mathcal{D}_2$ is tangent to M_1.

The above product structure of M implies the existence of local coordinates (u,p) for M based on an open subset containing the origin of $\mathbb{R}^n$, such that $\mathcal{D}_1$ is given by $dp = 0$ and $\mathcal{D}_2$ is given by $du = 0$. We may assume that $T = \lambda_2 \frac{\partial}{\partial u}$. Put

$$\psi_1 = f\big(T - \tfrac{\varepsilon}{\lambda_2}\psi\big), \quad \psi_2 = g(\lambda_2\psi - T), \tag{3.17}$$

where f and g are assumed to be nonzero functions which depend only on the variable u, and are given by

$$f(u) = \tfrac{1}{2\lambda_2-\lambda_1}e^{-u}, \quad g(u) = \tfrac{1}{2\lambda_2-\lambda_1}e^{\frac{\lambda_2-\lambda_1}{\lambda_2}u}.$$

It follows from (3.17) that

$$\begin{aligned} D_T\psi_1 &= -\lambda_2 f\big(T - \tfrac{\varepsilon}{\lambda_2}\psi\big) + f\big(D_T T - \tfrac{\varepsilon}{\lambda_2}D_T\psi\big) \\ &= f\big(-\lambda_2 + \lambda_1 - \tfrac{\varepsilon}{\lambda_2}\big)T \\ &= 0. \end{aligned}$$

Similarly

$$\begin{aligned} &D_T\psi_2 = D_V\psi_2 = 0, \\ &d\psi_1(V) = D_V\psi_1 = (2\lambda_2 - \lambda_1)fV. \end{aligned}$$

The above relations imply that ψ_1 reduces to a map of M_1 in $\mathbb{R}^{n+1}$. Whereas ψ_2 is a constant vector in $\mathbb{R}^{n+1}$. Moreover, denoting by ∇^1 the $\mathcal{D}_2$ component of ∇, we find that

$$\begin{aligned} D_V d\psi_1(\tilde{V}) &= f(2\lambda_2 - \lambda_1)D_V\tilde{V} \\ &= f(2\lambda_2 - \lambda_1)\Big(\nabla^1_V\tilde{V} - \varepsilon h(V,\tilde{V})\psi + \lambda_2 h(V,\tilde{V})T\Big) \\ &= d\psi_1(\nabla^1_V\tilde{V}) + (2\lambda_2 - \lambda_1)\lambda_2 h(V,\tilde{V})\psi_1. \end{aligned}$$

Hence ψ_1 can be interpreted as a centroaffine immersion contained in an n-dimensional vector subspace of $\mathbb{R}^{n+1}$ with induced connection ∇^1 and affine metric

$$h^1 = \lambda_2(2\lambda_2 - \lambda_1)h. \tag{3.18}$$

As ψ_2 is transversal to the immersion ψ_1, we may assume by a linear transformation that ψ_1 lies in the space spanned by the first n coordinates of $\mathbb{R}^{n+1}$, whereas the constant vector ψ_2 lies in the direction of the last coordinate.

Solving (3.17) for the immersion ψ, we have

$$\psi = \left(e^u\psi_1, e^{\frac{\lambda_1-\lambda_2}{\lambda_2}u}\psi_2\right).$$

From Proposition 3.2 we see that ψ is given as the Calabi product of the immersion ψ_1 and a point. Moreover, from (3.18), we know that ψ_1 is a locally strongly convex centroaffine hypersurface.

This completes the proof of Theorem 3.3. □

Similarly, if M in Theorem 3.3 is assumed additionally having parallel cubic form, then as deriving Theorem 4 in [11], we can prove the following theorem.

Theorem 3.4. *Let $\psi : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface. Assume that $\hat{\nabla} K = 0$ and there exist h-orthogonal distributions $\mathcal{D}_1$ (of dimension 1, spanned by a unit vector field T) and $\mathcal{D}_2$ (of dimension $n-1$) such that*

$$K(T,T) = \lambda_1 T,\ K(T,V) = \lambda_2 V,\ \forall V \in D_2;\quad \lambda_1 \neq 2\lambda_2. \tag{3.19}$$

Then $\psi : M^n \to \mathbb{R}^{n+1}$ can be locally decomposed as the Calabi product of a locally strongly convex centroaffine hypersurface $\psi_1 : M_1^{n-1} \to \mathbb{R}^n$ with parallel cubic form and a point.

4. Elementary Discussions in Terms of a Typical Basis

In this section, we consider an n-dimensional ($n \geq 2$) locally strongly convex centroaffine hypersurface M^n in $\mathbb{R}^{n+1}$ with $\hat{\nabla} C = 0$ and we choose ε such that the centroaffine metric h is positive definite. Our method here follows closely that of [10,12].

Since $\hat{\nabla} C = 0$ implies that $h(C,C)$ is constant, there are two cases. First, if $h(C,C) = 0$, as h being positive definite we have $C = 0$ and M^n is an open part of a hyperquadric which is centered at the origin. If otherwise, $h(C,C) \neq 0$, then C never vanishes. We assume this for the remainder of this section.

4.1. The Construction of the Typical Basis

Let $p \in M$ and $UM_p = \{u \in T_pM \mid h(u,u) = 1\}$. We define a function on UM_p by $f(u) = h(K_u u, u)$. Let e_1 be an element of UM_p at which the function $f(u)$ attains an absolute maximum. The following lemma about the construction of the typical basis can be proved totally similar to that of [10] (see also [22] for its earlier version).

Lemma 4.1 (see p. 191 of [10]). *There exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM satisfying:*

(i) *$K_{e_1} e_i = \lambda_i e_i$, for $i = 1, \ldots, n$, where λ_1 ($\lambda_1 > 0$) is the maximum of f. Moreover, for $i \geq 2$, the value of λ_i satisfies*

$$(\lambda_1 - 2\lambda_i)(\varepsilon - \lambda_1\lambda_i + \lambda_i^2) = 0. \tag{4.1}$$

(ii) *for $i \geq 2$, if $\lambda_1 = 2\lambda_i$, then $f(e_i) = 0$; if $\lambda_1 \neq 2\lambda_i$, then $\lambda_1^2 - 4\varepsilon > 0$ and $\lambda_i = \mu := \frac{1}{2}(\lambda_1 - \sqrt{\lambda_1^2 - 4\varepsilon})$.*

According to Lemma 4.1, for a locally strongly convex centroaffine hypersurface with parallel cubic form, we have to deal with the following $(n+1)$-cases:

Case $\mathfrak{C}_1$. $\lambda_1^2 - 4\varepsilon > 0$ and $\lambda_2 = \cdots = \lambda_n = \mu$.
Case $\mathfrak{C}_m$. $\lambda_1^2 - 4\varepsilon > 0$ and for some m ($2 \leq m \leq n-1$),

$$\lambda_2 = \cdots = \lambda_m = \tfrac{1}{2}\lambda_1,\quad \lambda_{m+1} = \cdots = \lambda_n = \mu.$$

Case $\mathfrak{C}_n$. $\lambda_1^2 - 4\varepsilon \neq 0$ and $\lambda_2 = \cdots = \lambda_n = \frac{1}{2}\lambda_1$.
Case $\mathfrak{B}$. $\lambda_1^2 - 4\varepsilon = 0$ and $\lambda_2 = \cdots = \lambda_n = \frac{1}{2}\lambda_1$.
In sequel of this paper, we are going to discuss these cases separately.

4.2. The Settlement of the Cases $\mathfrak{C}_1$ and $\mathfrak{C}_n$

First of all, about Case $\mathfrak{C}_1$, we have the following

Theorem 4.1. *If Case $\mathfrak{C}_1$ occurs, then M^n can be locally decomposed as the Calabi product of an $(n-1)$-dimensional locally strongly convex centroaffine hypersurface in $\mathbb{R}^n$ with parallel cubic form and a point.*

Proof. In Case $\mathfrak{C}_1$, the difference tensor takes the following form:

$$K(e_1, e_1) = \lambda_1 e_1, \quad K(e_1, e_i) = \mu e_i, \quad i = 2, \ldots, n.$$

By parallel translation along geodesics (with respect to $\hat{\nabla}$) through p, we extend $\{e_1, \ldots, e_n\}$ to obtain a local h-orthonormal basis denoted by $\{E_1, \ldots, E_n\}$. Then

$$K(E_1, E_1) = \lambda_1 E_1, \quad K(E_1, E_i) = \mu E_i, \quad i = 2, \ldots, n, \ \lambda_1 \neq 2\mu,$$

where both λ_1 and μ are defined in Lemma 4.1. Applying Theorem 3.4, we conclude that M^n can be decomposed as the Calabi product of a locally strongly convex centroaffine hypersurface with parallel cubic form and a point. □

Theorem 4.2. *Case $\mathfrak{C}_n$ does not occur.*

Proof. Suppose on the contrary that Case $\mathfrak{C}_n$ does occur. From (ii) of Lemma 4.1, we have $f(v) = 0$ for any $v \in \operatorname{span}\{e_2, \ldots, e_n\}$. Then, by polarization, we can show that

$$h(K_{e_i} e_j, e_k) = 0, \quad 2 \leq i, j, k \leq n. \tag{4.2}$$

Then, for any unit vector $v \in \operatorname{span}\{e_2, \ldots, e_n\}$, we have

$$K_{e_1} e_1 = \lambda_1 e_1, \ K_{e_1} v = \tfrac{1}{2}\lambda_1 v, \ K_v v = \tfrac{1}{2}\lambda_1 e_1.$$

Accordingly, by taking $X = e_1, \ Y = Z = U = v$ in (2.5), we will get $\lambda_1 = 0$. This contradiction completes the proof of Theorem 4.2. □

4.3. Intermediary Cases $\{\mathfrak{C}_m\}_{2 \leq m \leq n-1}$ and an Isotropic Mapping L

Now, we consider the cases $\{\mathfrak{C}_m\}_{2 \leq m \leq n-1}$. In these cases, we denote by $\mathcal{D}_2$ and $\mathcal{D}_3$ the two subspaces of T_pM:

$$\mathcal{D}_2 = \operatorname{span}\{e_2, \ldots, e_m\} \ \text{ and } \ \mathcal{D}_3 = \operatorname{span}\{e_{m+1}, \ldots, e_n\}.$$

First of all, we have the following

Lemma 4.2. *Associated with the direct sum decomposition $T_pM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$, where $\mathcal{D}_1 = \operatorname{span}\{e_1\}$, there hold the relations:*

(i) $K_{e_1} v = \frac{1}{2}\lambda_1 v$, *for any* $v \in \mathcal{D}_2$; $K_{e_1} w = \mu w$, *for any* $w \in \mathcal{D}_3$.
(ii) $K_{v_1} v_2 - \frac{1}{2}\lambda_1 h(v_1, v_2) e_1 \in \mathcal{D}_3$, *for any* $v_1, v_2 \in \mathcal{D}_2$.
(iii) $K_v w \in \mathcal{D}_2$, *for any* $v \in \mathcal{D}_2$, $w \in \mathcal{D}_3$.

Proof. By definition we have (i). The claim (ii) follows from (ii) of Lemma 4.1 or directly (4.2). In order to prove the third claim, we take $X = v \in \mathcal{D}_2$, $Y = w \in \mathcal{D}_3$ and $Z = U = e_1$ in (2.5) to obtain that

$$\lambda_1 \hat{R}(v, w)e_1 = 2K(\hat{R}(v, w)e_1, e_1).$$

Thus we have $\hat{R}(v, w)e_1 \in \mathcal{D}_2$.

On the other hand, a direct calculation by (2.3) gives

$$\hat{R}(v, w)e_1 = -K_v K_w e_1 + K_w K_v e_1 = \big(\tfrac{1}{2}\lambda_1 - \mu\big)K_v w.$$

Therefore, as $\mu \neq \frac{1}{2}\lambda_1$, combining with the preceding result we get $K_v w \in \mathcal{D}_2$. □

With the remarkable conclusions of Lemma 4.2, similar to that in [12], we can now introduce a bilinear map $L : \mathcal{D}_2 \times \mathcal{D}_2 \to \mathcal{D}_3$, defined by

$$L(v_1, v_2) := K_{v_1} v_2 - \tfrac{1}{2}\lambda_1 h(v_1, v_2)e_1, \ \ v_1, v_2 \in \mathcal{D}_2. \tag{4.3}$$

The following lemmas show that the operator L enjoys remarkable properties and it becomes an important tool for exploring information of the difference tensor. As we have $\lambda_1^2 - 4\varepsilon > 0$, for simplicity, from now on we denote $\eta := \frac{1}{2}\sqrt{\lambda_1^2 - 4\varepsilon}$.

Lemma 4.3. *The bilinear map L is isotropic in the sense that*

$$h(L(v, v), L(v, v)) = \tfrac{1}{2}\lambda_1 \eta h(v, v)^2, \quad \forall\, v \in \mathcal{D}_2. \tag{4.4}$$

Moreover, linearizing (4.4), it follows for arbitrary $v_1, v_2, v_3, v_4 \in \mathcal{D}_2$ that

$$\begin{aligned} &h(L(v_1, v_2), L(v_3, v_4)) + h(L(v_1, v_3), L(v_2, v_4)) + h(L(v_1, v_4), L(v_2, v_3)) \\ &\quad = \tfrac{1}{2}\lambda_1 \eta (h(v_1, v_2)h(v_3, v_4) + h(v_1, v_3)h(v_2, v_4) + h(v_1, v_4)h(v_2, v_3)). \end{aligned} \tag{4.5}$$

Proof. We use (2.5) and take $X = e_1$ and $Y = v_1,\ Z = v_2,\ U = v_3$ in $\mathcal{D}_2$. By using (2.3) and the definition of L, it follows immediately that

$$\begin{aligned} &K(L(v_1, v_2), v_3) + K(L(v_1, v_3), v_2) + K(L(v_2, v_3), v_1) \\ &\quad = \tfrac{1}{2}\lambda_1 \eta (h(v_1, v_2)v_3 + h(v_1, v_3)v_2 + h(v_2, v_3)v_1). \end{aligned} \tag{4.6}$$

Taking the product of (4.6) with $v_4 \in \mathcal{D}_2$, we can obtain (4.5). Finally, we choose $v_1 = v_2 = v_3 = v_4 = v$ in (4.5), then we get (4.4). □

Since $L : \mathcal{D}_2 \times \mathcal{D}_2 \to \mathcal{D}_3$ is isotropic, we see from (4.4) that, if $\dim \mathcal{D}_2 \geq 1$, then the image space of L has positive dimension, i.e. $\dim(\operatorname{Im} L) \geq 1$. Moreover, the following well-known properties hold.

Lemma 4.4 (cf. [10,12]). *If* $\dim \mathcal{D}_2 \geq 1$*, for orthonormal vectors v_1, v_2, v_3 and $v_4 \in \mathcal{D}_2$, there hold*

$$h(L(v_1, v_1), L(v_1, v_2)) = 0, \tag{4.7}$$

$$h(L(v_1, v_1), L(v_2, v_2)) + 2h(L(v_1, v_2), L(v_1, v_2)) = \tfrac{1}{2}\lambda_1 \eta, \tag{4.8}$$

$$h(L(v_1, v_1), L(v_2, v_3)) + 2h(L(v_1, v_2), L(v_1, v_3)) = 0, \tag{4.9}$$

$$\begin{aligned} &h(L(v_1,v_2),L(v_3,v_4))+h(L(v_1,v_3),L(v_2,v_4))\\ &\quad+h(L(v_1,v_4),L(v_2,v_3))=0. \end{aligned} \tag{4.10}$$

Lemma 4.5. *In Cases* $\{\mathfrak{C}_m\}_{2\le m\le n-1}$*, if it occurs that* $\operatorname{Im} L\neq \mathcal{D}_3$*, then for any* $v_1,\ v_2\in\mathcal{D}_2$ *and* $w\in\mathcal{D}_3$ *with* $w\perp \operatorname{Im} L$*, we have*

$$K(L(v_1,v_2),w)=\eta\mu h(v_1,v_2)w. \tag{4.11}$$

Proof. For every $v\in\mathcal{D}_2$ and $w\perp\operatorname{Im} L$, we apply (iii) of Lemma 4.2 and (2.3) to obtain

$$\begin{aligned} K(v,w)&=\sum_{i=2}^{m}h(K(v,w),e_i)e_i=\sum_{i=2}^{m}h(K(v,e_i),w)e_i\\ &=\sum_{i=2}^{m}h(L(v,e_i),w)e_i=0,\\ \hat{R}(e_1,v)w&=-K(K(v,w),e_1)+K(v,K(e_1,w))=0. \end{aligned}$$

Then, for v_1,v_2 and w as in the assumptions, the following equation

$$\hat{R}(e_1,v_1)K(v_2,w)=K(\hat{R}(e_1,v_1)v_2,w)+K(v_2,\hat{R}(e_1,v_1)w)$$

becomes equivalent to $K(\hat{R}(e_1,v_1)v_2,w)=0$. On the other hand, direct calculation gives that

$$\begin{aligned} \hat{R}(e_1,v_1)v_2&=\varepsilon h(v_1,v_2)e_1-K(e_1,K(v_1,v_2))+K(v_1,K(e_1,v_2))\\ &=\varepsilon h(v_1,v_2)e_1-K(e_1,L(v_1,v_2)+\tfrac{1}{2}\lambda_1 h(v_1,v_2)e_1)\\ &\quad+\tfrac{1}{2}\lambda_1(L(v_1,v_2)+\tfrac{1}{2}\lambda_1 h(v_1,v_2)e_1)\\ &=-\eta^2 h(v_1,v_2)e_1+\eta L(v_1,v_2). \end{aligned}$$

Then (4.11) immediately follows. □

Lemma 4.6. *In Cases* $\{\mathfrak{C}_m\}_{2\le m\le n-1}$*, let* $v_1,v_2,v_3,v_4\in\mathcal{D}_2$ *and* $\{u_1,\dots,u_{m-1}\}$ *be an orthonormal basis of* $\mathcal{D}_2$*, then we have*

$$\begin{aligned} K(L(v_1,v_2),L(v_3,v_4))&=\mu h(L(v_1,v_2),L(v_3,v_4))e_1+\mu\eta h(v_1,v_2)L(v_3,v_4)\\ &\quad+\sum_{i=1}^{m-1}h(L(v_1,u_i),L(v_3,v_4))L(u_i,v_2)\\ &\quad+\sum_{i=1}^{m-1}h(L(v_2,u_i),L(v_3,v_4))L(u_i,v_1). \end{aligned} \tag{4.12}$$

Proof. By (2.5), we have, for $v_1,v_2,v_3,v_4\in\mathcal{D}_2$, that

$$\begin{aligned} &\hat{R}(e_1,v_1)K(v_2,L(v_3,v_4))\\ &\quad=K(\hat{R}(e_1,v_1)v_2,L(v_3,v_4))+K(v_2,\hat{R}(e_1,v_1)L(v_3,v_4)). \end{aligned} \tag{4.13}$$

Applying (2.3) for $v_1, v_2 \in \mathcal{D}_2$, we obtain that

$$\hat{R}(e_1, v_1)v_2 = -\eta^2 h(v_1, v_2)e_1 + \eta L(v_1, v_2). \tag{4.14}$$

Similarly, for $v \in \mathcal{D}_2$ and $w \in \mathcal{D}_3$, we have that

$$\hat{R}(e_1, v)w = -\eta K(v, w). \tag{4.15}$$

By Lemma 4.2, $K(v_2, L(v_3, v_4)) \in \mathcal{D}_2$ and we can write

$$K(v_2, L(v_3, v_4)) = \sum_{i=1}^{m-1} h(L(v_2, u_i), L(v_3, v_4))u_i. \tag{4.16}$$

Now, we can compute both sides of (4.13) to obtain

$$\begin{aligned}
\text{LHS} &= -\eta^2 h(L(v_1, v_2), L(v_3, v_4))e_1 + \eta \sum_{i=1}^{m-1} h(L(v_2, u_i), L(v_3, v_4))L(u_i, v_1),\\
\text{RHS} &= -\mu\eta^2 h(v_1, v_2)L(v_3, v_4) + \eta K(L(v_1, v_2), L(v_3, v_4))\\
&\quad - \tfrac{1}{2}\lambda_1 \eta h(L(v_1, v_2), L(v_3, v_4))e_1\\
&\quad - \eta \sum_{i=1}^{m-1} h(L(v_1, u_i), L(v_3, v_4))L(u_i, v_2).
\end{aligned}$$

From these computations we immediately get (4.12). □

We note that (4.12) has very important consequences which will be used in sequel sections. For example, we have

Lemma 4.7. *For Case $\mathfrak{C}_m$ with $m \geq 3$, let $\{u_1, \ldots, u_{m-1}\}$ be an orthonormal basis of $\mathcal{D}_2$, then for $p \neq j$, we have*

$$\begin{aligned}
0 = &\big(\eta(\eta + \tfrac{1}{2}\lambda_1) - 4h(L(u_j, u_p), L(u_j, u_p))\big)L(u_j, u_p)\\
&- \sum_{i \neq p} 4h(L(u_j, u_i), L(u_j, u_p))L(u_i, u_j).
\end{aligned} \tag{4.17}$$

In particular, if $L(u_1, u_2) \neq 0$ and $L(u_1, u_i)$ is orthogonal to $L(u_1, u_2)$ for all $i \neq 2$, then

$$h(L(u_1, u_2), L(u_1, u_2)) = \tfrac{1}{4}\eta(\eta + \tfrac{1}{2}\lambda_1) =: \tau. \tag{4.18}$$

Proof. By (4.12), interchanging the couples of indices $\{1,2\}$ and $\{3,4\}$ we find the following condition:

$$\begin{aligned} 0 = {} & \mu\eta\big(h(v_1,v_2)L(v_3,v_4) - h(v_3,v_4)L(v_1,v_2)\big) \\ & + \sum_{i=1}^{m-1} h(L(v_1,u_i),L(v_3,v_4))L(u_i,v_2) \\ & + \sum_{i=1}^{m-1} h(L(v_2,u_i),L(v_3,v_4))L(u_i,v_1) \\ & - \sum_{i=1}^{m-1} h(L(v_3,u_i),L(v_1,v_2))L(u_i,v_4) \\ & - \sum_{i=1}^{m-1} h(L(v_4,u_i),L(v_1,v_2))L(u_i,v_3). \end{aligned} \tag{4.19}$$

If we take $v_2 = v_3 = v_4 = u_j$ and $v_1 = u_p$ with $j \neq p$, then by using also the isotropy condition, (4.19) reduces to (4.17). Taking $j = 1$ and $p = 2$ in (4.17), we obtain (4.18). □

4.4. The Mapping $P_v : \mathcal{D}_2 \to \mathcal{D}_2$ with Unit Vector $v \in \mathcal{D}_2$

We now define for any given unit vector $v \in \mathcal{D}_2$ a linear map $P_v : \mathcal{D}_2 \to \mathcal{D}_2$ by

$$P_v\tilde{v} = K_vL(v,\tilde{v}), \quad \forall\, \tilde{v} \in \mathcal{D}_2. \tag{4.20}$$

It is easily seen that P_v is a symmetric linear operator satisfying

$$h(P_v\tilde{v},v') = h(L(v,\tilde{v}),L(v,v')) = h(P_vv',\tilde{v}), \tag{4.21}$$

for any $\tilde{v}$, $v' \in \mathcal{D}_2$. Moreover, we have

Lemma 4.8. *For any unit vector $v \in \mathcal{D}_2$, the operator $P_v : \mathcal{D}_2 \to \mathcal{D}_2$ has $\sigma = \frac{1}{2}\lambda_1\eta$ as an eigenvalue with eigenvector v. In the orthogonal complement $\{v\}^\perp$ of $\{v\}$ in $\mathcal{D}_2$ the operator P_v has at most two eigenvalues, namely* 0 *and τ, defined as in* (4.18).

Proof. By (4.4), we have

$$h(P_vv,v) = h(L(v,v),L(v,v)) = \tfrac{1}{2}\lambda_1\eta. \tag{4.22}$$

Taking $v' \perp v$, we get

$$h(P_vv,v') = h(L(v,v'),L(v,v)) = 0. \tag{4.23}$$

(4.22) and (4.23) imply that $P_vv = \frac{1}{2}\lambda_1\eta v$.

Next, we take an orthonormal basis $\{u_1,\ldots,u_{m-1}\}$ of $\mathcal{D}_2$ consisting of eigenvectors of P_v such that $P_vu_i = \sigma_iu_i$, $i = 1,\ldots,m-1$, with $u_1 = v$ and $\sigma_1 = \sigma$. We take the inner product of (4.17) with $L(u_1,u_p)$ for $j = 1$ and any $p \geq 2$. We obtain that

$$h(L(u_1,u_p),L(u_1,u_p))\big(\tau - h(L(u_1,u_p),L(u_1,u_p))\big) = 0, \quad p \geq 2. \tag{4.24}$$

Here, to derive (4.24), we have used that

$$h(L(u_1, u_p), L(u_1, u_i)) = h(u_p, P_{u_1} u_i) = 0,\ i \neq p.$$

From (4.24), we immediately get the remaining assertion. □

In the following we denote by $V_v(0)$ and $V_v(\tau)$ the eigenspaces of P_v (in the orthogonal complement of $\{v\}$) with respect to the eigenvalues 0 and τ, respectively. Note that in exceptional cases it can happen that $\sigma = \tau$.

Lemma 4.9. *Let $v, u \in \mathcal{D}_2$ be two unit orthogonal vectors. Then the following statements are equivalent:*

(i) $u \in V_v(0)$.
(ii) $L(u, v) = 0$.
(iii) $L(u, u) = L(v, v)$.
(iv) $v \in V_u(0)$.

Moreover, any of the previous statements implies that

(v) $P_v = P_u$ *on* $\{u, v\}^{\perp}$.

Proof. As $h(P_v u, u) = h(L(v, u), L(v, u)) = h(P_u v, v)$, the equivalence of (i), (ii) and (iv) follows immediately. As u and v are orthogonal, (4.4) and (4.8) imply that

$$h(L(v, v) - L(u, u), L(v, v) - L(u, u)) = 4h(L(v, u), L(v, u)).$$

It follows that (ii) is equivalent to (iii).

Now we assume that (i), (ii), (iii) and (iv) are satisfied. In order to prove (v), we see that the space spanned by $\{u, v\}$ is invariant by P_v and P_u, also its orthogonal complement is invariant. By taking $v_1, v_2 \in \{u, v\}^{\perp}$ and using (4.6), we find

$$\begin{aligned} h(v_1, P_v v_2) &= h(L(v, v_1), L(v, v_2)) \\ &= -\tfrac{1}{2} h(L(v, v), L(v_1, v_2)) + \tfrac{1}{4}\lambda_1 \eta h(v_1, v_2) \\ &= -\tfrac{1}{2} h(L(u, u), L(v_1, v_2)) + \tfrac{1}{4}\lambda_1 \eta h(v_1, v_2) \\ &= h(v_1, P_u v_2). \end{aligned}$$

This completes the proof. □

Lemma 4.10. *Let $v, \tilde{v} \in \mathcal{D}_2$ be two unit orthogonal vectors, then*

$$h(L(v, \tilde{v}), L(v, \tilde{v})) = \tau \tag{4.25}$$

holds if and only if $\tilde{v} \in V_v(\tau)$. Moreover, if we assume $u \in V_v(0)$ and the equality in (4.25) holds, then $u \in V_{\tilde{v}}(\tau)$.

Proof. If $\tilde{v} \in V_v(\tau)$, then $h(L(v, \tilde{v}), L(v, \tilde{v})) = h(\tilde{v}, P_v \tilde{v}) = \tau$.

Conversely, if $h(L(v, \tilde{v}), L(v, \tilde{v})) = \tau$, we should consider the following three cases:

(i) $V_v(0) = \emptyset$. From Lemma 4.8, it is easily seen that $\tilde{v} \in V_v(\tau)$.

(ii) $V_v(\tau) = \emptyset$. In this case, Lemma 4.8 implies that $\tilde{v} \in V_v(0)$. By Lemma 4.9, we have $h(L(v, \tilde{v}), L(v, \tilde{v})) = 0$. This is a contradiction.

(iii) $V_v(0) \neq \emptyset$ and $V_v(\tau) \neq \emptyset$. We can write

$$\tilde{v} = \cos\theta v_0 + \sin\theta v_1, \quad h(v_0, v_0) = h(v_1, v_1) = 1,$$

where $v_0 \in V_v(0)$ and $v_1 \in V_v(\tau)$. Then we get

$$\tau = h(L(v, \tilde{v}), L(v, \tilde{v})) = \sin^2\theta\tau,$$

which means that $\sin\theta = \pm 1$ and $\cos\theta = 0$. Therefore, $\tilde{v} \in V_v(\tau)$.

Taking unit vector $u \in V_v(0)$, we have $L(u, u) = L(v, v)$. Consequently,

$$\begin{aligned} &h(L(\tilde{v}, u), L(\tilde{v}, u)) \\ &\quad = -\tfrac{1}{2}h(L(\tilde{v}, \tilde{v}), L(u, u)) + \tfrac{1}{4}\lambda_1\eta \\ &\quad = -\tfrac{1}{2}h(L(v, v), L(\tilde{v}, \tilde{v})) + \tfrac{1}{4}\lambda_1\eta \\ &\quad = h(v, P_{\tilde{v}}v) = \tau. \end{aligned}$$

Applying the first assertion of Lemma 4.10, we have $u \in V_{\tilde{v}}(\tau)$. □

Lemma 4.11. *Let $v_1, v_2, v_3 \in \mathcal{D}_2$ be orthonormal vectors satisfying $v_1, v_2 \in V_{v_3}(\tau)$, then for any vector $v \in \mathcal{D}_2$, we have $h(L(v_1, v_2), L(v, v_3)) = 0$.*

Proof. Using the linearity of the assertion with v, we may assume that v is an eigenvector of P_{v_3}. Let $\{u_1, \ldots, u_{m-1}\}$ be an orthonormal basis of $\mathcal{D}_2$ consisting of eigenvectors of P_{v_3} such that $u_1 = v_1,\ u_2 = v_2$ and $u_3 = v_3$. We now use (4.19) for $v_3 = v_4$ to obtain

$$\begin{aligned} 0 = &- \mu\eta L(v_1, v_2) + \sum_{i=1}^{m-1} h(L(v_1, u_i), L(v_3, v_3))L(u_i, v_2) \\ &+ \sum_{i=1}^{m-1} h(L(v_2, u_i), L(v_3, v_3))L(u_i, v_1) \\ &- 2\sum_{i=1}^{m-1} h(L(v_3, u_i), L(v_1, v_2))L(u_i, v_3). \end{aligned} \tag{4.26}$$

On the other hand, from (4.7)–(4.9), we have

$$\begin{aligned} &h(L(v_1, u_i), L(v_3, v_3)) = h(L(v_2, u_j), L(v_3, v_3)) = 0,\ i \neq 1, j \neq 2, \\ &h(L(v_1, v_1), L(v_3, v_3)) = h(L(v_2, v_2), L(v_3, v_3)) = \tfrac{1}{2}\lambda_1\eta - 2\tau. \end{aligned}$$

Inserting the above into (4.26), we obtain

$$0 = \sum_{i=1}^{m-1} h(L(v_3, u_i), L(v_1, v_2))L(u_i, v_3). \tag{4.27}$$

Since $h(L(u_i,v_3),L(u_j,v_3)) = h(P_{v_3}u_i,u_j) = 0$ if $i \neq j$, the equation (4.27) implies that $h(L(v_3,u_i),L(v_1,v_2)) = 0$ holds for all $u_i \notin V_{v_3}(0)$. Combining with Lemma 4.9, this immediately shows that for any vector $v \in \mathcal{D}_2$, we have $h(L(v_1,v_2),L(v,v_3)) = 0$. $\square$

4.5. Direct Sum Decomposition for $\mathcal{D}_2$

For our purpose, a crucial matter is to introduce a direct sum decomposition for $\mathcal{D}_2$ based on the preceding Lemmas. First, pick any unit vector $v_1 \in \mathcal{D}_2$ and recall that $\tau = \frac{1}{4}\eta(\eta + \frac{1}{2}\lambda_1)$, then by Lemma 4.8, we have a direct sum decomposition for $\mathcal{D}_2$:

$$\mathcal{D}_2 = \{v_1\} \oplus V_{v_1}(0) \oplus V_{v_1}(\tau),$$

where, here and later on, we denote also by $\{\cdot\}$ the vector space spanned by its elements. If $V_{v_1}(\tau) \neq \emptyset$, we take an arbitrary unit vector $v_2 \in V_{v_1}(\tau)$. Then by Lemma 4.10 we have:

$$v_1 \in V_{v_2}(\tau), \quad V_{v_1}(0) \subset V_{v_2}(\tau) \ \text{ and } \ V_{v_2}(0) \subset V_{v_1}(\tau).$$

From this we deduce that

$$\mathcal{D}_2 = \{v_1\} \oplus V_{v_1}(0) \oplus \{v_2\} \oplus V_{v_2}(0) \oplus \big(V_{v_1}(\tau) \cap V_{v_2}(\tau)\big).$$

If $V_{v_1}(\tau) \cap V_{v_2}(\tau) \neq \emptyset$, we further pick a unit vector $v_3 \in V_{v_1}(\tau) \cap V_{v_2}(\tau)$. Then

$$\mathcal{D}_2 = \{v_3\} \oplus V_{v_3}(0) \oplus V_{v_3}(\tau),$$

and by Lemma 4.10 we have

$$v_1, v_2 \in V_{v_3}(\tau); \ V_{v_1}(0), \ V_{v_2}(0) \subset V_{v_3}(\tau).$$

It follows that

$$\mathcal{D}_2 = \{v_1\} \oplus V_{v_1}(0) \oplus \{v_2\} \oplus V_{v_2}(0) \oplus \{v_3\} \oplus V_{v_3}(0) \oplus \big(V_{v_1}(\tau) \cap V_{v_2}(\tau) \cap V_{v_3}(\tau)\big).$$

Considering that $\dim(\mathcal{D}_2) = m - 1$ is finite, by induction, we get

Proposition 4.1. *In Cases $\{\mathfrak{C}_m\}_{2\le m\le n-1}$, there exists an integer k_0 and unit vectors $v_1, \ \ldots, \ v_{k_0} \in \mathcal{D}_2$ such that*

$$\mathcal{D}_2 = \{v_1\} \oplus V_{v_1}(0) \oplus \cdots \oplus \{v_{k_0}\} \oplus V_{v_{k_0}}(0). \tag{4.28}$$

In what follows, we will study the decomposition (4.28) in more details.

Lemma 4.12. (i) *For any unit vector $u_1 \in \{v_1\} \oplus V_{v_1}(0)$, we have*

$$\{v_1\} \oplus V_{v_1}(0) = \{u_1\} \oplus V_{u_1}(0).$$

(ii) *For any orthonormal vectors $u_1, \tilde{u}_1 \in \{v_1\} \oplus V_{v_1}(0)$, we have $L(u_1,\tilde{u}_1) = 0$.*

Proof. (i) We first assume the special case that $u_1 \perp v_1$. Then we have $u_1 \in V_{v_1}(0)$ and thus $L(u_1, v_1) = 0$, hence $v_1 \in V_{u_1}(0)$. Let $u \in V_{v_1}(0)$ and write $u = x_1 u_1 + u'$ with $u' \perp u_1$. By (v) in Lemma 4.9 we have $P_{u_1} u' = P_{v_1} u' = P_{v_1}(u - x_1 u_1) = 0$. Therefore, $u' \in V_{u_1}(0)$ and $\{v_1\} \oplus V_{v_1}(0) \subset \{u_1\} \oplus V_{u_1}(0)$. Similarly, we obtain $\{u_1\} \oplus V_{u_1}(0) \subset \{v_1\} \oplus V_{v_1}(0)$.

Next we consider the general case in three subcases. (a) If $V_{v_1}(0) = \emptyset$, there is nothing to prove. (b) If $\dim(V_{v_1}(0)) \geq 2$, we can take a vector $\tilde{u} \in V_{v_1}(0)$ which is orthogonal to both u_1 and v_1. Applying twice the previous result then completes the proof. (c) If $\dim(V_{v_1}(0)) = 1$, there exists $v_0 \in V_{v_1}(0)$ such that $V_{v_1}(0) = \{v_0\}$. Denote $u_1 = \cos\theta v_1 + \sin\theta v_0$. By Lemma 4.9, we see that

$$L(\cos\theta v_1 + \sin\theta v_0, \cos\theta v_0 - \sin\theta v_1) = 0,$$

thus $\cos\theta\, v_0 - \sin\theta\, v_1 \in V_{u_1}(0)$. Therefore, $\{v_1\} \oplus V_{v_1}(0) \subset \{u_1\} \oplus V_{u_1}(0)$. If $\{v_1\} \oplus V_{v_1}(0) \subsetneq \{u_1\} \oplus V_{u_1}(0)$, we have a unit vector $x \in \{u_1\} \oplus V_{u_1}(0)$ which is orthogonal to both u_1 and v_1. As $\{v_1\} \oplus V_{v_1}(0) = \{x\} \oplus V_x(0) = \{u_1\} \oplus V_{u_1}(0)$, we get a contradiction.

(ii) From (i) we have that $\{v_1\} \oplus V_{v_1}(0) = \{u_1\} \oplus V_{u_1}(0)$. As u_1 and $\tilde{u}_1$ are orthogonal, this implies that $\tilde{u}_1 \in V_{u_1}(0)$. Consequently, we have $L(u_1, \tilde{u}_1) = 0$. □

Lemma 4.13. *In the decomposition* (4.28)*, if we pick a unit vector* $u_2 \in V_{v_2}(0)$*, then there exists a unique vector* $u_1 \in \{v_1\} \oplus V_{v_1}(0)$ *such that* $L(u_1, v_2) = L(v_1, u_2)$*. Moreover,* u_1 *is a unit vector in* $V_{v_1}(0)$ *and* $L(v_1, v_2) = -L(u_1, u_2)$*.*

Proof. Let $u_1^l, \dots, u_{p_l}^l$ be an orthonormal basis of $V_{v_l}(0)$, $1 \leq l \leq k_0$, such that $u_1^2 = u_2$. Then

$$\{v_1, \dots, v_{k_0}, u_1^1, \dots, u_{p_1}^1, \dots, u_1^{k_0}, \dots, u_{p_{k_0}}^{k_0}\} =: \{\tilde{u}_i\}_{1 \leq i \leq m-1}$$

forms an orthonormal basis of $\mathcal{D}_2$. Now we use (4.12) with the vectors v_2, u_2, v_1, v_2. As by Lemma 4.9 $L(v_2, u_2) = 0$ and by our decomposition $v_1 \in V_{v_2}(\tau)$, we obtain

$$\begin{aligned}
0 &= K(L(v_2, u_2), L(v_1, v_2)) \\
&= \mu h(L(v_2, u_2), L(v_1, v_2)) e_1 + \sum_{i=1}^{m-1} h(L(v_2, \tilde{u}_i), L(v_1, v_2)) L(\tilde{u}_i, u_2) \\
&\quad + \sum_{i=1}^{m-1} h(L(u_2, \tilde{u}_i), L(v_1, v_2)) L(v_2, \tilde{u}_i) \\
&= \tau L(v_1, u_2) + \sum_{i=1}^{m-1} h(L(u_2, \tilde{u}_i), L(v_1, v_2)) L(v_2, \tilde{u}_i).
\end{aligned}$$

Let us take

$$u_1 = -\frac{1}{\tau}\sum_{i=1}^{m-1} h(L(u_2, \tilde{u}_i), L(v_1, v_2))\tilde{u}_i.$$

By Lemma 4.11, we have

$$h(L(u_2, \tilde{u}_i), L(v_1, v_2)) = 0, \quad \tilde{u}_i \notin \{v_1\} \oplus V_{v_1}(0) \oplus \{v_2\} \oplus V_{v_2}(0). \tag{4.29}$$

Applying (4.7) and Lemma 4.9, we get

$$h(L(u_2, \tilde{u}_i), L(v_1, v_2)) = 0, \quad \tilde{u}_i \in \{v_2\} \oplus V_{v_2}(0). \tag{4.30}$$

Moreover, note that $v_2 \in V_{v_1}(\tau)$, thus we have

$$h(L(u_2, v_1), L(v_1, v_2)) = 0. \tag{4.31}$$

It follows from (4.29), (4.30) and (4.31) that $u_1 \in V_{v_1}(0)$.

In order to prove the uniqueness of $u_1 \in \{v_1\} \oplus V_{v_1}(0)$, suppose that $\tilde{u}_1 \in \{v_1\} \oplus V_{v_1}(0)$ such that $L(\tilde{u}_1, v_2) = L(v_1, u_2)$, then we have $L(u_1 - \tilde{u}_1, v_2) = 0$. It follows from Lemma 4.9 that $u_1 - \tilde{u}_1 \in V_{v_2}(0)$. On the other hand, we also have $u_1 - \tilde{u}_1 \in \{v_1\} \oplus V_{v_1}(0)$; so we must have $u_1 = \tilde{u}_1$.

From the following fact

$$V_{v_1}(0) \subset V_{v_2}(\tau), \quad V_{v_2}(0) \subset V_{v_1}(\tau)$$

we have $h(u_1, u_1)\tau = h(L(u_1, v_2), L(u_1, v_2)) = h(L(v_1, u_2), L(v_1, u_2)) = \tau$. Hence, u_1 is a unit vector.

In order to prove the fact that $L(u_1, v_2) = L(v_1, u_2)$ and $L(v_1, v_2) = -L(u_1, u_2)$ are equivalent, we use (4.5) and the Cauchy-Schwarz inequality. In fact, if we first suppose $L(u_1, v_2) = L(v_1, u_2)$, then applying (4.5) we get

$$h(L(v_1, v_2), -L(u_1, u_2)) = h(L(v_1, u_2), L(v_2, u_1)) = h(L(v_2, u_1), L(v_2, u_1)) = \tau.$$

On the other hand, Lemma 4.12 implies that $v_1, u_1 \in V_{v_2}(\tau) = V_{u_2}(\tau)$ and thus

$$h(L(v_1, v_2), L(v_1, v_2)) = h(L(u_1, u_2), L(u_1, u_2)) = \tau.$$

Then, by Cauchy-Schwarz inequality we immediately have $L(v_1, v_2) = -L(u_1, u_2)$.

The converse can be proved in a similar way. □

To state the next lemma, we denote $V_l = \{v_l\} \oplus V_{v_l}(0)$ in the decomposition (4.28) for each $1 \le l \le k_0$. Then we have

Lemma 4.14. *With respect to the decomposition* (4.28)*, the following hold.*

(1) *For any unit vector* $a \in V_j$*,*

$$K(L(a, a), L(a, a)) = \tfrac{1}{2}\lambda_1\mu\eta e_1 + \eta(\mu + \lambda_1)L(a, a). \tag{4.32}$$

(2) *For $j \neq l$ and any unit vector $a \in V_j,\ b \in V_l$,*

$$K(L(a,a),L(a,b)) = \tfrac{1}{2}\eta(\mu+\lambda_1)L(a,b), \tag{4.33}$$

$$K(L(a,a),L(b,b)) = \tfrac{1}{2}\eta\mu^2 e_1 + \eta\mu(L(a,a)+L(b,b)), \tag{4.34}$$

$$K(L(a,b),L(a,b)) = \mu\tau e_1 + \tau(L(a,a)+L(b,b)). \tag{4.35}$$

(3) *For distinct $j,\ l,\ q,\ s$ and any unit vector $a \in V_j,\ b,\ b' \in V_l,\ c \in V_q,\ d \in V_s$, where b and b' are orthogonal, the following relations hold*

$$K(L(a,b),L(a,c)) = \tau L(b,c), \tag{4.36}$$

$$K(L(a,a),L(b,c)) = \eta\mu L(b,c), \tag{4.37}$$

$$K(L(a,b),L(a,b')) = 0, \tag{4.38}$$

$$K(L(a,b),L(c,d)) = 0. \tag{4.39}$$

(4) *For distinct $j,\ l,\ q$ and orthogonal unit vector $a_1,\ a_2 \in V_j$ and unit vectors $b \in V_l,\ c \in V_q$, it holds*

$$K(L(a_1,b),L(a_2,c)) = \tau L(b,c'), \tag{4.40}$$

where $c' \in V_q$ is the unique unit vector satisfying $L(a_1,c') = L(a_2,c)$.

Proof. Take an orthonormal basis of $\mathcal{D}_2$ such that it consists of the orthonormal basis of all $V_l,\ 1 \leq l \leq k_0$, the assertions are direct consequences of Lemma 4.6. Take for example, from the fact $h(L(a,b),L(a,c)) = h(P_a b,c) = 0$, eq. (4.6) and Lemma 4.6, we immediately get (4.36). As another example, from (4.36), Lemmas 4.12 and 4.13 we can get (4.40). □

Proposition 4.2. *In the decomposition* (4.28)*, if $k_0 = 1$, then* $\dim(\operatorname{Im} L) = 1$*. If $k_0 \geq 2$, then* $\dim V_{v_1}(0) = \cdots = \dim V_{v_{k_0}}(0)$ *and the dimension which we denote by $\mathfrak{p}$ can only be equal to 0, 1, 3 or 7.*

Proof. If $k_0 = 1$, from Lemmas 4.9 and 4.12 we see that $L(v_1,v_1)$ is a basis of the image $\operatorname{Im} L$, so we have $\dim(\operatorname{Im} L) = 1$. As a direct consequence of Lemma 4.13, for any $j \neq l$, we can define a one-to-one linear map from $V_{v_j}(0)$ to $V_{v_l}(0)$, which preserves the length of vectors. Hence $V_{v_j}(0)$ and $V_{v_l}(0)$ are isomorphic and have the same dimension which we denote by $\mathfrak{p}$. To make the following discussion meaningful, we now assume $\mathfrak{p} \geq 1$.

Let $\{v_l, u_1^l, \ldots, u_{\mathfrak{p}}^l\}$ be an orthonormal basis of V_l. For each $j = 1,\ldots,\mathfrak{p}$, Lemmas 4.12 and 4.13 show that we can define a linear map $\mathfrak{T}_j : V_1 \to V_1$ such that, for any unit vector v, the image $\mathfrak{T}_j(v)$ satisfies

$$L(v,u_j^2) = L(v_2,\mathfrak{T}_j(v)). \tag{4.41}$$

The linear map $\mathfrak{T}_j :\ V_1 \to V_1$ has the following properties:

(P1) For any $v \in V_1$, $h(\mathfrak{T}_j(v),\mathfrak{T}_j(v)) = h(v,v)$, i.e., $\mathfrak{T}_j$ preserves the length of vectors.

(P2) For all $v \in V_1$, we have $\mathfrak{T}_j(v) \perp v$.

(P3) $\mathfrak{T}_j^2 = -\mathrm{id}$.

(P4) For any $j \neq l$, we have $h(\mathfrak{T}_j(v), \mathfrak{T}_l(v)) = 0$ for all $v \in V_1$.

(P1) and (P2) can be easily seen from Lemma 4.13 and the definition of $\mathfrak{T}_j(v)$. We now verify (P3) and (P4). For any unit vector $v \in V_1$, we have

$$L(v_2, \mathfrak{T}_j^2(v)) = L(u_j^2, \mathfrak{T}_j(v)). \tag{4.42}$$

Using the fact $\{\mathfrak{T}_j(v)\} \oplus V_{\mathfrak{T}_j(v)}(0) = V_1$ and $u_j^2 \in V_{v_2}(0) \subset V_{\mathfrak{T}_j(v)}(\tau)$, we have

$$\begin{aligned} h(L(u_j^2, \mathfrak{T}_j(v)), L(u_j^2, \mathfrak{T}_j(v))) &= h(L(v_2, \mathfrak{T}_j(v)), L(v_2, \mathfrak{T}_j(v))) \\ &= h(L(v, v_2), L(v, v_2)) = \tau. \end{aligned}$$

Since v, $\mathfrak{T}_j(v)$, v_2, u_j^2 are orthonormal vectors, by (4.10), (4.41) and $L(v_2, u_j^2) = 0$, we see that

$$\begin{aligned} 0 &= h(L(v, v_2), L(u_j^2, \mathfrak{T}_j(v))) + h(L(v, \mathfrak{T}_j(v)), L(v_2, u_j^2)) \\ &\quad + h(L(v, u_j^2), L(\mathfrak{T}_j(v), v_2)) \\ &= h(L(v, v_2), L(u_j^2,\ \mathfrak{T}_j(v))) + h(L(v_2, \mathfrak{T}_j(v)), L(v_2, \mathfrak{T}_j(v))). \end{aligned}$$

Applying the Cauchy-Schwarz inequality we deduce

$$L(u_j^2, \mathfrak{T}_j(v)) = -L(v, v_2). \tag{4.43}$$

Combining (4.42) and (4.43), we get $L(v_2, \mathfrak{T}_j^2(v) + v) = 0$, which implies that $\mathfrak{T}_j^2(v) + v \in V_{v_2}(0)$. As $\mathfrak{T}_j^2(v) + v \in V_1 \subset V_{v_2}(\tau)$, it follows that $\mathfrak{T}_j^2(v) = -v$ for a unit vector v and then by linearity for all $v \in V_1$, as claimed by (P3).

To verify (P4), we note that, if $j \neq l$, and $\mathfrak{T}_j(v)$, $\mathfrak{T}_l(v) \in V_v(0)$, then by definition

$$L(v_2, \mathfrak{T}_j(v)) = L(v, u_j^2) \perp L(v, u_l^2) = L(v_2, \mathfrak{T}_l(v)).$$

If we assume $\mathfrak{T}_l(v) = a\mathfrak{T}_j(v) + x$, where $x \perp \mathfrak{T}_j(v)$, then

$$\begin{aligned} 0 &= h(L(v_2, \mathfrak{T}_j(v)), L(v_2, \mathfrak{T}_l(v))) \\ &= h(L(v_2, \mathfrak{T}_j(v)), aL(v_2, \mathfrak{T}_j(v)) + L(v_2, x)) \\ &= a\tau. \end{aligned}$$

Hence, $a = 0$ and $\mathfrak{T}_l(v) \perp \mathfrak{T}_j(v)$.

We look at the unit hypersphere $S^{\mathfrak{p}}(1) \subset V_1$, the above properties (P1)–(P4) show that at $v \in S^{\mathfrak{p}}(1)$ one has

$$T_v S^{\mathfrak{p}}(1) = \text{span}\,\{\mathfrak{T}_1(v), \ldots, \mathfrak{T}_{\mathfrak{p}}(v)\}.$$

Hence, by the properties (P1)–(P4), the $\mathfrak{p}$-dimensional sphere $S^{\mathfrak{p}}(1)$ is parallelizable. Then, according to Bott and Milnor [3] and Kervaire [13], the dimension $\mathfrak{p}$ can only be equal to 1, 3 or 7. $\square$

5. The Exceptional Case $\mathfrak{B}$

In this section, we shall study an n-dimensional ($n \geq 2$) locally strongly convex centroaffine hypersurface M^n which has parallel cubic form, such that Case $\mathfrak{B}$ occurs. The main result of this section is the following theorem.

Theorem 5.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ ($n \geq 2$) be a locally strongly convex centroaffine hypersurface which has parallel cubic form. If Case $\mathfrak{B}$ occurs, then M^n is locally centroaffinely equivalent to the hypersurface:*

$$x_{n+1} = \tfrac{1}{2x_1} \sum_{k=2}^{n} x_k^2 + x_1 \ln x_1. \tag{5.1}$$

To begin with, we prove the following lemma.

Lemma 5.1. *In Case $\mathfrak{B}$, there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM such that the difference tensor K satisfies*

$$K_{e_1}e_1 = 2e_1,\ K_{e_1}e_i = e_i,\ K_{e_i}e_j = \delta_{ij}e_1, \quad i,j = 2,\ldots,n. \tag{5.2}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be the orthonormal basis determined in Lemma 4.1. By assumption, $\lambda_1^2 - 4\varepsilon = 0$, we have

$$\varepsilon = 1, \quad \lambda_1 = 2. \tag{5.3}$$

Similar to the proof of (4.2), we now have

$$h(K_{e_i}e_j, e_k) = 0, \quad 2 \leq i,j,k \leq n. \tag{5.4}$$

From these results we easily get the assertion of Lemma 5.1. □

Next, as an extension of Lemma 5.1 we can prove the following lemma.

Lemma 5.2. *If Case $\mathfrak{B}$ occurs, then around p there exists a local orthonormal basis $\{E_1, \ldots, E_n\}$ such that $\hat{\nabla}_X E_1 = 0$ for all $X \in TM^n$, and*

$$K_{E_1}E_1 = 2E_1,\ K_{E_1}E_i = E_i,\ K_{E_i}E_j = \delta_{ij}E_1, \quad i,j = 2,\ldots,n. \tag{5.5}$$

Moreover, (M^n, h) is locally isometric to the Euclidean space $\mathbb{R}^n$.

Proof. Let $\{e_1, \ldots, e_n\}$ be the orthonormal basis of T_pM, given by Lemma 5.1. By parallel translation of $\{e_i\}_{i=1}^n$ along geodesics through p, we can obtain an h-orthonormal basis, denoted by $\{E_1, \ldots, E_n\}$, in a normal neighbourhood around p. Since $\hat{\nabla}K = 0$, the difference tensor K takes the form of (5.5).

It follows from (2.3), (5.3) and (5.5) that (M^n, h) satisfies $\hat{R}(E_i, E_j)E_j = 0$ for any i, j, i.e., (M^n, h) is flat and it is locally isometric to the Euclidean space $\mathbb{R}^n$.

To show that $\hat{\nabla}_X E_1 = 0$ for any $X \in TM^n$, we denote $\hat{\nabla}_{E_j} E_i = \sum_k \Gamma_{ij}^k E_k$, where $\Gamma_{ij}^k = -\Gamma_{kj}^i$, $1 \leq i,j,k \leq n$. By using $\hat{\nabla}K = 0$ and (5.5), straightforward calculations of the equations

$$0 = (\hat{\nabla}_{E_i}K)(E_i, E_i) = (\hat{\nabla}_{E_1}K)(E_i, E_i),\ i \neq 1$$

give that $\Gamma^1_{ij} = 0$ for $1 \le i, j \le n$. It follows that

$$\hat{\nabla}_{E_i} E_1 = 0, \quad 1 \le i \le n. \tag{5.6}$$

This completes the proof of Lemma 5.2. □

Now we will prove Theorem 5.1.

Proof of Theorem 5.1. As proved in Lemma 5.2, $\hat{\nabla}_X E_1 = 0$ and (M, h) is locally isometric to $\mathbb{R}^n$, we may choose local coordinates $(u_1, u_2, \ldots, u_n)$ on M^n such that the metric h has the following expression:

$$h = du_1^2 + du_2^2 + du_3^2 + \cdots + du_n^2, \tag{5.7}$$

and that $\frac{\partial}{\partial u_1} = E_1$. It follows from (5.7) that

$$\hat{\nabla}_{\partial u_i} \partial u_j = 0, \quad 1 \le i, j \le n, \tag{5.8}$$

where, and also later on, we use the notations $\partial u_k = \frac{\partial}{\partial u_k}, \ k = 1, \ldots, n.$

By using (5.5), we get that

$$K_{\partial u_1} X = X, \quad K_X Y = h(X, Y)\partial u_1, \quad X, Y \in \{\partial u_1\}^{\perp}. \tag{5.9}$$

By using (5.5), (5.7) and (5.9), we get that

$$\begin{cases} K_{\partial u_1} \partial u_1 = 2\partial u_1, \quad K_{\partial u_1} \partial u_k = \partial u_k, \quad 2 \le k \le n, \\ K_{\partial u_k} \partial u_j = \delta_{kj} \partial u_1, \quad 2 \le j, k \le n. \end{cases} \tag{5.10}$$

Write $x = x(u_1, \ldots, u_n) \in \mathbb{R}^{n+1}$. From (5.10), (5.8), and using (1.1) with the fact $\varepsilon = 1$, we have

$$x_{u_1 u_1} = 2x_{u_1} - x, \tag{5.11}$$

$$x_{u_1 u_k} = x_{u_k}, \quad 2 \le k \le n, \tag{5.12}$$

$$x_{u_k u_k} = x_{u_1} - x, \quad 2 \le k \le n, \tag{5.13}$$

$$x_{u_k u_j} = 0, \quad 2 \le j, k \le n \text{ and } j \ne k. \tag{5.14}$$

First of all, we can solve (5.11) to obtain that

$$x = P_1(u_2, \ldots, u_n)e^{u_1} + P_2(u_2, \ldots, u_n)u_1 e^{u_1}, \tag{5.15}$$

where $P_1(u_2, \ldots, u_n)$ and $P_2(u_2, \ldots, u_n)$ are $\mathbb{R}^{n+1}$-valued functions.

Inserting (5.15) into (5.12), we obtain $\frac{\partial P_2}{\partial u_k} = 0$, $2 \le k \le n$, which shows that $P_2(u_2, \ldots, u_n)$ is a constant vector denoted by A_1. Hence, we have

$$x = P_1(u_2, \ldots, u_n)e^{u_1} + A_1 u_1 e^{u_1}. \tag{5.16}$$

Putting (5.16) into (5.13) for $k = 2$, we further obtain that

$$\frac{\partial^2 P_1}{\partial u_2 \partial u_2} = A_1. \tag{5.17}$$

Thus, we can write

$$x = \Big(\tfrac{1}{2} u_2^2 A_1 + P_3(u_3, \ldots, u_n)u_2 + P_4(u_3, \ldots, u_n)\Big)e^{u_1} + u_1 e^{u_1} A_1. \tag{5.18}$$

From (5.14) and (5.18), we can derive that $P_3(u_3,\dots,u_n)$ is a constant vector denoted by A_2. Hence, we have

$$x=\left(\tfrac{1}{2}u_2^2A_1+u_2A_2+P_4(u_3,\dots,u_n)\right)e^{u_1}+u_1e^{u_1}A_1.$$

If we carry out such procedure by induction for other u_k with $k\geq 3$, we can finally obtain constant vectors $\{A_1,A_2,\dots,A_{n+1}\}$ such that $x(u_1,\dots,u_n)$ has the following expression:

$$x=\Big(\frac{1}{2}\sum_{k=2}^{n}u_k^2+u_1\Big)e^{u_1}A_1+\sum_{k=2}^{n}u_ke^{u_1}A_k+e^{u_1}A_{n+1}. \tag{5.19}$$

The nondegeneracy of x implies that it lies linearly full in $\mathbb{R}^{n+1}$ and thus A_1, ..., A_{n+1} are linearly independent vectors. Thus, up to a centroaffine transformation, x can be written as

$$x=\Big(e^{u_1},u_2e^{u_1},\dots,u_ne^{u_1},\big(\tfrac{1}{2}\sum_{k=2}^{n}u_k^2+u_1\big)e^{u_1}\Big),$$

which is easily seen to be locally centroaffinely equivalent to the hypersurface given in Theorem 5.1.

We have completed the proof of Theorem 5.1. □

6. Centroaffine Surfaces in $\mathbb{R}^3$ with $\hat{\nabla}C=0$

Although Theorem 1.1 gives a complete classification of locally strongly convex centroaffine hypersurfaces in $\mathbb{R}^{n+1}$ with parallel cubic form, its statement involving the Calabi product constructions actually makes use of the induction procedure. Therefore, in order to guarantee the validity of such induction procedure, we need first consider the lowest dimension case (i.e. $n=2$). This problem will be settled by the following theorem.

Theorem 6.1. *Let $x:M^2\to\mathbb{R}^3$ be a locally strongly convex centroaffine surface which has parallel cubic form. Then x is locally centroaffinely equivalent to one of the following hypersurfaces:*

(i) *quadrics with $C=0$;*
(ii) $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}=1$, *where $\{\alpha_i\}$ are real numbers which satisfy*

$$\alpha_i>0,\ i=1,2,3;\ \text{or}\ \alpha_1<0,\ \alpha_2,\alpha_3>0,\ \alpha_1+\alpha_2+\alpha_3<0;$$

(iii) $x_1^{\alpha_1}(x_2^2+x_3^2)^{\alpha_2}\exp(\alpha_3\arctan\frac{x_2}{x_3})=1,\quad \alpha_1<0,\ \alpha_1+2\alpha_2>0;$
(iv) $x_3=x_1(\ln x_1-\alpha_2\ln x_2),\quad 0<\alpha_2<1;$
(v) $x_3=\frac{1}{2x_1}x_2^2+x_1\ln x_1,$

where $\alpha_1,\alpha_2,\alpha_3$ are constants and (x_1,x_2,x_3) is the coordinate of $\mathbb{R}^3$.

Remark 6.1. Centroaffine surfaces with parallel cubic form have been studied in [19], where the authors made use of Theorem 1.3 in [17]. Comparing our theorem with the result in [19], one can see that the surface (v) of Theorem 6.1 is missing in [19]. This appearance is because in [17] the authors only obtained the classification of canonical centroaffine hypersurfaces for $N(h) \leq 1$, hence in [19] the conclusion for the case $N(h) = 2$ is unfortunately not correct stated. Here, the fact that the surface (v) corresponds to $n = 2,\ v = 3$ and $N(h) = 2$ in corollary 1.1 should be emphasized.

In order to prove Theorem 6.1, we first notice that, for $n = 2$, it follows from Theorem 5.1 that in Case $\mathfrak{B}$ the surface M^2 is centroaffinely equivalent to the surface (v). Thus, taking into consideration of Theorem 4.2, we see that what we need to consider is Case $\mathfrak{C}_1$ with $n = 2$ in a more explicit way, rather than like the sketchy statement of Theorem 4.1.

To begin with, we state the following lemma which is a direct consequence of Lemma 4.1.

Lemma 6.1 (cf. Lemma 4.1). *If Case $\mathfrak{C}_1$ occurs, then there exists an orthonormal basis $\{e_1, e_2\}$ of T_pM^2 such that the difference tensor K takes the following form:*

$$K_{e_1}e_1 = \lambda_1 e_1,\quad K_{e_1}e_2 = \mu e_2,\quad K_{e_2}e_2 = \mu e_1 + a_1 e_2,$$
$$\varepsilon - \lambda_1\mu + \mu^2 = 0,\ \lambda_1 > 0,\ \lambda_1^2 - 4\varepsilon > 0,\ \lambda_1 > 2\mu.$$

To prove Theorem 6.1, we also need the following lemma.

Lemma 6.2. *If Case $\mathfrak{C}_1$ occurs, then there exists a local orthonormal basis $\{E_1, E_2\}$ around p, such that the difference tensor takes the following form:*

$$K_{E_1}E_1 = \lambda_1 E_1,\ K_{E_1}E_2 = \mu E_2,\ K_{E_2}E_2 = \mu E_1 + a_1 E_2, \tag{6.1}$$
$$\varepsilon - \lambda_1\mu + \mu^2 = 0,\ \lambda_1 > 0,\ \lambda_1^2 - 4\varepsilon > 0,\ \lambda_1 > 2\mu,$$

where λ_1, μ, a_1 are constant numbers and $\hat{\nabla}_{E_i}E_j = 0,\ i, j = 1, 2$. Moreover, (M^2, h) is locally isometric to the Euclidean space $\mathbb{R}^2$.

Proof. Let $\{e_1, e_2\}$ be the orthonormal basis of T_pM^2 given by Lemma 6.1. By parallel translation of $\{e_1, e_2\}$ along geodesics (with respect to $\hat{\nabla}$) through p, we can obtain an h-orthonormal basis, denoted by $\{E_1, E_2\}$, in a normal neighbourhood around p such that, thanks to $\hat{\nabla}K = 0$, the difference tensor K takes the form stated in (6.1).

First, from the calculation

$$0 = (\hat{\nabla}_{E_i}K)(E_1, E_1) = \lambda_1\hat{\nabla}_{E_i}E_1 - 2K(\hat{\nabla}_{E_i}E_1, E_1),\quad i = 1, 2,$$

and noting that $\hat{\nabla}_{E_i}E_1$ is h-orthogonal to E_1, we have $\hat{\nabla}_{E_i}E_1 = 0,\ i = 1, 2$.

Next, by computation of $0 = h((\hat{\nabla}_{E_2}K)(E_1, E_2), E_1)$ we obtain that

$$h(\hat{\nabla}_{E_2}E_2, E_1) = 0. \tag{6.2}$$

This, together with $h(\hat{\nabla}_{E_i}E_2,E_2)=0$ and $h(\hat{\nabla}_{E_1}E_2,E_1)=-h(\hat{\nabla}_{E_1}E_1,E_2)=0$, we will obtain

$$\hat{\nabla}_{E_i}E_j=0,\quad i,j=1,2. \tag{6.3}$$

It follows that $\hat{R}(E_i,E_j)E_k=0$ and (M^2,h) is locally isometric to the Euclidean space $\mathbb{R}^2$. □

Proof of Theorem 6.1. According to Lemma 6.2, we can choose local coordinates (u_1,u_2) for M^2 such that the centroaffine metric h has the following expression:

$$h=du_1^2+du_2^2, \tag{6.4}$$

and $E_i=\frac{\partial}{\partial u_i}$ for $i=1,2$. It follows from (6.4) that

$$\hat{\nabla}_{\partial u_i}\partial u_j=0,\quad 1\le i,j\le 2. \tag{6.5}$$

For $x=x(u_1,u_2)\in\mathbb{R}^3$, using (6.1), (6.4), (6.5) and (1.1) we can obtain:

$$x_{u_1u_1}=\lambda_1x_{u_1}-\varepsilon x, \tag{6.6}$$

$$x_{u_1u_2}=\mu x_{u_2}, \tag{6.7}$$

$$x_{u_2u_2}=\mu x_{u_1}+a_1x_{u_2}-\varepsilon x. \tag{6.8}$$

We first solve the equation (6.6) to obtain that

$$x=P_1(u_2)\exp\{(\lambda_1-\mu)u_1\}+P_2(u_2)\exp(\mu u_1), \tag{6.9}$$

where $P_1(u_2)$ and $P_2(u_2)$ are $\mathbb{R}^3$-valued functions.

Inserting (6.9) into (6.7), we obtain $\frac{\partial P_1}{\partial u_2}=0$, showing that $P_1(u_2)$ is a constant vector, denoted by A_1. Hence, we have

$$x=\exp\{(\lambda_1-\mu)u_1\}A_1+P_2(u_2)\exp(\mu u_1). \tag{6.10}$$

Combining (6.10) and (6.8), we get

$$\frac{\mathrm{d}^2P_2}{\mathrm{d}u_2\mathrm{d}u_2}=a_1\frac{\mathrm{d}P_2}{\mathrm{d}u_2}+(\mu^2-\varepsilon)P_2. \tag{6.11}$$

To solve (6.11), we will consider the following three cases, separately:

(a) $a_1^2+4(\mu^2-\varepsilon)>0$.
(b) $a_1^2+4(\mu^2-\varepsilon)<0$.
(c) $a_1^2+4(\mu^2-\varepsilon)=0$.

(a) In this case, the solution of (6.11) is

$$P_2=\exp\left\{\tfrac{1}{2}\Big(a_1+\sqrt{a_1^2+4(\mu^2-\varepsilon)}\Big)u_2\right\}A_2 +\exp\left\{\tfrac{1}{2}\Big(a_1-\sqrt{a_1^2+4(\mu^2-\varepsilon)}\Big)u_2\right\}A_3,$$

where A_2, A_3 are constant vectors.

It follows that, up to a centroaffine transformation, x can be written as

$$x = \Big(\exp\{(\lambda_1-\mu)u_1\}, \exp\Big\{\tfrac{1}{2}\big(a_1+\sqrt{a_1^2+4(\mu^2-\varepsilon)}\,\big)u_2+\mu u_1\Big\},\\ \exp\Big\{\tfrac{1}{2}\big(a_1-\sqrt{a_1^2+4(\mu^2-\varepsilon)}\,\big)u_2+\mu u_1\Big\}\Big), \tag{6.12}$$

which, due to its locally strongly convexity, is easily seen locally on the hypersurface (ii) of Theorem 6.1.

(b) In this case, we have $\varepsilon = 1$. The solution of (6.11) is given by

$$P_2 = \cos\Big(\tfrac{1}{2}\sqrt{-a_1^2-4(\mu^2-1)}u_2\Big)\exp(\tfrac{1}{2}a_1u_2)A_2\\ +\sin\Big(\tfrac{1}{2}\sqrt{-a_1^2-4(\mu^2-1)}u_2\Big)\exp(\tfrac{1}{2}a_1u_2)A_3,$$

where A_2, A_3 are constant vectors.

It follows that, up to a centroaffine transformation, x can be written as

$$x = \Big(\exp\{(\lambda_1-\mu)u_1\}, \sin\big(\tfrac{1}{2}\sqrt{-a_1^2-4(\mu^2-1)}u_2\big)\exp(\tfrac{1}{2}a_1u_2+\mu u_1),\\ \cos\big(\tfrac{1}{2}\sqrt{-a_1^2-4(\mu^2-1)}u_2\big)\exp(\tfrac{1}{2}a_1u_2+\mu u_1)\Big), \tag{6.13}$$

which, due to its locally strongly convexity, is locally on the hypersurface (iii) of Theorem 6.1.

(c) In this case, from the fact that $a_1^2+4(\mu^2-\varepsilon)=0$ and Lemma 6.2, we have

$$a_1 \neq 0, \quad \varepsilon = 1. \tag{6.14}$$

The solution of (6.11) is given by

$$P_2 = \exp\big(\tfrac{1}{2}a_1u_2\big)A_2 + u_2\exp\big(\tfrac{1}{2}a_1u_2\big)A_3,$$

where A_2, A_3 are constant vectors.

It follows that, up to a centroaffine transformation, x can be written as

$$x = \Big(\exp(\tfrac{1}{2}a_1u_2+\mu u_1), \exp\{(\lambda_1-\mu)u_1\}, \tfrac{1}{2}a_1u_2\exp(\tfrac{1}{2}a_1u_2+\mu u_1)\Big), \tag{6.15}$$

which, according to (6.14), (6.15) and due to its locally strongly convexity, is locally on the hypersurface (iv) of Theorem 6.1.

We have completed the proof of Theorem 6.1. □

7. Case $\{\mathfrak{C}_m\}_{2\leq m\leq n-1}$ with $k_0 = 1$

In this section, we consider Case $\mathfrak{C}_m$ $(2 \leq m \leq n-1)$ with the condition that in the decomposition (4.28), $k_0 = 1$. We will prove the following theorem.

Theorem 7.1. *Let M^n be a locally strongly convex centroaffine hypersurface in $\mathbb{R}^{n+1}$ which has parallel and non-vanishing cubic form. If $\mathfrak{C}_m$ with $2 \leq m \leq n-1$ occurs and the integer k_0, as defined in Sect. 4.5, satisfies $k_0 = 1$, then M^n can be decomposed as the Calabi product of two locally strongly convex centroaffine hypersurfaces with parallel cubic form, or the Calabi product of a locally strongly convex centroaffine hypersurface with parallel cubic form and a point.*

To prove Theorem 7.1, we first note that if $k_0 = 1$ then by Proposition 4.2 we have $\dim(\operatorname{Im} L) = 1$. Moreover, we can prove the following result.

Lemma 7.1. *If $\dim(\operatorname{Im} L) = 1$, then there is a unit vector $w_1 \in \operatorname{Im} L \subset \mathcal{D}_3$ such that L has the expression*

$$L(v_1, v_2) = \sqrt{\tfrac{1}{2}\lambda_1\eta}\, h(v_1, v_2) w_1, \quad \forall\, v_1, v_2 \in \mathcal{D}_2. \tag{7.1}$$

Proof. The fact $\dim(\operatorname{Im} L) = 1$ implies that we have a unit vector $\bar{w} \in \operatorname{Im} L \subset \mathcal{D}_3$ and a symmetric bilinear form α over $\mathcal{D}_2$ such that

$$L(v_1, v_2) = \alpha(v_1, v_2)\bar{w}, \quad \forall\, v_1, v_2 \in \mathcal{D}_2. \tag{7.2}$$

We define $Q : \mathcal{D}_2 \to \mathcal{D}_2$ by $h(Qv_1, v_2) := \alpha(v_1, v_2)$. From Lemma 4.12 we have

$$L(v_1, v_2) = 0, \quad \text{if } h(v_1, v_2) = 0. \tag{7.3}$$

Now we see that $h(Qv_1, v_2) = 0$ if $h(v_1, v_2) = 0$. Hence, $Qv = \sqrt{\frac{1}{2}\lambda_1\eta}\,\varepsilon(v)v$ for all $v \in \mathcal{D}_2$ and $\varepsilon(v) = \pm 1$. It follows that

$$L(v_1, v_2) = \alpha(v_1, v_2)\bar{w} = \sqrt{\tfrac{1}{2}\lambda_1\eta}\,\varepsilon(v_1) h(v_1, v_2)\bar{w}. \tag{7.4}$$

This, together with the fact that both L and h are symmetric, implies that, for any $v_1, v_2 \in \mathcal{D}_2$, $\varepsilon(v_1) = \varepsilon(v_2)$ holds, i.e., $\varepsilon(v)$ is independent of v.

We finally get the assertion by putting $w_1 := \varepsilon(v_1)\bar{w}$. □

In sequel of this section, we will fix the unit vector $w_1 \in \mathcal{D}_3$ as in Lemma 7.1. Then, besides $K_{e_1} w_1 = \mu w_1$, the next three lemmas give all informations about the difference tensor K.

Lemma 7.2. *There exists an orthonormal basis $\{v_1, \ldots, v_{m-1}\}$ of $\mathcal{D}_2$ such that*

$$K(e_1, v_i) = \tfrac{1}{2}\lambda_1 v_i, \;\; K(w_1, v_i) = \sqrt{\tfrac{1}{2}\lambda_1\eta}\, v_i, \quad 1 \leq i \leq m-1, \tag{7.5}$$

$$K(v_i, v_j) = \left(\tfrac{1}{2}\lambda_1 e_1 + \sqrt{\tfrac{1}{2}\lambda_1\eta}\, w_1\right)\delta_{ij}, \quad 1 \leq i, j \leq m-1. \tag{7.6}$$

Proof. From Lemma 4.2, we see that K_{w_1} maps $\mathcal{D}_2$ to $\mathcal{D}_2$. Note that K_{w_1} is self-adjoint, then there exists an orthonormal basis $\{v_1, \ldots, v_{m-1}\}$ of $\mathcal{D}_2$ such that $K_{w_1} v_i = \alpha_i v_i$ with eigenvalues α_i. As $v_i \in \mathcal{D}_2$, we have $K_{e_1} v_i = \frac{1}{2}\lambda_1 v_i$. By Lemma 7.1 we get

$$\alpha_i = h(K_{w_1} v_i, v_i) = h(L(v_i, v_i), w_1) = \sqrt{\tfrac{1}{2}\lambda_1\eta}.$$

Since

$$L(v_i, v_j) = \sqrt{\tfrac{1}{2}\lambda_1\eta}\,h(v_i, v_j)w_1 = \sqrt{\tfrac{1}{2}\lambda_1\eta}\,\delta_{ij}w_1,$$

we get

$$K(v_i, v_j) = \left(\tfrac{1}{2}\lambda_1 e_1 + \sqrt{\tfrac{1}{2}\lambda_1\eta}\,w_1\right)\delta_{ij}.$$

This completes the proof of Lemma 7.2. □

Next, by (4.32) and Lemma 7.1 we get the following result.

Lemma 7.3. $K(w_1, w_1) = \mu e_1 + (\lambda_1 + \mu)\sqrt{\frac{2\eta}{\lambda_1}}w_1$.

Finally, in case $\mathcal{D}_3 \neq \mathbb{R}w_1$ and let $\{w_2, \ldots, w_{n-m}\}$ be an orthonormal basis of $\mathcal{D}_3 \backslash \mathbb{R}w_1$, by Lemmas 4.5 and 7.1, we immediately have:

Lemma 7.4. $K(w_1, w_i) = \tilde{\mu}w_i,\ 2 \leq i \leq n - m$, *where* $\tilde{\mu} = \sqrt{\frac{2\eta}{\lambda_1}}\mu$.

Now, we are ready to complete the proof of Theorem 7.1.

Proof of Theorem 7.1. Based on Lemmas 7.1, 7.2, 7.3 and 7.4, by putting

$$t = \sqrt{\tfrac{\lambda_1}{\lambda_1+2\eta}}e_1 + \sqrt{\tfrac{2\eta}{\lambda_1+2\eta}}w_1, \quad v = -\sqrt{\tfrac{2\eta}{\lambda_1+2\eta}}e_1 + \sqrt{\tfrac{\lambda_1}{\lambda_1+2\eta}}w_1,$$

we see that if $\mathcal{D}_3 = \mathbb{R}w_1$, then $\{t, v, v_1, \ldots, v_{m-1}\}$ (or, resp. if $\mathcal{D}_3 \neq \mathbb{R}w_1$, then $\{t, v, v_1, \ldots, v_{m-1}, w_2, \ldots, w_{n-m}\}$) forms an orthonormal basis of T_pM^n, with respect to which, the difference tensor K takes the following form:

$$\begin{cases} K(t,t) = \sigma_1 t; \quad K(t,v) = \sigma_2 v; \quad K(t,v_i) = \sigma_2 v_i, \quad 1 \leq i \leq m-1; \\ \text{if } \mathcal{D}_3 \neq \mathbb{R}w_1, \quad K(t,w_i) = \sigma_3 w_i, \quad 2 \leq i \leq n-m, \end{cases} \tag{7.7}$$

where

$$\sigma_1 = \tfrac{\lambda_1^2+2\eta\mu}{\sqrt{\lambda_1(\lambda_1+2\eta)}}, \quad \sigma_2 = \tfrac{\frac{1}{2}\lambda_1^2+\lambda_1\eta}{\sqrt{\lambda_1(\lambda_1+2\eta)}}, \quad \sigma_3 = \tfrac{\lambda_1\mu+2\eta\mu}{\sqrt{\lambda_1(\lambda_1+2\eta)}}. \tag{7.8}$$

It is easy to show that the constants σ_1, σ_2 and σ_3 satisfy the relations:

$$\sigma_1 \neq 2\sigma_2, \quad \sigma_1 \neq 2\sigma_3, \quad \sigma_2 \neq \sigma_3. \tag{7.9}$$

By parallel translation along geodesics (with respect to $\hat{\nabla}$) through p, we can extend $\{t, v, v_1, \ldots, v_{m-1}\}$ (if $\mathcal{D}_3 = \mathbb{R}w_1$), or, resp. $\{t, v, v_1, \ldots, v_{m-1}, w_2, \ldots, w_{n-m}\}$ (if $\mathcal{D}_3 \neq \mathbb{R}w_1$) to obtain a local h-orthonormal basis $\{T, V, V_1, \ldots, V_{m-1}\}$, or, resp. $\{T, V, V_1, \ldots, V_{m-1}, W_2, \ldots, W_{n-m}\}$ such that

$$\begin{cases} K(T,T) = \sigma_1 T; \quad K(T,V) = \sigma_2 V; \quad K(T,V_i) = \sigma_2 V_i, \quad 1 \leq i \leq m-1; \\ \text{if } \mathcal{D}_3 \neq \mathbb{R}w_1, \quad K(T,W_i) = \sigma_3 W_i, \quad 2 \leq i \leq n-m. \end{cases}$$

Now, the above fact implies that, if $\mathcal{D}_3 \neq \mathbb{R}w_1$ we can apply Theorem 3.2 to conclude that M^n is decomposed as the Calabi product of two locally strongly convex centroaffine hypersurfaces with parallel cubic form. If $\mathcal{D}_3 =$

$\mathbb{R}w_1$, then we can apply Theorem 3.4 to conclude that M can be decomposed as the Calabi product of a locally strongly convex centroaffine hypersurface with parallel cubic form and a point. □

8. Case $\{\mathfrak{C}_m\}_{2\le m\le n-1}$ with $k_0 \ge 2$ and $\mathfrak{p} = 0$

In this section, we will prove the following theorem.

Theorem 8.1. *Let M^n be a locally strongly convex centroaffine hypersurface in $\mathbb{R}^{n+1}$ which has parallel and non-vanishing cubic form. If $\mathfrak{C}_m$ with $2 \le m \le n-1$ occurs and the integers k_0 and $\mathfrak{p}$, as defined in Sect. 4.5, satisfy $k_0 \ge 2$ and $\mathfrak{p} = 0$, then $n \ge \frac{1}{2}m(m+1) - 1$. Moreover, we have either*

- (i) *$n = \frac{1}{2}m(m+1)$, M^n can be decomposed as the Calabi product of a locally strongly convex centroaffine hypersurface with parallel cubic form and a point, or*
- (ii) *$n > \frac{1}{2}m(m+1)$, M^n can be decomposed as the Calabi product of two locally strongly convex centroaffine hypersurfaces with parallel cubic form, or*
- (iii) *$n = \frac{1}{2}m(m+1) - 1$, M^n is centroaffinely equivalent to the standard embedding of $\mathrm{SL}(m,\mathbb{R})/\mathrm{SO}(m;\mathbb{R}) \hookrightarrow \mathbb{R}^{n+1}$.*

In the present situation, the decomposition (4.28) reduces to $\mathcal{D}_2 = \{v_1\} \oplus \cdots \oplus \{v_{k_0}\}$. Then $\dim \mathcal{D}_2 = k_0 = m-1,\ m \ge 3$, and $\{v_1, \ldots, v_{k_0}\}$ forms an orthonormal basis of $\mathcal{D}_2$.

According to (4.5), Lemma 4.11 and the fact that for $j \ne l,\ v_j \in V_{v_l}(\tau)$, we have

$$h(L(v_j, v_l), L(v_j, v_l)) = \tau, \quad j \ne l, \tag{8.1}$$

$$h(L(v_j, v_{l_1}), L(v_j, v_{l_2})) = 0, \quad j, l_1, l_2 \text{ distinct}, \tag{8.2}$$

$$h(L(v_{j_1}, v_{j_2}), L(v_{j_3}, v_{j_4})) = 0, \quad j_1, j_2, j_3, j_4 \text{ distinct}, \tag{8.3}$$

$$h(L(v_j, v_j), L(v_j, v_j)) = \tfrac{1}{2}\lambda_1\eta, \tag{8.4}$$

$$h(L(v_j, v_j), L(v_l, v_l)) = \tfrac{1}{2}\mu\eta, \quad j \ne l, \tag{8.5}$$

$$h(L(v_j, v_j), L(v_j, v_l)) = 0, \quad j \ne l, \tag{8.6}$$

$$h(L(v_j, v_j), L(v_{l_1}, v_{l_2})) = 0, \quad j,\ l_1,\ l_2 \text{ distinct}. \tag{8.7}$$

Denote $L_j := L(v_1, v_1) + \cdots + L(v_j, v_j) - jL(v_{j+1}, v_{j+1}),\ 1 \le j \le k_0 - 1$. Then it is easy to check $h(L_j, L_j) = 2j(j+1)\tau \ne 0$, and that

$$\begin{cases} w_j = \frac{1}{\sqrt{2j(j+1)\tau}} L_j, & 1 \le j \le k_0 - 1, \\ w_{kl} = \frac{1}{\sqrt{\tau}} L(v_k, v_l), & 1 \le k < l \le k_0 \end{cases} \tag{8.8}$$

give $\frac{1}{2}(m+1)(m-2)$ orthonormal vectors in $\operatorname{Im} L \subset \mathcal{D}_3$. Thus, we have the estimate of the dimension

$$\begin{aligned} n &= 1 + \dim(\mathcal{D}_2) + \dim(\mathcal{D}_3) \\ &\geq 1 + m - 1 + \tfrac{1}{2}(m+1)(m-2) = \tfrac{1}{2}m(m+1) - 1. \end{aligned} \tag{8.9}$$

Direct computations show that $\operatorname{Tr} L = L(v_1, v_1) + \cdots + L(v_{k_0}, v_{k_0})$ is orthogonal to all vectors in (8.8), and by using (4.4), (4.8) and the fact that $v_i \in V_{v_j}(\tau),\ i \neq j$, we get

$$\begin{aligned} h(\operatorname{Tr} L, \operatorname{Tr} L) &= \tfrac{1}{2}k_0\eta(\lambda_1 + (k_0 - 1)\mu) \\ &= \tfrac{1}{8}(m-1)\sqrt{\lambda_1^2 - 4\varepsilon}\left(m\lambda_1 - (m-2)\sqrt{\lambda_1^2 - 4\varepsilon}\,\right) \\ &=: \rho^2, \end{aligned} \tag{8.10}$$

where $\rho \geq 0$. From (8.10) and that $\lambda_1^2 - 4\varepsilon > 0$, the following result is obvious.

Lemma 8.1. $\operatorname{Tr} L = 0$ *if and only if* $\lambda_1 = \frac{m-2}{\sqrt{m-1}}$ *and* $\varepsilon = -1$.

On the other hand, an implicit fact can be said about the statement $\operatorname{Tr} L = 0$.

Lemma 8.2. $\operatorname{Tr} L = 0$ *if and only if* $n = \frac{1}{2}m(m+1) - 1$.

Proof. If $\operatorname{Tr} L = 0$, then we claim that $\mathcal{D}_3 = \operatorname{Im} L$. In fact, if $\mathcal{D}_3 \neq \operatorname{Im} L$, we have a unit vector $w \in \mathcal{D}_3$ which is orthogonal to $\operatorname{Im} L$. Then by Lemma 4.5 we get the contradiction

$$0 = K(\operatorname{Tr} L, w) = k_0\eta\mu w. \tag{8.11}$$

Thus, according to this claim and (8.9), we have $n = \frac{1}{2}m(m+1) - 1$, provided that $\operatorname{Tr} L = 0$.

Conversely, if $n = \frac{1}{2}m(m+1) - 1$, then by (8.9) we have

$$\dim(\mathcal{D}_3) = \tfrac{1}{2}(m+1)(m-2)$$

which implies that $\mathcal{D}_3 = \operatorname{Im} L$. This further implies that $\operatorname{Tr} L = 0$ due to the fact that the vector $\operatorname{Tr} L$, which belongs to $\mathcal{D}_3$, is orthogonal to all vectors in (8.8). □

Now, we are ready to complete the proof of Theorem 8.1.

Proof of Theorem 8.1. We need to consider three cases:

Case (i) $n = \frac{1}{2}m(m+1)$.
Case (ii) $n > \frac{1}{2}m(m+1)$.
Case (iii) $n = \frac{1}{2}m(m+1) - 1$.

For Cases (i) and (ii), as $\operatorname{Tr} L \neq 0$, we can define a unit vector $t := \frac{1}{\rho}\operatorname{Tr} L$.

In Case (i), from the previous discussions we see that

$$\{t,\ w_j\,|_{1\leq j\leq k_0-1},\ w_{kl}\,|_{1\leq k<l\leq k_0}\}$$

forms an orthonormal basis of $\operatorname{Im} L = \mathcal{D}_3$. By direct calculations with the use of Lemmas 4.2, 4.14 and (8.1)–(8.7), we have the following fact which we state as

Lemma 8.3. *In Case (i), the difference tensor K satisfies*

$$\begin{cases} K(t,e_1)=\mu t, \quad K(t,v_i)=\frac{\rho}{k_0}v_i, \quad 1\le i\le m-1,\\ K(t,w_j)=\frac{2\rho}{k_0}w_j, \quad 1\le j\le k_0-1,\\ K(t,w_{kl})=\frac{2\rho}{k_0}w_{kl}, \quad 1\le k<l\le k_0,\\ K(t,t)=\mu e_1+\left(\frac{2\rho}{k_0}+\frac{k_0\mu\eta}{\rho}\right)t. \end{cases} \tag{8.12}$$

Put

$$T=\frac{\rho}{\sqrt{\rho^2+k_0^2\eta^2}}e_1+\frac{k_0\eta}{\sqrt{\rho^2+k_0^2\eta^2}}t, \quad T^*=-\frac{k_0\eta}{\sqrt{\rho^2+k_0^2\eta^2}}e_1+\frac{\rho}{\sqrt{\rho^2+k_0^2\eta^2}}t. \tag{8.13}$$

It is easily to see that $\{T,\ T^*,\ v_j|_{1\le j\le k_0},\ w_j\ |_{1\le j\le k_0-1},\ w_{kl}\ |_{1\le k<l\le k_0}\}$ is an orthonormal basis of T_pM. By Lemmas 4.2 and 8.3 we have the following lemma.

Lemma 8.4. *In Case (i), under the above notations, we have*

$$\begin{cases} K(T,T)=\sigma_1 T, \qquad K(T,v_j)=\sigma_2 v_j, \quad 1\le j\le k_0;\\ K(T,T^*)=\sigma_2 T^*, \quad K(T,w_j)=\sigma_2 w_j, \quad 1\le j\le k_0-1;\\ K(T,w_{kl})=\sigma_2 w_{kl}, \quad 1\le k<l\le k_0, \end{cases} \tag{8.14}$$

where σ_1 and σ_2 are defined by

$$\sigma_1=\frac{\rho^2\lambda_1+k_0^2\eta^2\mu}{\rho\sqrt{\rho^2+k_0^2\eta^2}}, \quad \sigma_2=\frac{(\frac{1}{2}\lambda_1+\eta)\rho}{\sqrt{\rho^2+k_0^2\eta^2}}, \tag{8.15}$$

which satisfy $\sigma_1\neq 2\sigma_2$.

Given the parallelism of the difference tensor K, Lemma 8.4 and Theorem 3.4, we conclude that in Case (i), M is locally the Calabi product of a lower-dimensional locally strongly convex centroaffine hypersurface with parallel cubic form with a point.

In Case (ii), we proceed in the same way as in Case (i). We first see that

$$\{t,\ w_{kl}\ |_{1\le k<l\le k_0},\ w_j\ |_{1\le j\le k_0-1}\}$$

is still an orthonormal basis of $\operatorname{Im} L$, even though we have $\operatorname{Im} L\subsetneq \mathcal{D}_3$.

Denote $\tilde n=n-\frac{1}{2}m(m+1)$ and choose $\tilde w_1,\ldots,\tilde w_{\tilde n}$ in the orthogonal complement of $\operatorname{Im} L$ in $\mathcal{D}_3$ such that

$$\{t,\ w_{kl}\ |_{1\le k<l\le k_0},\ w_j\ |_{1\le j\le k_0-1},\ \tilde w_r\ |_{1\le r\le \tilde n}\}$$

is an orthonormal basis of $\mathcal{D}_3$. By Lemma 4.5, we obtain that

$$K(t,\tilde w_r)=k_0\eta\mu\rho^{-1}\tilde w_r. \tag{8.16}$$

We define T and T^* as in (8.13). Then

$$\{T,\ T^*,\ v_j|_{1\le j\le k_0},\ w_{kl}\ |_{1\le k<l\le k_0},\ w_j\ |_{1\le j\le k_0-1},\ \tilde w_r|_{1\le r\le \tilde n}\}$$

is an orthonormal basis of T_pM. Similar to Lemma 8.4, we can easily show the following

Lemma 8.5. *In Case (ii), under the previous notations, we have*

$$\begin{cases} K(T,T)=\sigma_1 T, & K(T,v_j)=\sigma_2 v_j, \quad 1\le j\le k_0;\\ K(T,T^*)=\sigma_2 T^*, & K(T,w_j)=\sigma_2 w_j, \quad 1\le j\le k_0-1;\\ K(T,w_{kl})=\sigma_2 w_{kl}, & 1\le k<l\le k_0;\\ K(T,\tilde{w}_r)=\sigma_3\tilde{w}_r, & 1\le r\le \tilde{n}, \end{cases} \tag{8.17}$$

where σ_1 and σ_2 are defined by (8.15), and

$$\sigma_3=\rho^{-1}\mu\sqrt{\rho^2+k_0^2\eta^2}, \tag{8.18}$$

which satisfy the relations $\sigma_1\neq 2\sigma_2$, $\sigma_1\neq 2\sigma_3$ and $\sigma_2\neq\sigma_3$.

Given the parallelism of the difference tensor K, Lemma 8.5 and Theorem 3.2, we conclude that in Case (ii), M is locally the Calabi product of two lower-dimensional locally strongly convex centroaffine hypersurfaces with parallel cubic form.

In Case (iii), we take the following basis of T_pM:

$$\{e_1,\ v_i|_{1\le i\le k_0},\ w_j\ |_{1\le j\le k_0-1},\ w_{jk}\ |_{1\le j<k\le k_0-1}\}. \tag{8.19}$$

By Lemmas 8.1, 8.2, 4.14 and a direct computation, we obtain that

$$K(e_1,e_1)+\sum_{j=1}^{k_0}K(v_j,v_j)+\sum_{j=1}^{k_0-1}K(w_j,w_j)+\sum_{1\le i<j\le k_0}K(w_{ij},w_{ij})=0. \tag{8.20}$$

This implies that in Case (iii) it holds $\operatorname{Tr} K_X=0$ for any vector X. Thus M is a proper affine hypersphere centered at the origin O. Then, according to previous computations and the proof of Theorem 5.1 in [12], we can easily show that in Case (iii) M^n is centroaffinely equivalent to the standard embedding $\mathrm{SL}(m,\mathbb{R})/\mathrm{SO}(m;\mathbb{R})\hookrightarrow\mathbb{R}^{n+1}$.

The combination of the preceding three cases' discussion then completes the proof of Theorem 8.1. □

9. Case $\{\mathfrak{C}_m\}_{2\le m\le n-1}$ with $k_0\ge 2$ and $\mathfrak{p}=1$

In this section, we will prove the following theorem.

Theorem 9.1. *Let M^n be a locally strongly convex centroaffine hypersurface in $\mathbb{R}^{n+1}$ which has parallel and non-vanishing cubic form. If $\mathfrak{C}_m$ with $2\le m\le n-1$ occurs and the integers k_0 and $\mathfrak{p}$, as defined in Sect. 4.5, satisfy $k_0\ge 2$ and $\mathfrak{p}=1$, then $n\ge\frac{1}{4}(m+1)^2-1$. Moreover, we have either*

(i) *$n=\frac{1}{4}(m+1)^2$, M^n can be decomposed as the Calabi product of a locally strongly convex centroaffine hypersurface with parallel cubic form and a point, or*

(ii) $n > \frac{1}{4}(m+1)^2$, *M^n can be decomposed as the Calabi product of two locally strongly convex centroaffine hypersurfaces with parallel cubic form, or*

(iii) $n = \frac{1}{4}(m+1)^2 - 1$, *$M^n$ is centroaffinely equivalent to the standard embedding* $\mathrm{SL}(\frac{m+1}{2}, \mathbb{C})/\mathrm{SU}(\frac{m+1}{2}) \hookrightarrow \mathbb{R}^{n+1}$.

Now we have $\dim \mathcal{D}_2 = m - 1 = 2k_0$ and $m \geq 5$. Similar to Lemma 6.1 of [12], we will prove the following

Lemma 9.1. *In the decomposition* (4.28), *if we have* $k_0 \geq 2$ *and* $\mathfrak{p} = 1$, *then there exist unit vectors* $u_j \in V_{v_j}(0)$ $(1 \leq j \leq k_0)$ *such that the orthonormal basis* $\{v_1, u_1, \ldots, v_{k_0}, u_{k_0}\}$ *of* $\mathcal{D}_2$ *satisfies the relations*

$$L(u_l, v_j) = -L(v_l, u_j), \quad L(v_l, v_j) = L(u_l, u_j), \quad 1 \leq j,\ l \leq k_0. \tag{9.1}$$

Proof. As for each $1 \leq j \leq k_0$ it holds $\dim (V_{v_j}(0)) = 1$, we assume $V_{v_2}(0) = \{u_2\}$ for a unit vector u_2. Then, for each $j \neq 2$, by Lemma 4.13, we have a unique unit vector $u_j \in V_{v_j}(0)$ satisfying

$$L(u_2, v_j) = -L(v_2, u_j), \quad L(v_2, v_j) = L(u_2, u_j), \quad 1 \leq j \leq k_0,\ j \neq 2. \tag{9.2}$$

Moreover, Lemma 4.9 implies that (9.2) also holds for $j = 2$. Next, we state

Claim 1. $L(u_l, v_j) = -L(v_l, u_j)$, $L(v_l, v_j) = L(u_l, u_j)$, $1 \leq j, l \leq k_0$, j, $l \neq 2$.

To verify the claim, as $u_j \in V_{v_j}(0)$, we first see by Lemma 4.9 that $L(u_j, v_j) = 0$ and $L(v_j, v_j) = L(u_j, u_j)$. Hence the claim is true for $j = l$.

Now we fix $j \neq l$ such that $j, l \neq 2$. By Lemma 4.13, there exists a unique unit vector $u_j^{(l)} \in V_{v_j}(0)$, such that

$$L(u_l, v_j) = -L(v_l, u_j^{(l)}), \quad L(v_l, v_j) = L(u_l, u_j^{(l)}). \tag{9.3}$$

Noting that $\dim (V_{v_j}(0)) = 1$ and $u_j^{(l)}$, $u_j \in V_{v_j}(0)$ are unit vectors, we have $u_j^{(l)} = \epsilon u_j$ with $\epsilon = \pm 1$. Hence from (9.3) we have

$$L(u_l, v_j) = -\epsilon L(v_l, u_j), \quad L(v_l, v_j) = \epsilon L(u_l, u_j). \tag{9.4}$$

On the other hand, by using (4.36), (9.2) and (9.4), we get

$$K(L(v_j, v_l), L(v_2, u_j)) = K(L(v_j, v_l), -L(v_j, u_2)) = -\tau L(v_l, u_2),$$
$$K(L(v_j, v_l), L(v_2, u_j)) = K(\epsilon L(u_j, u_l), L(v_2, u_j)) = -\epsilon\tau L(v_l, u_2).$$

From the comparison of the above two equations we get $\epsilon = 1$.

From (9.4) we have verified Claim 1 and the proof of Lemma 9.1 is fulfilled. □

To continue the proof of Theorem 9.1, we now assume that $k_0 \geq 2$ and let $\{v_1, u_1, \ldots, v_{k_0}, u_{k_0}\}$ be the orthonormal basis of $\mathcal{D}_2$ as constructed in Lemma 9.1.

Given (4.5), Lemmas 4.9, 4.11 and that for $j \neq l, v_j, u_j \in V_{v_l}(\tau) = V_{u_l}(\tau)$, we have the following calculations:

$$
\begin{aligned}
& h(L(v_j,u_l),L(v_j,u_l)) = h(L(v_j,v_l),L(v_j,v_l)) = \tau,\ j \neq l, && (9.5)\\
& h(L(u_j,v_{l_1}),L(u_j,v_{l_2})) = h(L(v_j,u_{l_1}),L(v_j,u_{l_2})) \\
& \quad = h(L(v_j,v_{l_1}),L(v_j,v_{l_2})) = 0, \quad j,l_1,l_2 \text{ distinct}, && (9.6)\\
& h(L(v_{j_1},v_{j_2}),L(v_{j_3},v_{j_4})) = 0, \quad j_1,j_2,j_3,j_4 \text{ distinct}, && (9.7)\\
& h(L(v_j,v_l),L(v_{j_1},u_{l_1})) = 0, \quad j \neq l \text{ and } j_1 \neq l_1, && (9.8)\\
& h(L(v_j,v_j),L(v_j,v_j)) = \tfrac{1}{2}\lambda_1\eta, \quad 1 \le j \le k_0, && (9.9)\\
& h(L(v_j,v_j),L(v_l,v_l)) = \tfrac{1}{2}\mu\eta, \quad 1 \le j \neq l \le k_0, && (9.10)\\
& h(L(v_j,v_j),L(v_j,v_l)) = h(L(v_j,v_j),L(v_j,u_l)) \\
& \quad = h(L(v_j,v_j),L(v_l,u_j)) = 0, \quad 1 \le j \neq l \le k_0; && (9.11)\\
& h(L(v_j,v_j),L(v_{l_1},v_{l_2})) = h(L(v_j,v_j),L(v_{l_1},u_{l_2})) = 0, && (9.12)\\
& \quad 1 \le j,l_1,l_2 \text{ distinct} \le k_0.
\end{aligned}
$$

Similar as in the proof of Theorem 8.1, we denote

$$L_j := L(v_1,v_1) + \cdots + L(v_j,v_j) - jL(v_{j+1},v_{j+1}),\ 1 \le j \le k_0 - 1.$$

Then direct calculations show that $h(L_j,L_j) = 2j(j+1)\tau \neq 0$ for each j, and

$$
\begin{cases}
w_j = \frac{1}{\sqrt{2j(j+1)\tau}} L_j, & 1 \le j \le k_0 - 1,\\
w_{kl} = \frac{1}{\sqrt{\tau}} L(v_k,v_l), & 1 \le k < l \le k_0,\\
w'_{kl} = \frac{1}{\sqrt{\tau}} L(v_k,u_l), & 1 \le k < l \le k_0
\end{cases}
\tag{9.13}
$$

give $\frac{1}{4}(m+1)(m-3)$ mutually orthogonal unit vectors in $\operatorname{Im} L \subset \mathcal{D}_3$. Thus, we have the estimate of the dimension

$$
\begin{aligned}
n &= 1 + \dim(\mathcal{D}_2) + \dim(\mathcal{D}_3)\\
&\ge 1 + m - 1 + \tfrac{1}{4}(m+1)(m-3) = \tfrac{1}{4}(m+1)^2 - 1.
\end{aligned}
\tag{9.14}
$$

Moreover, direct computations show that $\operatorname{Tr} L = 2[L(v_1,v_1) + \cdots + L(v_{k_0},v_{k_0})]$ is orthogonal to all vectors in (9.13), and by using (4.4), (4.8) and the fact that $v_i \in V_{v_j}(\tau)$ for $i \neq j$, we get

$$
\begin{aligned}
\tfrac{1}{4}h(\operatorname{Tr} L, \operatorname{Tr} L) &= \tfrac{1}{2}k_0\eta(\lambda_1 + (k_0-1)\mu)\\
&= \tfrac{1}{32}(m-1)\sqrt{\lambda_1^2 - 4\varepsilon}\Big[(m+1)\lambda_1 - (m-3)\sqrt{\lambda_1^2 - 4\varepsilon}\Big]\\
&=: \rho^2
\end{aligned}
\tag{9.15}
$$

for $\rho \ge 0$. From (9.15) and that $\lambda_1^2 - 4\varepsilon > 0$, the following result is obvious.

Lemma 9.2. $\operatorname{Tr}L = 0$ *if and only if* $\lambda_1 = \frac{m-3}{\sqrt{2(m-1)}}$ *and* $\varepsilon = -1$.

On the other hand, the statement $\operatorname{Tr} L = 0$ has an implicit characterization with a proof totally similar to that of Lemma 8.2.

Lemma 9.3. $\operatorname{Tr} L = 0$ *if and only if* $n = \frac{1}{4}(m+1)^2 - 1$.

Now, we are ready to complete the proof of Theorem 9.1.

Proof of Theorem 9.1. First, if $n \neq \frac{1}{4}(m+1)^2 - 1$, we define a unit vector $t = \frac{1}{2\rho}\operatorname{Tr} L$. We separate the discussions into three cases:

(i) If $n = \frac{1}{4}(m+1)^2$, the previous results show that

$$\{t,\ w_j\ |_{1\le j\le k_0-1},\ w_{kl}\ |_{1\le k<l\le k_0},\ w'_{kl}\ |_{1\le k<l\le k_0}\}$$

is an orthonormal basis of $\operatorname{Im} L = \mathcal{D}_3$.

(ii) If $n > \frac{1}{4}(m+1)^2$, we still have that $\{t,\ w_{kl}\ |_{1\le k<l\le k_0},\ w_j\ |_{1\le j\le k_0-1}\}$ is an orthonormal basis of $\operatorname{Im} L$, but now $\operatorname{Im} L \subsetneq \mathcal{D}_3$. Denote $\tilde{n} = n - \frac{1}{4}(m+1)^2$ and let $\{\tilde{w}_1, \ldots, \tilde{w}_{\tilde{n}}\}$ be an orthonormal basis of $\mathcal{D}_3 \setminus \operatorname{Im} L$ such that

$$\{t,\ w_j\ |_{1\le j\le k_0-1},\ w_{kl}\ |_{1\le k<l\le k_0},\ w'_{kl}\ |_{1\le k<l\le k_0},\ \tilde{w}_r\ |_{1\le r\le \tilde{n}}\}$$

is an orthonormal basis of $\mathcal{D}_3$.

(iii) If $n = \frac{1}{4}(m+1)^2 - 1$, then an orthonormal basis of $\operatorname{Im} L = \mathcal{D}_3$ is given by

$$\{w_j\ |_{1\le j\le k_0-1},\ w_{kl}\ |_{1\le k<l\le k_0},\ w'_{kl}\ |_{1\le k<l\le k_0}\}.$$

Now, following the proof of Theorem 6.1 in [12], we can proceed in the same way as in the proof of Theorem 8.1 to obtain the following conclusions:

If $n = \frac{1}{4}(m+1)^2$, we can apply Theorem 3.4 to conclude that M^n can be decomposed as the Calabi product of a locally strongly convex centroaffine hypersurface with parallel cubic form and a point.

If $n > \frac{1}{4}(m+1)^2$, we can apply Theorem 3.2 to conclude that M^n can be decomposed as the Calabi product of two locally strongly convex centroaffine hypersurfaces with parallel cubic form.

If $n = \frac{1}{4}(m+1)^2 - 1$, then M^n is centroaffinely equivalent to the standard embedding $\mathrm{SL}(\frac{m+1}{2}, \mathbb{C})/\mathrm{SU}(\frac{m+1}{2}) \hookrightarrow \mathbb{R}^{n+1}$. □

10. Case $\{\mathfrak{C}_m\}_{2\le m\le n-1}$ with $k_0 \geq 2$ and $\mathfrak{p} = 3$

In this section, we will prove the following theorem.

Theorem 10.1. *Let M^n be a locally strongly convex centroaffine hypersurface in $\mathbb{R}^{n+1}$ which has parallel and non-vanishing cubic form. If $\mathfrak{C}_m$ with $2 \le m \le n-1$ occurs and the integers k_0 and $\mathfrak{p}$, as defined in Sect. 4.5, satisfy $k_0 \ge 2$ and $\mathfrak{p} = 3$, then $n \ge \frac{1}{8}(m+1)(m+3) - 1$. Moreover, we have either*

(i) *$n = \frac{1}{8}(m+1)(m+3)$, M^n can be decomposed as the Calabi product of a locally strongly convex centroaffine hypersurface with parallel cubic form and a point, or*

(ii) $n > \frac{1}{8}(m+1)(m+3)$, *$M^n$ can be decomposed as the Calabi product of two locally strongly convex centroaffine hypersurfaces with parallel cubic form, or*

(iii) $n = \frac{1}{8}(m+1)(m+3) - 1$, *$M^n$ is centroaffinely equivalent to the standard embedding* $\mathrm{SU}^*(\frac{m+3}{2})/\mathrm{Sp}(\frac{m+3}{4}) \hookrightarrow \mathbb{R}^{n+1}$.

Now we have $\dim \mathcal{D}_2 = m - 1 = 4k_0$ and $m \geq 9$. Similar to Lemma 7.1 of [12], we will prove the following lemma.

Lemma 10.1. *In the decomposition* (4.28)*, if we have $k_0 \geq 2$ and $\mathfrak{p} = 3$, then there exist unit vectors $x_j, y_j, z_j \in V_{v_j}(0)$ $(1 \leq j \leq k_0)$ such that the orthonormal basis $\{v_1, x_1, y_1, z_1; \dots; v_{k_0}, x_{k_0}, y_{k_0}, z_{k_0}\}$ of $\mathcal{D}_2$ satisfies the relations*

$$\begin{cases} L(v_j, v_l) = L(x_j, x_l) = L(y_j, y_l) = L(z_j, z_l), \\ L(v_j, x_l) = -L(x_j, v_l) = -L(y_j, z_l) = L(z_j, y_l), \\ L(v_j, y_l) = -L(y_j, v_l) = -L(z_j, x_l) = L(x_j, z_l), \\ L(v_j, z_l) = -L(z_j, v_l) = -L(x_j, y_l) = L(y_j, x_l), \end{cases} \quad 1 \leq j, \ l \leq k_0. \tag{10.1}$$

Proof. As doing before, we denote $V_j = \{v_j\} \oplus V_{v_j}(0), \ 1 \leq l \leq k_0$. Let us fix two orthogonal unit vectors $x_1, y_1 \in V_{v_1}(0)$. By using Lemmas 4.12 and 4.13, for each $j \neq 1$, we have two unit vectors $x_j, \ y_j \in V_{v_j}(0)$ such that

$$\begin{cases} L(v_j, v_1) = L(x_j, x_1) = L(y_j, y_1), \\ L(v_j, x_1) = -L(x_j, v_1), \quad L(v_j, y_1) = -L(y_j, v_1). \end{cases} \tag{10.2}$$

Then, according to Lemma 4.13, we further have unit vectors $z_1^j \in V_{x_1}(0)$ and $z_j \in V_{x_j}(0)$ such that

$$\begin{cases} L(v_j, z_1^j) = L(y_j, x_1), \ L(v_j, x_1) = -L(y_j, z_1^j), \\ L(z_j, v_1) = L(x_j, y_1), \ L(z_j, y_1) = -L(x_j, v_1). \end{cases} \tag{10.3}$$

The important is that we have the following

Claim 1. *For each $j \neq 2$, $\{x_1, y_1, z_1^j\}$ is an orthonormal basis of $V_{v_1}(0)$ and $\{x_j, y_j, z_j\}$ is an orthonormal basis of $V_{v_j}(0)$.*

To verify this claim, it suffices to show that

$$z_1^j \perp v_1, \quad z_1^j \perp y_1, \quad x_j \perp y_j, \quad z_j \perp y_j, \quad z_j \perp v_j.$$

In fact, by using (10.2) and (10.3), we obtain that

$$\begin{aligned} &\tau h(z_1^j, v_1) = h(L(z_1^j, v_j), \ L(v_j, v_1)) = h(L(y_j, x_1), \ L(y_j, y_1)) = 0, \\ &\tau h(z_1^j, y_1) = h(L(z_1^j, v_j), \ L(v_j, y_1)) = h(L(y_j, x_1), \ -L(y_j, v_1)) = 0, \\ &\tau h(x_j, y_j) = h(L(x_j, v_1), \ L(y_j, v_1)) = h(L(v_j, x_1), \ L(v_j, y_1)) = 0, \\ &\tau h(z_j, y_j) = h(L(z_j, v_1), \ L(y_j, v_1)) = h(L(x_j, y_1), \ -L(v_j, y_1)) = 0, \\ &\tau h(z_j, v_j) = h(L(z_j, v_1), \ L(v_j, v_1)) = h(L(x_j, y_1), \ L(x_j, x_1)) = 0. \end{aligned}$$

From these relations, we immediately get the claim.

Next, by using Lemmas 4.12 and 4.13, (10.2) and (10.3), we have

$$\begin{cases} L(v_j, v_1) = L(x_j, x_1) = L(y_j, y_1) = L(z_j, z_1^j), \\ L(v_j, x_1) = -L(x_j, v_1) = -L(y_j, z_1^j) = L(z_j, y_1), \\ L(v_j, y_1) = -L(y_j, v_1) = -L(z_j, x_1) = L(x_j, z_1^j), \\ L(v_j, z_1^j) = -L(z_j, v_1) = -L(x_j, y_1) = L(y_j, x_1), \end{cases} \quad 2 \leq j \leq k_0. \tag{10.4}$$

From these relations we can prove the following assertion:

Claim 2. $z_1^2 = \cdots = z_1^{k_0} =: z_1$.

In fact, by Claim 1, we know that for $j \neq l$ $(j, l \geq 2)$ we have $z_1^j = \varepsilon_{jl} z_1^l$ with $\varepsilon_{jl} = \pm 1$. From Lemma 4.14 and (10.4) we get

$$\begin{aligned} \varepsilon_{jl} \tau L(v_j, v_l) &= K(L(z_1^j, v_j), L(z_1^l, v_l)) \\ &= K(L(y_j, x_1), L(y_l, x_1)) = \tau L(y_j, y_l). \end{aligned} \tag{10.5}$$

Similarly, we get

$$\varepsilon_{jl} L(x_j, x_l) = L(y_j, y_l) = L(z_j, z_l) = L(v_j, v_l). \tag{10.6}$$

From (10.5) and (10.6) we have $\varepsilon_{jl} = 1$. Thus Claim 2 is verified.

Moreover, the following relations hold

$$L(v_j, v_l) = L(x_j, x_l) = L(y_j, y_l) = L(z_j, z_l), \; j \neq l, \; j, \; l \geq 2. \tag{10.7}$$

From (10.4) and apply Lemma 4.14, we get

$$\begin{aligned} \tau L(x_j, y_l) &= K(L(y_1, x_j), L(y_1, y_l)) \\ &= K(L(z_j, v_1), L(v_1, v_l)) = \tau L(z_j, v_l). \end{aligned} \tag{10.8}$$

Similarly, we have the following relations:

$$L(z_j, x_l) = L(y_j, v_l), \quad L(y_j, z_l) = L(x_j, v_l). \tag{10.9}$$

Combination of (10.4), Claim 2 and (10.7)–(10.9), we get (10.1) immediately. □

To continue the proof of Theorem 10.1, we now assume that $k_0 \geq 2$ and let $\{v_1, x_1, y_1, z_1; \ldots; v_{k_0}, x_{k_0}, y_{k_0}, z_{k_0}\}$ be the orthonormal basis of $\mathcal{D}_2$ as constructed in Lemmas 4.9 and 10.1. According to (4.5), Lemma 4.11 and the fact that for $j \neq l$, $v_j, \, x_j, \, y_j, \, z_j \in V_{v_l}(\tau) = V_{x_l}(\tau) = V_{y_l}(\tau) = V_{z_l}(\tau)$, we have

$$\begin{aligned}
&h(L(v_j,x_l),L(v_j,x_l)) = h(L(v_j,y_l),L(v_j,y_l)) = h(L(v_j,z_l),L(v_j,z_l))\\
&\quad = h(L(v_j,v_l),L(v_j,v_l)) = \tau, \quad j \neq l, &(10.10)\\
&h(L(v_j,v_{l_1}),L(v_j,v_{l_2})) = h(L(v_j,x_{l_1}),L(v_j,x_{l_2}))\\
&\quad = h(L(x_j,v_{l_1}),L(x_j,v_{l_2})) = h(L(y_j,v_{l_1}),L(y_j,v_{l_2}))\\
&\quad = h(L(v_j,y_{l_1}),L(v_j,y_{l_2})) = h(L(z_j,v_{l_1}),L(z_j,v_{l_2}))\\
&\quad = h(L(v_j,z_{l_1}),L(v_j,z_{l_2})) = 0, \quad j,l_1,l_2 \text{ distinct}, &(10.11)\\
&h(L(v_{j_1},v_{j_2}),L(v_{j_3},v_{j_4})) = h(L(v_{j_1},x_{j_2}),L(v_{j_3},x_{j_4}))\\
&\quad = h(L(v_{j_1},y_{j_2}),L(v_{j_3},y_{j_4})) = h(L(v_{j_1},z_{j_2}),L(v_{j_3},z_{j_4}))\\
&\quad = 0, \quad j_1,j_2,j_3,j_4 \text{ distinct}, &(10.12)\\
&h(L(v_j,v_l),L(v_{j_1},x_{l_1})) = h(L(v_j,v_l),L(v_{j_1},y_{l_1}))\\
&\quad = h(L(v_j,v_l),L(v_{j_1},z_{l_1})) = 0, \quad j \neq l \text{ and } j_1 \neq l_1, &(10.13)\\
&h(L(v_j,v_j),L(v_j,v_j)) = \tfrac{1}{2}\lambda_1\eta, \quad 1 \leq j \leq k_0, &(10.14)\\
&h(L(v_j,v_j),L(v_l,v_l)) = \tfrac{1}{2}\mu\eta, \quad j \neq l, &(10.15)\\
&h(L(v_j,v_j),L(v_j,v_l)) = h(L(v_j,v_j),L(v_j,x_l)) = h(L(v_j,v_j),L(v_j,y_l))\\
&\quad = h(L(v_j,v_j),L(v_j,z_l)) = h(L(v_j,v_j),L(v_l,x_j))\\
&\quad = h(L(v_j,v_j),L(v_l,y_j)) = h(L(v_j,v_j),L(v_l,z_j))\\
&\quad = 0, \quad j \neq l, &(10.16)\\
&h(L(v_j,v_j),L(v_{l_1},v_{l_2})) = h(L(v_j,v_j),L(v_{l_1},x_{l_2}))\\
&\quad = h(L(v_j,v_j),L(v_{l_1},y_{l_2})) = h(L(v_j,v_j),L(v_{l_1},z_{l_2}))\\
&\quad = 0, \quad j,l_1,l_2 \text{ distinct}. &(10.17)
\end{aligned}$$

As in preceding sections we denote

$$L_j := L(v_1,v_1) + \cdots + L(v_j,v_j) - jL(v_{j+1},v_{j+1}), \quad 1 \leq j \leq k_0 - 1.$$

Then we have $h(L_j,L_j) = 2j(j+1)\tau \neq 0$ for each j. Moreover,

$$\begin{cases}
w_j = \frac{1}{\sqrt{2j(j+1)\tau}}L_j, & 1 \leq j \leq k_0 - 1,\\
w_{kl} = \frac{1}{\sqrt{\tau}}L(v_k,v_l), & 1 \leq k < l \leq k_0,\\
w'_{kl} = \frac{1}{\sqrt{\tau}}L(v_k,x_l), & 1 \leq k < l \leq k_0,\\
w''_{kl} = \frac{1}{\sqrt{\tau}}L(v_k,y_l), & 1 \leq k < l \leq k_0,\\
w'''_{kl} = \frac{1}{\sqrt{\tau}}L(v_k,z_l), & 1 \leq k < l \leq k_0
\end{cases} \tag{10.18}$$

give $\frac{1}{8}(m+1)(m-5)$ mutually orthogonal unit vectors in $\operatorname{Im} L \subset \mathcal{D}_3$. Thus we have the estimate of the dimension

$$\begin{aligned}
n &= 1 + \dim(\mathcal{D}_2) + \dim(\mathcal{D}_3)\\
&\geq 1 + m - 1 + \tfrac{1}{8}(m+1)(m-5) = \tfrac{1}{8}(m+1)(m+3) - 1. &(10.19)
\end{aligned}$$

Further direct computations show that $\operatorname{Tr} L = 4[L(v_1,v_1) + \cdots + L(v_{k_0},v_{k_0})]$ is orthogonal to all vectors in (10.18), and by using the fact that $v_i \in V_{v_j}(\tau)$ $(i \neq j)$, (4.4) and (4.8) we have the calculation

$$\begin{aligned}\tfrac{1}{16}h(\operatorname{Tr} L, \operatorname{Tr} L) &= \tfrac{1}{2}k_0\eta(\lambda_1 + (k_0-1)\mu)\\ &= \tfrac{1}{128}(m-1)\sqrt{\lambda_1^2 - 4\varepsilon}\Big((m+3)\lambda_1 - (m-5)\sqrt{\lambda_1^2-4\varepsilon}\,\Big)\\ &=: \rho^2 \end{aligned} \qquad (10.20)$$

for $\rho \geq 0$. From (10.20) and that $\lambda_1^2 - 4\varepsilon > 0$, the following result is obvious.

Lemma 10.2. $\operatorname{Tr} L = 0$ *if and only if* $\lambda_1 = \frac{m-5}{2\sqrt{m-1}}$ *and* $\varepsilon = -1$.

On the other hand, by similar proof of Lemmas 8.2 and 9.3, we also obtain the following implicit characterization of the statement $\operatorname{Tr} L = 0$.

Lemma 10.3. $\operatorname{Tr} L = 0$ *if and only if* $n = \frac{1}{8}(m+1)(m+3) - 1$.

Now, we are ready to complete the proof of Theorem 10.1.

Proof of Theorem 10.1. We consider three cases:

(i) $n = \frac{1}{8}(m+1)(m+3)$.
(ii) $n > \frac{1}{8}(m+1)(m+3)$.
(iii) $n = \frac{1}{8}(m+1)(m+3) - 1$.

For Cases (i) and (ii), as $\operatorname{Tr} L \neq 0$, we can define a unit vector $t := \frac{1}{4\rho}\operatorname{Tr} L$.

For Case (i), from previous discussions we see that

$$\{t,\, w_j|_{1\le j\le k_0-1},\, w_{kl}|_{1\le k<l\le k_0},\, w'_{kl}|_{1\le k<l\le k_0},\, w''_{kl}|_{1\le k<l\le k_0},\, w'''_{kl}\,|_{1\le k<l\le k_0}\}$$

forms an orthonormal basis of $\operatorname{Im} L = \mathcal{D}_3$.

For Case (ii), as $\operatorname{Im} L \subsetneq \mathcal{D}_3$, we choose $\{\tilde w_1, \ldots, \tilde w_{\tilde n}\}$ in $\mathcal{D}_3 \backslash \operatorname{Im} L$ such that

$$\begin{aligned}\{t,\, w_j\,|_{1\le j\le k_0-1},\, w_{kl}\,|_{1\le k<l\le k_0},\, w'_{kl}\,|_{1\le k<l\le k_0},\, w''_{kl}\,|_{1\le k<l\le k_0},\\ w'''_{kl}\,|_{1\le k<l\le k_0},\, \tilde w_r\,|_{1\le r\le \tilde n}\}\end{aligned}$$

is an orthonormal basis of $\mathcal{D}_3$.

For Case (iii), we see that

$$\{w_j|_{1\le j\le k_0-1},\, w_{kl}|_{1\le k<l\le k_0},\, w'_{kl}|_{1\le k<l\le k_0},\, w''_{kl}|_{1\le k<l\le k_0},\, w'''_{kl}|_{1\le k<l\le k_0}\}$$

is an orthonormal basis of $\operatorname{Im} L = \mathcal{D}_3$.

Now, following the proof of Theorem 7.1 in [12], we can proceed in the same way as in the proof of Theorem 8.1 to obtain the following conclusions:

If $n = \frac{1}{8}(m+1)(m+3)$, we can apply Theorem 3.4 to conclude that M^n can be decomposed as the Calabi product of a locally strongly convex centroaffine hypersurface with parallel cubic form and a point.

If $n > \frac{1}{8}(m+1)(m+3)$, we can apply Theorem 3.2 to conclude that M^n can be decomposed as the Calabi product of two locally strongly convex centroaffine hypersurfaces with parallel cubic form.

If $n = \frac{1}{8}(m+1)(m+3) - 1$, then M^n is centroaffinely equivalent to the standard embedding $\mathrm{SU}^*(\frac{m+3}{2})/\mathrm{Sp}(\frac{m+3}{4}) \hookrightarrow \mathbb{R}^{n+1}$. □

11. Case $\{\mathfrak{C}_m\}_{2\le m\le n-1}$ with $k_0 \ge 2$ and $\mathfrak{p} = 7$

In this section, we will prove the following theorem.

Theorem 11.1. *Let M^n be a locally strongly convex centroaffine hypersurface in $\mathbb{R}^{n+1}$ which has parallel and non-vanishing cubic form. If $\mathfrak{C}_m$ with $2 \le m \le n-1$ occurs and the integers k_0 and $\mathfrak{p}$, as defined in Sect. 4.5, satisfy $k_0 \ge 2$ and $\mathfrak{p} = 7$, then $k_0 = 2$, $m = 17$ and $n \ge 26$. Moreover, we have either*

(i) *$n = 27$, M^n can be decomposed as the Calabi product of a locally strongly convex centroaffine hypersurface with parallel cubic form and a point, or*
(ii) *$n > 27$, M^n can be decomposed as the Calabi product of two locally strongly convex centroaffine hypersurfaces with parallel cubic form, or*
(iii) *$n = 26$, M^n is centroaffinely equivalent to the standard embedding $\mathrm{E}_{6(-26)}/\mathrm{F}_4 \hookrightarrow \mathbb{R}^{27}$.*

To prove Theorem 11.1, a key ingredient is the following lemma whose proof is similar to that of Lemma 8.1 in [12].

Lemma 11.1. *If in the decomposition (4.28), $k_0 \ge 2$ and $\mathfrak{p} = 7$, then we can choose an orthonormal basis $\{x_j\}_{1\le j\le 7}$ for $V_{v_1}(0)$ and an orthonormal basis $\{y_j\}_{1\le j\le 7}$ for $V_{v_2}(0)$ so that by identifying $e_j(v_1) = x_j$ and $e_j(v_2) = y_j$, we have the relations*

$$L(e_j(v_1), e_l(v_2)) = -L(v_1, e_je_l(v_2)) = -L(e_le_j(v_1), v_2), \tag{11.1}$$

for $1 \le j, l \le 7$, where e_je_l denotes a product defined by the following multiplication table.

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-\mathrm{id}$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	$-\mathrm{id}$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	$-\mathrm{id}$	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	$-\mathrm{id}$	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	$-\mathrm{id}$	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-\mathrm{id}$	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-\mathrm{id}$

Proof. As before we denote $V_j = \{v_j\} \oplus V_{v_j}(0)$, $1 \le j \le k_0$. First we fix any two orthogonal unit vectors $x_1, x_2 \in V_{v_1}(0)$. Then, by Lemmas 4.12 and 4.13, we can consecutively find unit vectors $y_1, y_2 \in V_{v_2}(0)$ and $x_3 \in V_{x_2}(0)$, such that

$$L(y_1, v_1) = -L(x_1, v_2), \quad L(y_1, x_1) = L(v_1, v_2), \tag{11.2}$$

$$L(y_2, v_1) = -L(x_2, v_2), \quad L(y_2, x_2) = L(v_1, v_2), \tag{11.3}$$

$$L(y_1, x_2) = -L(x_3, v_2), \quad L(y_1, x_3) = L(x_2, v_2). \tag{11.4}$$

From the computation

$$\tau h(x_3, v_1) = h(L(x_3, v_2), L(v_1, v_2)) = h(-L(y_1, x_2), L(y_1, x_1)) = 0, \quad (11.5)$$

we get $x_3 \in V_{v_1}(0)$. Thus, we can further take unit vector $y_3 \in V_{v_2}(0)$ such that

$$L(y_3, v_1) = -L(x_3, v_2),\ L(y_3, x_3) = L(v_1, v_2). \quad (11.6)$$

Claim 1. $\{x_1, x_2, x_3, v_1\}$ are orthonormal vectors. Similarly, $\{y_1, y_2, y_3, v_2\}$ are orthonormal vectors.

In fact, by using (11.2) and (11.4), we have

$$\tau h(x_3, x_1) = h(L(x_3, v_2), L(x_1, v_2)) = h(L(y_1, x_2), L(y_1, v_1)) = 0,$$

so we have $x_3 \perp x_1$, and the mutual orthogonality of $\{x_1, x_2, x_3, v_1\}$ immediately follows. The assertion that $\{y_1, y_2, y_3\}$ are mutually orthogonal vectors can be proved using Lemmas 4.12 and 4.13. Hence we have the Claim 1.

By (11.2), (11.3) and (11.6), we get the relation

$$L(y_1, x_1) = L(y_2, x_2) = L(y_3, x_3) = L(v_1, v_2), \quad (11.7)$$

which together with Lemmas 4.12, 4.13, Claim 1 and (11.4), imply that

$$L(y_1, x_3) = -L(x_1, y_3) = L(x_2, v_2), \quad (11.8)$$

$$L(x_1, y_2) = -L(y_1, x_2) = L(x_3, v_2), \quad (11.9)$$

$$L(y_3, x_2) = -L(x_3, y_2) = -L(y_1, v_1). \quad (11.10)$$

Now we pick an arbitrary unit vector $x_4 \in V_{v_1}(0)$ such that it is orthogonal to all $x_1,\ x_2$ and x_3. Then, inductively and following the preceding argument, we can find unit vectors $x_5, x_6, x_7 \in V_{v_1}(0)$ and $y_4, y_5, y_6, y_7 \in V_{v_2}(0)$ such that the following relations hold:

$$\begin{aligned} &L(x_4, y_1) = -L(x_1, y_4) = -L(x_5, v_2) = L(y_5, v_1),\\ &L(x_4, y_4) = L(x_1, y_1) = L(x_5, y_5) = L(v_1, v_2),\ L(x_4, v_2) = L(x_5, y_1), \end{aligned} \quad (11.11)$$

$$\begin{cases} L(x_4, y_2) = -L(x_6, v_2) = L(y_6, v_1),\\ L(x_4, v_2) = L(x_6, y_2),\ L(x_6, y_6) = L(v_1, v_2), \end{cases} \quad (11.12)$$

$$\begin{cases} L(x_4, y_3) = -L(x_7, v_2) = L(y_7, v_1),\\ L(x_4, v_2) = L(x_7, y_3),\ L(x_7, y_7) = L(v_1, v_2). \end{cases} \quad (11.13)$$

Similar to Claim 1, applying Lemmas 4.12, 4.13, (11.2)–(11.4) and (11.6)–(11.13), we obtain:

Claim 2. $\{x_1, \ldots, x_7, v_1\}$ are orthonormal vectors. Similarly, $\{y_1, \ldots, y_7, v_2\}$ are orthonormal vectors.

From (11.7), (11.11)–(11.13), it follows immediately that

$$L(x_i, y_i) = L(v_1, v_2), \quad i = 1, \ldots, 7, \quad (11.14)$$

and therefore, by Lemmas 4.12 and 4.13, we obtain

$$L(x_i,y_j)=-L(y_i,x_j),\ L(x_i,v_2)=-L(y_i,v_1),\ 1\le i\ne j\le 7. \tag{11.15}$$

Finally, based on the relations (11.2)–(11.4) and (11.6)–(11.15), the following relations can be established (cf. proof of Lemma 8.1 in [12]):

$$\begin{aligned} &L(x_4,y_5)=-L(v_1,y_1), \quad L(x_4,y_6)=-L(v_1,y_2),\\ &L(x_4,y_7)=-L(v_1,y_3), \end{aligned} \tag{11.16}$$

$$\begin{aligned} &L(x_5,y_1)=-L(v_1,y_4), \quad L(x_5,y_2)=L(v_1,y_7),\\ &L(x_5,y_3)=-L(v_1,y_6), \quad L(x_5,y_6)=L(v_1,y_3),\\ &L(x_5,y_7)=-L(v_1,y_2), \end{aligned} \tag{11.17}$$

$$\begin{aligned} &L(x_6,y_1)=-L(v_1,y_7), \quad L(x_6,y_2)=-L(v_1,y_4),\\ &L(x_6,y_3)=L(v_1,y_5), \quad L(x_6,y_7)=L(v_1,y_1), \end{aligned} \tag{11.18}$$

$$\begin{aligned} &L(x_7,y_1)=L(v_1,y_6), \quad L(x_7,y_2)=-L(v_1,y_5),\\ &L(x_7,y_3)=-L(v_1,y_4). \end{aligned} \tag{11.19}$$

In a similar way as above, all relations in (11.1) can be verified, and thus we complete the proof of Lemma 11.1. □

Now, we can present the following crucial and remarkable lemma with a simplified proof (comparing to that of Lemma 8.2 in [12]) included.

Lemma 11.2. *Suppose that in the decomposition* (4.28) *we have* $k_0\ge 2$ *and* $\mathfrak{p}=7$. *Then it must be the case that* $k_0=2$.

Proof. Suppose on the contrary that $k_0\ge 3$. Following the same argument as in the proof of Lemma 11.1 for $V_{v_1}(0)$ and $V_{v_2}(0)$, we choose a basis $\{x_1,x_2,\tilde{x}_3,x_4,\tilde{x}_5,\tilde{x}_6,\tilde{x}_7\}$ of $V_{v_1}(0)$ and a basis $\{z_1,z_2,z_3,z_4,z_5,z_6,z_7\}$ of $V_{v_3}(0)$ such that all the following relations hold:

$$L(e_j(v_1),e_l(v_3))=-L(v_1,e_je_l(v_3))=-L(e_le_j(v_1),v_3),\ 1\le j,l\le 7. \tag{11.20}$$

Now, we have two orthonormal bases of $V_{v_1}(0)$, i.e. $\{x_1,x_2,\tilde{x}_3,x_4,\tilde{x}_5,\tilde{x}_6,\tilde{x}_7\}$ and $\{x_1,x_2,x_3,x_4,x_5,x_6,x_7\}$. We first show that $\tilde{x}_i=x_i$ for $i=3,5,6,7$:

By (4.36) and (11.20), we get

$$\tau L(y_1,z_1)=K(L(y_1,x_2),L(x_2,z_1))=K(-L(x_3,v_2),-L(x_3,v_3))=\tau L(v_2,v_3).$$

Thus, similarly, we can prove that

$$L(y_1,z_1)=\cdots=L(y_7,z_7)=L(v_2,v_3). \tag{11.21}$$

Since $\{x_1,x_2,\tilde{x}_3,x_4,\tilde{x}_5,\tilde{x}_6,\tilde{x}_7\}$ and $\{x_1,x_2,x_3,x_4,x_5,x_6,x_7\}$ are two orthonormal bases for $V_{v_1}(0)$, we may assume that $x_3=b_3\tilde{x}_3+b_5\tilde{x}_5+b_6\tilde{x}_6+b_7\tilde{x}_7$. Then we have the following calculation

$$\begin{aligned}\tau L(y_2, z_2) &= K(L(v_1, y_2), L(v_1, z_2)) = -K(L(x_3, y_1), L(v_1, z_2))\\ &= b_3 K(L(\tilde{x}_3, y_1), L(\tilde{x}_3, z_1)) + b_5 K(L(\tilde{x}_5, y_1), L(\tilde{x}_5, z_7))\\ &\quad - b_6 K(L(\tilde{x}_6, y_1), L(\tilde{x}_6, z_4)) - b_7 K(L(\tilde{x}_7, y_1), L(\tilde{x}_7, z_5))\\ &= b_3 \tau L(y_1, z_1) + b_5 \tau L(y_1, z_7) - b_6 \tau L(y_1, z_4) - b_7 \tau L(y_1, z_5).\end{aligned} \tag{11.22}$$

On the other hand, by (11.21) and that $L(y_1, z_1), L(y_1, z_4), L(y_1, z_5)$ and $L(y_1, z_7)$ are mutually orthogonal, (11.22) implies that $b_3 = 1, b_5 = b_6 = b_7 = 0$ and hence $x_3 = \tilde{x}_3$. Similarly, we can verify that $x_i = \tilde{x}_i$ for $i = 5, 6, 7$.

In order to complete the proof of Lemma 11.2, we will first use (11.1) and (11.20) to show that we have also similar relations between $V_{v_2}(0)$ and $V_{v_3}(0)$, i.e.,

$$L(e_j(v_2), e_l(v_3)) = -L(v_2, e_j e_l(v_3)) = -L(e_l e_j(v_2), v_3), \quad 1 \le j, l \le 7. \tag{11.23}$$

In fact, for $j = l$, by Lemma 4.14, (11.1) and (11.20), we have

$$\begin{aligned}\tau L(e_j(v_2), e_j(v_3)) &= K(L(e_j(v_2), e_k(v_1)), L(e_k(v_1), e_j(v_3)))\\ &= K(L(v_2, e_j e_k(v_1)), L(e_j e_k(v_1), v_3)) = \tau L(v_2, v_3).\end{aligned}$$

For $j \ne l$, according to the multiplication table in Lemma 11.1, there exists a unique k and $\epsilon = \pm 1$ such that $e_l e_j = \epsilon e_k, e_j e_k = \epsilon e_l, e_k e_l = \epsilon e_j$. It follows, by applying (4.36), (11.1) and (11.20), that

$$\begin{aligned}\tau L(e_j(v_2), e_l(v_3)) &= K(L(e_j(v_2), v_1), L(v_1, e_l(v_3)))\\ &= K(L(-\epsilon e_l e_k(v_2), v_1), L(v_1, e_l(v_3)))\\ &= \epsilon K(L(e_k(v_2), e_l(v_1)), -L(v_3, e_l(v_1)))\\ &= -\epsilon \tau L(e_k(v_2), v_3) = -\tau L(e_l e_j(v_2), v_3)\end{aligned}$$

and that

$$\begin{aligned}\tau L(v_2, e_j e_l(v_3)) &= K(L(e_k(v_1), v_2), L(e_j e_l(v_3), e_k(v_1)))\\ &= K(L(v_2, \epsilon e_l e_j(v_1)), L(-\epsilon e_k(v_3), e_k(v_1)))\\ &= K(L(v_1, -\epsilon e_l e_j(v_2)), L(-\epsilon v_3, v_1)) = \tau L(e_l e_j(v_2), v_3).\end{aligned}$$

Thus, (11.23) holds indeed.

From (11.1), (11.20), (11.23) and Lemma 4.14, we have

$$K(L(v_1, y_6) + L(x_1, y_7), L(x_2, v_3)) = 0. \tag{11.24}$$

On the other hand, we have

$$\begin{aligned}K(L(v_1, y_6), L(x_2, v_3)) &= K(L(v_1, y_6), -L(v_1, z_2)) = -\tau L(z_2, y_6),\\ K(L(x_1, y_7), L(x_2, v_3)) &= K(L(x_1, y_7), -L(x_1, z_3)) = -\tau L(z_3, y_7).\end{aligned}$$

These, together with (11.24), give that

$$L(z_2, y_6) + L(z_3, y_7) = 0. \tag{11.25}$$

(11.23) implies that $L(z_2, y_6) = L(z_3, y_7)$, and by (11.25) we get $L(z_2, y_6) = 0$.

However, we also have the relation $h(L(z_2, y_6), L(z_2, y_6)) = \tau$, which gives the contradiction.

This completes the proof of Lemma 11.2. □

Now, we are ready to complete the proof of Theorem 11.1.

Proof of Theorem 11.1. First, Lemma 11.2 implies that $k_0 = 2$ and $\dim(\mathcal{D}_2) = 16$.

Let $\{v_1, v_2, x_j, y_j,\ 1 \le j \le 7\}$ be the orthonormal basis of $\mathcal{D}_2$ as constructed in Lemma 11.1 such that all relations in (11.1) hold. Then we easily see that the image of L is spanned by

$$\{L(v_1, v_1),\ L(v_1, v_2),\ L(v_2, v_2);\ L(v_1, y_j)\ |_{1 \le j \le 7}\}.$$

Define $L_1 = L(v_1, v_1) - L(v_2, v_2)$, then we have

$$h(L_1, L_1) = 4\tau \neq 0. \tag{11.26}$$

We now easily see that there exist nine orthonormal vectors in $\operatorname{Im} L \subset \mathcal{D}_3$:

$$w_0 = \tfrac{1}{\sqrt{4\tau}} L_1,\ w_1 = \tfrac{1}{\sqrt{\tau}} L(v_1, v_2),\ w_{j+1} := \tfrac{1}{\sqrt{\tau}} L(v_1, y_j),\ 1 \le j \le 7.$$

Note that $\mathrm{Tr} L = 8(L(v_1, v_1) + L(v_2, v_2))$ is orthogonal to $\{w_0, w_1, w_{j+1}\ |_{1 \le j \le 7}\}$, by using (4.4), (4.8) and the fact $v_1 \in V_{v_2}(\tau)$, we obtain

$$\tfrac{1}{64} h(\mathrm{Tr}\, L, \mathrm{Tr}\, L) = \eta(\lambda_1 + \mu) = \tfrac{1}{4}\sqrt{\lambda_1^2 - 4\varepsilon}\Big(3\lambda_1 - \sqrt{\lambda_1^2 - 4\varepsilon}\Big) =: \rho^2 \tag{11.27}$$

for $\rho \ge 0$. Then we have the estimate of the dimension

$$n = 1 + \dim(\mathcal{D}_2) + \dim(\mathcal{D}_3) \ge 26. \tag{11.28}$$

From (11.27) and the fact $\lambda_1^2 - 4\varepsilon > 0$, we have the following result.

Lemma 11.3. $\mathrm{Tr}\, L = 0$ *if and only if* $2\lambda_1^2 = 1$ *and* $\varepsilon = -1$.

On the other hand, by similar proof of Lemma 8.2, we also obtain the following implicit characterization of the statement $\mathrm{Tr}\, L = 0$.

Lemma 11.4. $\mathrm{Tr}\, L = 0$ *if and only if* $n = 26$.

Then, if $n = 27$ or $n \ge 28$, we can define a unit vector $t = \frac{1}{8\rho}\mathrm{Tr}\, L$ so that we can construct an orthonormal basis for $\mathcal{D}_3$ and T_pM^n, respectively, and we get the similar expressions as in Lemmas 8.3, 8.4 and 8.5 which allows us to conclude that M^n can be decomposed as the Calabi product of a locally strongly convex centroaffine hypersurface with parallel cubic form and a point, or the Calabi product of two locally strongly convex centroaffine hypersurfaces with parallel cubic form.

If $n = 26$, by calculating the difference tensor K with respect to the preceding typical basis of T_pM^n totally similar to previous sections as in Sects. 8–10, we can also show that $\mathrm{Tr}\,(K_X) = 0$ for any $X \in T_pM^n$. Then, according to

Theorem 8.1 of [12], we can finally conclude that M^n is locally centroaffnely equivalent to the standard embedding $\mathrm{E}_{6(-26)}/\mathrm{F}_4 \hookrightarrow \mathbb{R}^{27}$ that was introduced in [1] and also [12].

In conclusion, we have completed the proof of Theorem 11.1. □

12. Completion of the Proof of Theorem 1.1

If $C = 0$, according to subsection 7.1.1 of [21], and also Lemma 2.1 of [15], we have (i).

For hypersurfaces with $C \neq 0$, according to Lemma 4.1, it is necessary and sufficient to consider the cases $\{\mathfrak{C}_m\}_{1\leq m\leq n}$ as well as the exceptional case $\mathfrak{B}$.

Firstly, by Theorems 4.1 and 4.2, we have settled the two cases, $\mathfrak{C}_1$ and $\mathfrak{C}_n$, from which we have (ii).

Next, case $\mathfrak{B}$ is settled by Theorem 5.1, from which we have (viii).

Then, being of independent meaning we have Theorem 6.1, by which a complete classification is given for the lowest dimension $n = 2$. Theorem 6.1 verifies the assertion of Theorem 1.1 explicitly for $n = 2$.

The remaining cases, i.e. $\mathfrak{C}_m$ with $2 \leq m \leq n-1$, are completely settled by Proposition 4.2 and subsequent five theorems, i.e. Theorems 7.1, 8.1, 9.1, 10.1 and 11.1. In these cases, we have (ii)–(vii).

From all of above discussions, we have completed the proof of Theorem 1.1.

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References

[1] Birembaux, O., Djorić, M.: Isotropic affine spheres. Acta Math. Sin. (Engl. Ser.) **28**, 1955–1972 (2012)

[2] Bokan, N., Nomizu, K., Simon, U.: Affine hypersurfaces with parallel cubic forms. Tôhoku Math. J. **42**, 101–108 (1990)

[3] Bott, R., Milnor, J.: On the parallelizability of the spheres. Bull. Am. Math. Soc. **64**, 87–89 (1958)

[4] Cheng, X.X., Hu, Z.J.: Classification of locally strongly convex isotropic centroaffine hypersurfaces, submitted. Preprint (2016)

[5] Dillen, F., Vrancken, L.: 3-Dimensional affine hypersurfaces in $\mathbb{R}^4$ with parallel cubic form. Nagoya Math. J. **124**, 41–53 (1991)

[6] Dillen, F., Vrancken, L., Yaprak, S.: Affine hypersurfaces with parallel cubic form. Nagoya Math. J. **135**, 153–164 (1994)

[7] Dillen, F., Verbouwe, G., Vrancken, L.: Cubic form geometry for immersions in centro-affine and graph hypersurfacess. Results Math. **43**, 88–95 (2003)

[8] Hildebrand, R.: Centro-affine hypersurface immersions with parallel cubic form. Beitr. Algebra Geom. **56**, 593–640 (2015)

[9] Hu, Z.J., Li, C.C., Li, H., Vrancken, L.: Lorentzian affine hypersurfaces with parallel cubic form. Results Math. **59**, 577–620 (2011)

[10] Hu, Z.J., Li, H., Simon, U., Vrancken, L.: On locally strongly convex affine hypersurfaces with parallel cubic form. Part I. Differ. Geom. Appl. **27**, 188–205 (2009)

[11] Hu, Z.J., Li, H., Vrancken, L.: Characterization of the Calabi product of hyperbolic affine hyperspheres. Results Math. **52**, 299–314 (2008)

[12] Hu, Z.J., Li, H., Vrancken, L.: Locally strongly convex affine hypersurfaces with parallel cubic form. J. Differ. Geom. **87**, 239–307 (2011)

[13] Kervaire, M.: Non-parallelizability of the n-sphere for $n > 7$. Proc. N. A. S. **44**(3), 280–283 (1958)

[14] Li, A.-M., Li, H., Simon, U.: Centroaffine Bernstein problems. Differ. Geom. Appl. **20**, 331–356 (2004)

[15] Li, A.-M., Liu, H.L., Schwenk-Schellschmidt, A., Simon, U., Wang, C.P.: Cubic form methods and relative Tchebychev hypersurfaces. Geom. Dedicata **66**, 203–221 (1997)

[16] Li, A.-M., Simon, U., Zhao, G., Hu, Z.J.: Global Affine Differential Geometry of Hypersurfaces, 2nd edn. W. de Gruyter, Berlin (2015)

[17] Li, A.-M., Wang, C.P.: Canonical centroaffine hypersurfaces in $\mathbb{R}^{n+1}$. Results Math. **20**, 660–681 (1991)

[18] Liu, H.L., Wang, C.P.: The centroaffine Tchebychev operator. Results Math. **27**, 77–92 (1995)

[19] Liu, H.L., Wang, C.P.: Centroaffine surfaces with parallel traceless cubic form. Bull. Belg. Math. Soc. **4**, 493–499 (1997)

[20] Nomizu, K., Sasaki, T.: Affine Differential Geometry. Geometry of Affine Immersions. Cambridge Tracts in Mathematics, vol. 111. Cambridge University Press, Cambridge (1994)

[21] Simon, U., Schwenk-Schellschmidt, A., Viesel, H.: Introduction to the Affine Differential Geometry of Hypersurfaces. Lecture Notes of the Science University of Tokyo. Science University of Tokyo, Tokyo (1991)

[22] Vrancken, L., Li, A.-M., Simon, U.: Affine spheres with constant affine sectional curvature. Math. Z. **206**, 651–658 (1991)

[23] Wang, C.P.: Centroaffine minimal hypersurfaces in $\mathbb{R}^{n+1}$. Geom. Dedicata **51**, 63–74 (1994)

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