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Results in Mathematics

Bezier variant of the Bernstein–Durrmeyer type operators

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Abstract. In the present paper, we introduce the Bezier-variant of Durrmeyer modification of the Bernstein operators based on a function τ, which is infinite times continuously differentiable and strictly increasing function on $[0, 1]$ such that $\tau(0) = 0$ and $\tau(1) = 1$. We give the rate of approximation of these operators in terms of usual modulus of continuity and K-functional. Next, we establish the quantitative Voronovskaja type theorem. In the last section we obtain the rate of convergence for functions having derivative of bounded variation.

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1. Introduction

In 1912, Bernstein [6] defined a sequence of positive linear operators for $f \in C[0, 1]$, as

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0, 1]$$

which preserves linear functions. To make convergence faster, King [12] introduced a modification of these operators as

$$((B_n f) \circ r_n)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} (r_n(x))^k (1 - r_n(x))^{n-k},$$

which depends on a sequence $r_n(x)$ of continuous functions on $[0, 1]$ with $0 \leq r_n(x) \leq 1$, for each $x \in [0, 1]$ and considered a particular case for the sequence

$r_n(x)$ such that the corresponding operators preserve the test function e_0 and e_2 of the Bohman–Korovkin theorem. Gonska et al. [10] constructed sequences of King-type operators which are based on a strictly increasing continuous function τ such that $\tau(0) = 0$ and $\tau(1) = 1$. These operators are defined by $V_n : C[0,1] \to C[0,1]$

$$V_n^{\tau} f = (B_n f) \circ \tau_n = (B_n f) \circ (B_n \tau)^{-1} \circ \tau.$$

Inspired by the above ideas, for any function τ being infinite times continuously differentiable on $[0,1]$, such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0,1]$, Morales et al. [13] defined the sequence of linear Bernstein type operators for $f \in C[0,1]$ by

$$B_n^{\tau}(f;x) = \sum_{k=0}^{n} \left(f \circ \tau^{-1}\right)\left(\frac{k}{n}\right)\binom{n}{k}\tau^k(x)\left(1-\tau(x)\right)^{n-k}, \tag{1.1}$$

and also investigated its shape preserving and convergence properties as well as its asymptotic behavior and saturation. This type of approximation operators generalizes the Korovkin set from $\{1, e_1, e_2\}$ to $\left\{1,\ \tau,\ \tau^2\right\}$ and also presents a better degree of approximation depending on τ. To approximate the Lebesgue integrable functions on $[0,1]$, Acar et al. [1] defined the Durrmeyer type modication for the operators (1.1) as

$$D_n^{\tau}(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{\tau}(x)\int_0^1 \left(f \circ \tau^{-1}\right)(t)\, p_{n,k}(t)\,\mathrm{d}t, \tag{1.2}$$

where

$$p_{n,k}^{\tau}(x) := \binom{n}{k}\tau^k(x)\left(1-\tau(x)\right)^{n-k}$$

and

$$p_{n,k}(x) := \binom{n}{k}x^k(1-x)^{n-k},$$

and studied Voronovskaya type asymptotic formula as well as its quantitative version and the local approximation properties of D_n^{τ} in quantitative form in terms of K-functional and Ditzian–Totik moduli of smoothness.

A Bezier curve is a parametric curve frequently used in computer graphics and image processing. These are mainly used in interpolation, approximation, curve fitting etc. Zeng and Piriou [16] pioneered the study of Bezier-variant of Bernstein operators. Subsequently, many researchers defined the Bezier-variant of several operators ([7,11,14,15] etc.).

Motivated by these ideas, we introduce the Bezier-variant of the operators given by (1.2) as

$$D_n^{\tau,\theta}(f;x) = (n+1)\sum_{k=0}^{n} Q_{n,k}^{\tau,\theta}(x)\int_0^1 \left(f \circ \tau^{-1}\right)(t)\, p_{n,k}(t)\,\mathrm{d}t, \tag{1.3}$$

where $Q^{\tau,\theta}_{n,k}(x) = \left[I^{\tau}_{n,k}(x)\right]^{\theta} - \left[I^{\tau}_{n,k+1}(x)\right]^{\theta}$, $\theta \geq 1$ with $I^{\tau}_{n,k}(x) = \sum_{j=k}^{n} p^{\tau}_{n,k}(x)$, when $k \leq n$ and 0 otherwise.

The aim of this paper is to study the degree of approximation in terms of the modulus of continuity and the K-functional for the operators given by (2.1). The quantitative Voronoskaja type theorem and the rate of convergence of the functions having derivatives of bounded variation for these operators is also investigated.

2. Auxiliary Results

Lemma 1 [1]. *For the operators D^{τ}_{n}, one has*

$$D^{\tau}_{n} e_0 = e_0, \quad D^{\tau}_{n} \tau = \frac{1 + \tau n}{n+2}, \quad D^{\tau}_{n} \tau^2 = \frac{\tau^2 n\,(n-1) + 4n\tau + 2}{(n+2)\,(n+3)}.$$

Consequently, for the m-th order central moment of operators D^{τ}_{n} defined as

$$\mu^{\tau}_{n,m}(x) = D^{\tau}_{n}\left((\tau(t) - \tau(x))^{m}; x\right), \ m \in \mathbb{N},$$

there follows

$$\mu^{\tau}_{n,0}(x) = 1, \ \mu^{\tau}_{n,1}(x) = \frac{1 - 2\tau(x)}{n+2} \tag{2.1}$$

$$\mu^{\tau}_{n,2}(x) = \frac{\tau(x)\,(1 - \tau(x))\,(2n-6) + 2}{(n+2)\,(n+3)}, \tag{2.2}$$

for all $n \in \mathbb{N}$.

By a simple calculation, we have

$$\mu^{\tau}_{n,4}(x) = \frac{4\varphi^2_{\tau}(x)\left\{(3n^2 + 25n - 210)\varphi^2_{\tau}(x) + (6n+12)\right\} + 24}{(n+2)(n+3)(n+4)(n+5)}.$$

Remark 1 [1]. *For all $n \in \mathbb{N}$ we have*

$$\mu^{\tau}_{n,2}(x) \leq \frac{2}{n+2}\delta^2_{n,\tau}(x), \tag{2.3}$$

where $\delta^2_{n,\tau}(x) := \varphi^2_{\tau}(x) + \frac{1}{n+3}$, $\varphi^2_{\tau}(x) := \tau(x)\,(1 - \tau(x))$, $x \in [0,1]$.

Lemma 2 [1]. *For every $f \in C[0,1]$, we have*

$$\|D^{\tau}_{n}(f;.)\| \leq \|f\|.$$

Applying Lemma 1 , the proof of this lemma easily follows. Hence the details are omitted.

Lemma 3 *Let $f \in C[0,1]$. Then, we have*

$$\left\|D^{\tau,\theta}_{n}(f;.)\right\| \leq \theta \parallel f \parallel.$$

Proof. Using the inequality $\mid a^{\theta}-b^{\theta}\mid\leq\theta\mid a-b\mid$ with $0\leq a,b\leq 1,\theta\geq 1$ and from the definition of $Q_{n,k}^{\tau,\theta}$, we have

$$0<\left[I_{n,k}^{\tau}(x)\right]^{\theta}-\left[I_{n,k+1}^{\tau}(x)\right]^{\theta}\leq\theta(I_{n,k}^{\tau}(x)-I_{n,k+1}^{\tau}(x)=\theta p_{n,k}^{\tau}(x).$$

Hence from the definition $D_{n}^{\tau,\theta}$ and Lemma 2, we obtain

$$\left\|D_{n}^{\tau,\theta}(f)\right\|\leq\theta\left\|D_{n}^{\tau}(f)\right\|\leq\theta\parallel f\parallel.$$

□

Remark 2. *We have*

$$\begin{aligned}D_{n}^{\tau,\theta}(e_{0};x)&=\sum_{k=0}^{n}Q_{n,k}^{(\theta)}(x)=\left[J_{n,0}(x)\right]^{\theta}\\&=\left[\sum_{j=0}^{n}p_{n,j}^{\tau}(x)\right]^{\theta}=1,\ \textit{since}\ \sum_{j=0}^{n}p_{n,k}^{\tau}(x)=1.\end{aligned}$$

3. Main Results

Throughout this paper we assume that $\inf\limits_{x\in[0,1]}\tau'(x)\geq a, a\in\mathbb{R}^{+}$.

Theorem 1. *For $f\in C[0,1]$ and $x\in[0,1]$, there holds*

$$\left|D_{n}^{\tau,\theta}(f;x)-f(x)\right|\leq\left\{1+\sqrt{2\theta\left(\varphi_{\tau}^{2}(x)+\frac{1}{n+3}\right)}\right\}\omega\left(f;\sqrt{\frac{1}{n}}\right),$$

where $\omega(f;\delta)$ is the usual modulus of continuity.

Proof. By linearity of the operators $D_{n}^{\tau,\theta}$, we get

$$\begin{aligned}\left|D_{n}^{\tau,\theta}(f;x)-f(x)\right|&\leq(n+1)\sum_{k=0}^{n}Q_{n,k}^{\tau,\theta}(x)\int_{0}^{1}p_{n,k}(t)\left|\left(f\circ\tau^{-1}\right)(t)-f(x)\right|\mathrm{d}t\\&\leq(n+1)\sum_{k=0}^{n}Q_{n,k}^{\tau,\theta}(x)\int_{0}^{1}p_{n,k}(t)\left|\left(1+\frac{|t-\tau(x)|}{\delta}\right)\mathrm{d}t.\right.\end{aligned}$$

By applying Holder's inequality and Lemma2, we obtain

$$\left|D_{n}^{\tau,\theta}(f;x)-f(x)\right|\leq\left\{1+\frac{1}{\delta}\left(D_{n}^{\tau,\theta}\left(\left(\tau(t)-\tau(x)\right)^{2};x\right)\right)^{1/2}\right\}\omega(f;\delta)$$

$$\leq \left\{1+\frac{1}{\delta}\left(\theta D_n^{\tau}\left((\tau(t)-\tau(x))^2;x\right)\right)^{1/2}\right\}\omega(f;\delta)$$
$$\leq \left\{1+\frac{1}{\delta}\sqrt{\frac{2\theta}{n+2}\left(\varphi_\tau^2(x)+\frac{1}{n+3}\right)}\right\}\omega\left(f;\delta\right).$$

Taking $\delta=\sqrt{\frac{1}{n}}$, we get the desired result. □

Now let us recall the definitions of the Ditzian-Totik first order modulus of smoothness and the K-functional [8]. Let $\varphi_\tau(x) := \sqrt{\tau(x)(1-\tau(x))}$ and $f\in C[0,1]$. The first order modulus of smoothness is given by

$$\omega_{\varphi_\tau}(f;t)=\sup_{0<h\leq t}\left\{\left|f\left(x+\frac{h\varphi_\tau(x)}{2}\right)-f\left(x-\frac{h\varphi_\tau(x)}{2}\right)\right|, x\pm\frac{h\varphi_\tau(x)}{2}\in[0,1]\right\}.$$

Further, the appropriate K-functional is defined by

$$K_{\varphi_\tau}(f;t)=\inf_{g\in W_{\varphi_\tau}[0,1]}\{||f-g||+t||\varphi_\tau g'||\}\ (t>0),$$

where $W_{\varphi_\tau}[0,1]=\{g: g\in AC_{loc}[0,1], \|\varphi_\tau g'\|<\infty\}$ and $g\in AC_{loc}[0,1]$ means that g is absolutely continuous on every interval $[a,b]\subset(0,1)$. It is well known [8, p. 11] that there exists a constant $C>0$ such that

$$K_{\varphi_\tau}(f;t)\leq C\omega_{\varphi_\tau}(f;t). \tag{3.1}$$

Theorem 2. *Let $f\in C[0,1]$. Then for every $x\in(0,1)$, we have*

$$\left|D_n^{\tau,\theta}(f;x)-f(x)\right|\leq C(\theta)\omega_{\varphi_\tau}\left(f;\frac{1}{a}\sqrt{\frac{\theta}{n+2}\left(1+\frac{1}{(n+3)\varphi_\tau^2(x)}\right)}\right).$$

Proof. Using the representation

$$h(t)=\left(h\circ\tau^{-1}\right)(\tau(t))=\left(h\circ\tau^{-1}\right)(\tau(x))+\int_{\tau(x)}^{\tau(t)}\left(h\circ\tau^{-1}\right)'(u)\mathrm{d}u$$

we get

$$\left|D_n^{\tau,\theta}(h;x)-h(x)\right|=\left|D_n^{\tau,\theta}\left(\int_{\tau(x)}^{\tau(t)}\left(h\circ\tau^{-1}\right)'(u)\mathrm{d}u\right)\right|. \tag{3.2}$$

But,

$$\left|\int_{\tau(x)}^{\tau(t)}\left(h\circ\tau^{-1}\right)'(u)\mathrm{d}u\right|=\left|\int_x^t\frac{h'(y)}{\tau'(y)}\tau'(y)\mathrm{d}y\right|=\left|\int_x^t\frac{\varphi_\tau(y)}{\varphi_\tau(y)}\cdot\frac{h'(y)}{\tau'(y)}\tau'(y)\mathrm{d}y\right|$$
$$\leq\frac{\|\varphi_\tau h'\|}{a}\left|\int_x^t\frac{\tau'(y)}{\varphi_\tau(y)}\mathrm{d}y\right|, \tag{3.3}$$

and

$$\begin{aligned}\left|\int_x^t \frac{\tau'(y)}{\varphi_\tau(y)}\mathrm{d}y\right| &\leq \left|\int_x^t \left(\frac{1}{\sqrt{\tau(y)}}+\frac{1}{\sqrt{1-\tau(y)}}\right)\tau'(y)\mathrm{d}y\right|\\ &\leq 2\left(\left|\sqrt{\tau(t)}-\sqrt{\tau(x)}\right|+\left|\sqrt{1-\tau(t)}-\sqrt{1-\tau(x)}\right|\right)\\ &= 2\left|\tau(t)-\tau(x)\right|\left(\frac{1}{\sqrt{\tau(t)}+\sqrt{\tau(x)}}+\frac{1}{\sqrt{1-\tau(t)}+\sqrt{1-\tau(x)}}\right)\\ &< 2|\tau(t)-\tau(x)|\left(\frac{1}{\sqrt{\tau(x)}}+\frac{1}{\sqrt{1-\tau(x)}}\right)\leq \frac{2\sqrt{2}|\tau(t)-\tau(x)|}{\varphi_\tau(x)},\end{aligned} \tag{3.4}$$

hence from relations (3.2)–(3.4) and using Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}|D_n^{\tau,\theta}(h;x)-h(x)| &\leq 2\sqrt{2}\frac{\|\varphi_\tau h'\|}{a\varphi_\tau(x)}D_n^{\tau,\theta}\left(|\tau(t)-\tau(x)|;x\right)\\ &\leq 2\sqrt{2}\frac{\|\varphi_\tau h'\|}{a\varphi_\tau(x)}\left[D_n^{\tau,\theta}\left((\tau(t)-\tau(x))^2;x\right)\right]^{1/2}\\ &\leq 2\sqrt{2}\frac{\|\varphi_\tau h'\|}{a\varphi_\tau(x)}\left[\theta D_n^{\tau}\left((\tau(t)-\tau(x))^2;x\right)\right]^{1/2}\\ &\leq \frac{4}{a}\|\varphi_\tau h'\|\sqrt{\frac{\theta}{n+2}\left(1+\frac{1}{(n+3)\varphi_\tau^2(x)}\right)}.\end{aligned} \tag{3.5}$$

Using Lemma 3 and (3.5) it follows that

$$\begin{aligned}&\left|D_n^{\tau,\theta}(f;x)-f(x)\right| \leq \left|D_n^{\tau,\theta}(f-h;x)\right|+|f(x)-h(x)|+\left|D_n^{\tau,\theta}(h;x)-h(x)\right|\\ &\quad\leq \left\{(\theta+1)\|f-h\|+\frac{4}{a}\|\varphi_\tau h'\|\sqrt{\frac{\theta}{n+2}\left(1+\frac{1}{(n+3)\varphi_\tau^2(x)}\right)}\right\}\\ &\quad\leq C_1(\theta)\left\{\|f-h\|+\frac{1}{a}\|\varphi_\tau h'\|\sqrt{\frac{\theta}{n+2}\left(1+\frac{1}{(n+3)\varphi_\tau^2(x)}\right)}\right\},\end{aligned}$$

where $C_1(\theta)=\max\{(\theta+1),4\}$.

Taking infimum on the right hand side of the above inequality over all $g\in W_{\varphi_\tau}[0,1]$, we get

$$\left|D_n^{\tau,\theta}(f;x)-f(x)\right|\leq C_1(\theta)K_{\varphi_\tau}\left(f;\frac{1}{a}\sqrt{\frac{\theta}{n+2}\left(1+\frac{1}{(n+3)\varphi_\tau^2(x)}\right)}\right).$$

Using the relation (3.1), the theorem is proved. □

4. Quantitative Voronovskaja Type Theorem

In this section we prove a quantitative Voronovskaja type theorem for the operator $D_n^{\tau,\theta}$. This result is given using the first order Ditzian–Totik modulus of smoothness. In the recent years, several researchers have made significant contribution in this area [1–4, 9].

Theorem 3. *For any $f \in C^2[0,1]$ and $x \in [0,1]$ the following inequalities hold*

$$\begin{aligned}|\sqrt{n}\left[D_n^{\tau,\theta}(f;x)-f(x)\right]| \leq &\sqrt{2\theta\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}||\left(f\circ\tau^{-1}\right)'||\\&+||\left(f\circ\tau^{-1}\right)''||\frac{\theta}{\sqrt{n}}\varphi_\tau^2(x)\\&+\frac{C}{\sqrt{n}}\omega_{\varphi_\tau}\left((f\circ\tau^{-1})'';\frac{2\sqrt{6}}{an^{1/2}}\varphi_\tau(x)\right)+o(n^{-1});\end{aligned}$$

$$\begin{aligned}|\sqrt{n}\left[D_n^{\tau,\theta}(f;x)-f(x)\right]| \leq &\sqrt{2\theta\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}||\left(f\circ\tau^{-1}\right)'||\\&+||\left(f\circ\tau^{-1}\right)''||\frac{\theta}{\sqrt{n}}\varphi_\tau^2(x)\\&+\frac{C}{\sqrt{n}}\varphi_\tau(x)\omega_{\varphi_\tau}\left((f\circ\tau^{-1})'';\frac{2\sqrt{6}}{an^{1/2}}\right)+o(n^{-1}),\end{aligned}$$

where C is a constant depending on θ.

Proof. Let $f \in C^2[0,1]$ and $x, t \in [0,1]$. Then by Taylor's expansion, we have

$$\begin{aligned}f(t) &= \left(f\circ\tau^{-1}\right)(\tau(t))\\&=\left(f\circ\tau^{-1}\right)(\tau(x))+\left(f\circ\tau^{-1}\right)'(\tau(x))\left(\tau(t)-\tau(x)\right)\\&\quad+\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(f\circ\tau^{-1}\right)''(u)\mathrm{d}u.\end{aligned}$$

Hence

$$\begin{aligned}f(t)-f(x) &= \left(f\circ\tau^{-1}\right)'(\tau(x))\left(\tau(t)-\tau(x)\right)-\frac{1}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))\left(\tau(t)-\tau(x)\right)^2\\&\quad+\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(f\circ\tau^{-1}\right)''(u)\mathrm{d}u\\&\quad-\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(f\circ\tau^{-1}\right)''(\tau(x))\mathrm{d}u\\&=\left(f\circ\tau^{-1}\right)'(\tau(x))\left(\tau(t)-\tau(x)\right)\\&\quad-\frac{1}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))\left(\tau(t)-\tau(x)\right)^2\end{aligned}$$

$$+\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left[\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right]\mathrm{d}u.$$

Applying $D_n^{\tau,\theta}$ to both sides of the above relation, we get

$$\begin{aligned}\left|D_n^{\tau,\theta}(f;x)-f(x)\right| &= \left(f\circ\tau^{-1}\right)'(\tau(x))D_n^{\tau,\theta}((\tau(t)-\tau(x));x)\\ &\quad -\frac{1}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))D_n^{\tau,\theta}((\tau(t)-\tau(x))^2;x)\\ &\quad +\left|D_n^{\tau,\theta}\left(\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left[\left(f\circ\tau^{-1}\right)''(u)\right.\right.\right.\\ &\quad \left.\left.\left.-\left(f\circ\tau^{-1}\right)''(\tau(x))\right]\mathrm{d}u;x\right)\right| \qquad (4.1)\end{aligned}$$

For $g\in W_{\varphi_\tau[0,1]}$, we have

$$\begin{aligned}&\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right|\mathrm{d}u\right|\\ &\le\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(g\circ\tau^{-1}\right)(u)\right|\mathrm{d}u\right|\\ &\quad+\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(g\circ\tau^{-1}\right)(u)-\left(g\circ\tau^{-1}\right)(\tau(x))\right|\mathrm{d}u\right|\\ &\quad+\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(g\circ\tau^{-1}\right)(\tau(x))-\left(f\circ\tau^{-1}\right)''(\tau(x))\right|\mathrm{d}u\right|\\ &=\left|\int_x^t\left|\left(f\circ\tau^{-1}\right)''(\tau(y))-g(y)\right||\tau(t)-\tau(y)|\,\tau'(y)\mathrm{d}y\right|\\ &\quad+\left|\int_x^t|g(y)-g(x)|\,|\tau(t)-\tau(y)|\,\tau'(y)\mathrm{d}y\right|\\ &\quad+\left|\int_x^t\left|g(x)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right||\tau(t)-\tau(y)|\,\tau'(y)\mathrm{d}y\right|\\ &\le 2\|\left(f\circ\tau^{-1}\right)''-g\|\left|\int_x^t|\tau(t)-\tau(y)|\tau'(y)\mathrm{d}y\right|\\ &\quad+\left|\int_x^t\left|\int_x^y|g'(v)|dv\right||\tau(t)-\tau(y)|\tau'(y)\mathrm{d}y\right|\\ &\le \|\left(f\circ\tau^{-1}\right)''-g\|(\tau(t)-\tau(x))^2\\ &\quad+\|\varphi_\tau g'\|\left|\int_x^t\left|\int_x^y\frac{dv}{\varphi_\tau(v)}\right||\tau(t)-\tau(y)|\tau'(y)\mathrm{d}y\right|.\end{aligned}$$

Using the inequality [8, p. 141]

$$\frac{|y-v|}{v(1-v)} \leq \frac{|y-x|}{x(1-x)},\ v \text{ is between } y \text{ and } x,$$

we can write

$$\frac{|\tau(y)-\tau(v)|}{\tau(v)(1-\tau(v))} \leq \frac{|\tau(y)-\tau(x)|}{\tau(x)(1-\tau(x))}.$$

Therefore,

$$\begin{aligned}
&\left|\int_{\tau(x)}^{\tau(t)} |\tau(t)-u| \left|\left(f\circ\tau^{-1}\right)''(u) - \left(f\circ\tau^{-1}\right)''(\tau(x))\right| \mathrm{d}u\right| \\
&\quad \leq \|\left(f\circ\tau^{-1}\right)'' - g\|(\tau(t)-\tau(x))^2 \\
&\qquad + \|\varphi_\tau g'\| \left|\int_x^t \left|\int_x^y \frac{|\tau(y)-\tau(x)|^{1/2}}{\tau'(v)\varphi_\tau(x)} \cdot \frac{\tau'(v)}{|\tau(y)-\tau(v)|^{1/2}} dv\right|\right. \\
&\qquad \left. \times |\tau(t)-\tau(y)|\tau'(y)\mathrm{d}y\right| \\
&\quad \leq \|\left(f\circ\tau^{-1}\right)'' - g\|(\tau(t)-\tau(x))^2 \\
&\qquad + 2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)\left|\int_x^t |\tau(y)-\tau(x)|\,|\tau(t)-\tau(y)|\tau'(y)\mathrm{d}y\right| \\
&\quad \leq \|\left(f\circ\tau^{-1}\right)'' - g\|(\tau(t)-\tau(x))^2 \\
&\qquad + 2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)\left|\int_x^t (\tau(t)-\tau(x))^2\tau'(y)\mathrm{d}y\right| \\
&\quad \leq \|\left(f\circ\tau^{-1}\right)'' - g\|(\tau(t)-\tau(x))^2 + 2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)|\tau(t)-\tau(x)|^3. \qquad (4.2)
\end{aligned}$$

Now combining the relations (4.1)–(4.2), applying Lemma 1, Remark 1, Lemma 3 and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
\left|D_n^{\tau,\theta}(f;x) - f(x)\right| &\leq |\left(f\circ\tau^{-1}\right)'(\tau(x))|D_n^{\tau,\theta}(|\tau(t)-\tau(x)|\,;x) \\
&\quad + \frac{1}{2}|\left(f\circ\tau^{-1}\right)''(\tau(x)|D_n^{\tau,\theta}\left((\tau(t)-\tau(x))^2\,;x\right) \\
&\quad + \|(f\circ\tau^{-1})'' - g\|D_n^{\tau,\theta}((\tau(t)-\tau(x))^2;x) \\
&\quad + 2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)D_n^{\tau,\theta}(|\tau(t)-\tau(x)|^3;x) \\
&\leq ||\left(f\circ\tau^{-1}\right)'||\left(D_n^{\tau,\theta}\left(((\tau(t)-\tau(x))^2;x\right)\right)^{1/2} \\
&\quad + \frac{1}{2}||\left(f\circ\tau^{-1}\right)''||D_n^{\tau,\theta}((\tau(t)-\tau(x))^2\,;x) \\
&\quad + \|(f\circ\tau^{-1})'' - g\|D_n^{\tau,\theta}\left((\tau(t)-\tau(x))^2;x\right)
\end{aligned}$$

$$
\begin{aligned}
&+\ \frac{2\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)\left(D_n^{\tau,\theta}\left((\tau(t)-\tau(x))^2;x\right)\right)^{1/2} \\
&\times\left(D_n^{\tau,\theta}\left((\tau(t)-\tau(x))^4;x\right)\right)^{1/2} \\
\leq\ & ||\left(f\circ\tau^{-1}\right)'||\left(\theta D_n^\tau(\left((\tau(t)-\tau(x))^2\right);x)\right)^{1/2} \\
&+\ \frac{1}{2}||\left(f\circ\tau^{-1}\right)''||\left(\theta D_n^\tau((\tau(t)-\tau(x))^2\,;x)\right) \\
&+\ \|(f\circ\tau^{-1})''-g\|\left(\theta D_n^\tau\left((\tau(t)-\tau(x))^2;x\right)\right) \\
&+\ \frac{2\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)\left(\theta D_n^\tau\left((\tau(t)-\tau(x))^2;x\right)\right)^{1/2} \\
&\times\left(\theta D_n^\tau\left((\tau(t)-\tau(x))^4;x\right)\right)^{1/2} \\
=\ & \sqrt{\frac{2\theta}{n+2}\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}||\left(f\circ\tau^{-1}\right)'|| \\
&+\ ||\left(f\circ\tau^{-1}\right)''||\frac{\theta}{n+2}\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\} \\
&+\ \frac{2\theta}{n+2}\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}\|\left(f\circ\tau^{-1}\right)''-g\|+\varphi_\tau^{-1}(x)\frac{2\|\varphi_\tau g'\|}{a}.
\end{aligned}
$$

$$
\begin{aligned}
&\sqrt{\frac{2\theta}{n+2}\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}\sqrt{\theta\left\{\frac{4\varphi_\tau^2(x)\left\{(3n^2+25n-210)\varphi_\tau^2(x)+(6n+12)\right\}+24}{(n+2)(n+3)(n+4)(n+5)}\right\}} \\
&\leq\sqrt{\frac{2\theta}{n+2}\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}||\left(f\circ\tau^{-1}\right)'||+||\left(f\circ\tau^{-1}\right)''||\frac{\theta}{n+2}\varphi_\tau^2(x) \\
&\quad+\ \frac{2\theta}{n+2}\left\{\varphi_\tau^2(x)\|\left(f\circ\tau^{-1}\right)''-g\|+\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau(x)\frac{2\sqrt{6}}{n^{1/2}}\right\}+o(n^{-3/2}).
\end{aligned}
$$

Because $\varphi_\tau^2(x)\leq\varphi_\tau(x)\leq 1$, $x\in[0,1]$ we obtain

$$
\begin{aligned}
\left|D_n^{\tau,\theta}(f;x)-f(x)\right|\leq\ & \sqrt{\frac{2\theta}{n+2}\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}||\left(f\circ\tau^{-1}\right)'|| \\
&+\ ||\left(f\circ\tau^{-1}\right)''||\frac{\theta}{n+2}\varphi_\tau^2(x) \\
&+\ \frac{2\theta}{n+2}\left\{\|(f\circ\tau^{-1})''-g\|+\frac{2\sqrt{6}}{an^{1/2}}\varphi_\tau(x)\|\varphi_\tau g'\|\right\}+o(n^{-3/2}); \quad (4.3)
\end{aligned}
$$

$$
\begin{aligned}
\left|D_n^{\tau,\theta}(f;x)-f(x)\right| &\leq \sqrt{\frac{2\theta}{n+2}\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}\,||\left(f\circ\tau^{-1}\right)'||\\
&\quad+||\left(f\circ\tau^{-1}\right)''||\frac{\theta}{n+2}\varphi_\tau^2(x)\\
&\quad+\frac{2\theta}{n+2}\varphi_\tau(x)\left\{\|(f\circ\tau^{-1})''-g\|+\frac{2\sqrt{6}}{an^{1/2}}\|\varphi_\tau g'\|\right\}+o(n^{-3/2}). \qquad (4.4)
\end{aligned}
$$

Taking the infimum on the right hand side of (4.3) and (4.4) over all $g \in W_{\varphi_\tau}[0,1]$, we get

$$
\begin{aligned}
|\sqrt{n}\left[D_n^{\tau,\theta}(f;x)-f(x)\right]| &\leq \sqrt{2\theta\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}\,||\left(f\circ\tau^{-1}\right)'||\\
&\quad+||\left(f\circ\tau^{-1}\right)''||\frac{\theta}{\sqrt{n}}\varphi_\tau^2(x)\\
&\quad+\frac{C}{\sqrt{n}}K_{\varphi_\tau}\left((f\circ\tau^{-1})'';\frac{2\sqrt{6}}{an^{1/2}}\varphi_\tau(x)\right)+o(n^{-1});\\
|\sqrt{n}\left[D_n^{\tau,\theta}(f;x)-f(x)\right]| &\leq \sqrt{2\theta\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}\,||\left(f\circ\tau^{-1}\right)'||\\
&\quad+||\left(f\circ\tau^{-1}\right)''||\frac{\theta}{\sqrt{n}}\varphi_\tau^2(x)\\
&\quad+\frac{C}{\sqrt{n}}\varphi_\tau(x)K_{\varphi_\tau}\left((f\circ\tau^{-1})'';\frac{2\sqrt{6}}{an^{1/2}}\right)+o(n^{-1}).
\end{aligned}
$$

Using (3.1), we reach the desired result. □

Lastly we discuss the approximation of functions with a derivative of bounded variation on $[0,1]$. Let $DBV[0,1]$ denote the class of differentiable functions f defined on $[0,1]$, whose derivative $f^{'}$ are of bounded variation on $[0,1]$. The functions
$f \in DBV[0,1]$ has the following representation

$$f(x)=\int_0^x g(t)\mathrm{d}t+f(0),$$

where $g \in BV[0,1]$, i.e., g is a function of bounded variation on $[0,1]$.
The operators $D_n^{\tau,\theta}$ can be expressed in an integral form as follows:

$$D_n^{\tau,\theta}(f;x)=\int_0^1 K_n^{\tau,\theta}(x,t)(f\circ\tau^{-1})(t)\mathrm{d}t, \qquad (4.5)$$

where the kernel $K_n^{\tau,\theta}$ is given by

$$K_n^{\tau,\theta}(x,t)=(n+1)\sum_{k=0}^{n}Q_{n,k}^{\tau,\theta}(x)\,p_{n,k}(t)\,.$$

Lemma 4. *For a fixed $x \in (0,1)$ and sufficiently large n, we have*

(i) $\xi_n^{\tau,\theta}(x,y) = \int_0^y K_n^{\tau,\theta}(x,t)\mathrm{d}t \leq \dfrac{2\theta}{n+2}\dfrac{\delta_{n,\tau}^2(x)}{(\tau(x)-y)^2}$, $0 \leq y < \tau(x)$,

(ii) $1-\xi_n^{\tau,\theta}(x,z) = \int_z^1 K_n^{\tau,\theta}(x,t)\mathrm{d}t \leq \dfrac{2\theta}{n+2}\dfrac{\delta_{n,\tau}^2(x)}{(z-\tau(x))^2}$, $\tau(x) < z < 1$,

where $\delta_{n,\tau}^2(x)$ is defined in Remark 1.

Proof. (i) Using Remark 1 and Lemma 3, we get

$$\begin{aligned}\xi_n^{\tau,\theta}(x,y) &= \int_0^y K_n^{\tau,\theta}(x,t)\mathrm{d}t \leq \int_0^y \left(\frac{\tau(x)-t}{\tau(x)-y}\right)^2 K_n^{\tau,\theta}(x,t)\mathrm{d}t\\ &\leq \frac{D_n^{\tau,\theta}((\tau(t)-\tau(x))^2;x)}{(\tau(x)-y)^2} \leq \theta\frac{D_n^{\tau}((\tau(t)-\tau(x))^2;x)}{(\tau(x)-y)^2} \leq \theta\frac{\delta_{n,\tau}^2(x)}{(\tau(x)-y)^2}.\end{aligned}$$

The proof of (ii) is similar hence the details are omitted. □

Theorem 4. *Let $f \in DBV[0,1]$. Then, for every $x \in (0,1)$ and sufficiently large n, we have*

$$\begin{aligned}&|D_n^{\tau,\theta}(f;x)-f(x)|\\ &\leq \left\{\frac{1}{\theta+1}\left|(f\circ\tau^{-1})'(\tau(x+))+\theta(f\circ\tau^{-1})'(\tau(x-))\right|\right.\\ &\quad\left.+\left|(f\circ\tau^{-1})'(\tau(x+))-(f\circ\tau^{-1})'(\tau(x-))\right|\right\}\sqrt{\frac{2\theta}{n+2}}\delta_{n,\tau}(x)\\ &\quad+\frac{2\theta}{n+2}\frac{\delta_{n,\tau}^2(x)}{\tau(x)}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{\tau(x)-\frac{\tau(x)}{k}}^{\tau(x)}(f\circ\tau^{-1})'_x\right)+\frac{\tau(x)}{\sqrt{n}}\left(\bigvee_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)}(f\circ\tau^{-1})'_x\right)\\ &\quad+\frac{2\theta}{n+2}\frac{\delta_{n,\tau}^2(x)}{(1-\tau(x))}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{\tau(x)}^{\tau(x)+\frac{(1-\tau(x))}{k}}(f\circ\tau^{-1})'_x\right)\\ &\quad+\frac{(1-\tau(x))}{\sqrt{n}}\left(\bigvee_{\tau(x)}^{\tau(x)+\frac{(1-\tau(x))}{\sqrt{n}}}(f\circ\tau^{-1})'_x\right),\end{aligned}$$

where $\bigvee_a^b f$ denotes the total variation of f on $[a,b]$ and $(f\circ\tau^{-1})'_x$ is defined by

$$(f\circ\tau^{-1})'_x(t) = \begin{cases}(f\circ\tau^{-1})'(t)-(f\circ\tau^{-1})'(\tau(x-)), & 0 \leq t < \tau(x)\\ 0, & t = \tau(x)\\ (f\circ\tau^{-1})'(t)-(f\circ\tau^{-1})'(\tau(x+)) & \tau(x) < t < 1.\end{cases} \tag{4.6}$$

Proof. Since $D_n^{\tau,\theta}(1;x)=1$, using (2.3), for every $x\in(0,1)$ we get

$$\begin{aligned} D_n^{\tau,\theta}(f;x)-f(x) &= \int_0^1 K_n^{\tau,\theta}(x,t)((f\circ\tau^{-1})(t)-(f\circ\tau^{-1})(\tau(x)))\mathrm{d}t \\ &= \int_0^1 K_n^{\tau,\theta}(x,t)\int_{\tau(x)}^t (f\circ\tau^{-1})'(u)\mathrm{d}u\mathrm{d}t. \end{aligned} \tag{4.7}$$

For any $f\in DBV[0,1]$, from (4.6) we may write

$$\begin{aligned} &\left(f\circ\tau^{-1}\right)'(u) \\ &= \left(f\circ\tau^{-1}\right)_x'(u)+\frac{1}{\theta+1}\left(\left(f\circ\tau^{-1}\right)'(\tau(x+))+\theta\left(f\circ\tau^{-1}\right)'(\tau(x-))\right) \\ &\quad+\frac{1}{2}\left(\left(f\circ\tau^{-1}\right)'(\tau(x+))-\left(f\circ\tau^{-1}\right)'(\tau(x-))\right)\left(sgn(u-\tau(x))+\frac{\theta-1}{\theta+1}\right) \\ &\quad+\delta_x(u)\left[\left(f\circ\tau^{-1}\right)'(u)-\frac{1}{2}\left(\left(f\circ\tau^{-1}\right)'(\tau(x+))+\left(f\circ\tau^{-1}\right)'(\tau(x-))\right)\right], \end{aligned} \tag{4.8}$$

where

$$\delta_x(u)=\begin{cases}1,\ u=\tau(x)\\ 0,\ u\neq\tau(x)\end{cases}.$$

Obviously,

$$\begin{aligned} &\int_0^1\left(\int_{\tau(x)}^t\left(\left(f\circ\tau^{-1}\right)'(u)-\frac{1}{2}\left(\left(f\circ\tau^{-1}\right)'(\tau(x+))\right.\right.\right. \\ &\qquad\left.\left.\left.+\left(f\circ\tau^{-1}\right)'(\tau(x-))\right)\right)\delta_x(u)\mathrm{d}u\right)K_n^{\tau,\theta}(x,t)\mathrm{d}t=0. \end{aligned} \tag{4.9}$$

Let us take

$$\begin{aligned} A_1 &= \int_0^1\left(\int_{\tau(x)}^t\frac{1}{\theta+1}((f\circ\tau^{-1})'(\tau(x+))+\theta(f\circ\tau^{-1})'(\tau(x-)))\mathrm{d}u\right)K_n^{\tau,\theta}(x,t)\mathrm{d}t \\ &= \frac{1}{\theta+1}\left((f\circ\tau^{-1})'(\tau(x+))+\theta(f\circ\tau^{-1})'(\tau(x-))\right)\int_0^1(t-\tau(x))K_n^{\tau,\theta}(x,t)\mathrm{d}t \\ &= \frac{1}{\theta+1}\left((f\circ\tau^{-1})'(\tau(x+))+\theta(f\circ\tau^{-1})'(\tau(x-))\right)D_n^{\tau,\theta}((\tau(t)-\tau(x));x) \end{aligned} \tag{4.10}$$

and

$$
\begin{aligned}
A_2 &= \int_0^1 K_n^{\tau,\theta}(x,t)\bigg(\int_{\tau(x)}^t \frac{1}{2}\Big((f\circ\tau^{-1})'(\tau(x+)) \\
&\quad - (f\circ\tau^{-1})'(\tau(x-))\Big)\Big(sgn(u-\tau(x)) + \frac{\theta-1}{\theta+1}\Big)\mathrm{d}u\bigg)\mathrm{d}t \\
&= \frac{1}{2}\Big((f\circ\tau^{-1})'(\tau(x+)) - (f\circ\tau^{-1})'(\tau(x-))\Big) \\
&\quad \times\bigg[-\int_0^{\tau(x)}\bigg(\int_t^{\tau(x)}\Big(sgn(u-\tau(x)) + \frac{\theta-1}{\theta+1}\Big)\mathrm{d}u\bigg)K_n^{\tau,\theta}(x,t)\mathrm{d}t \\
&\quad + \int_{\tau(x)}^1\bigg(\int\limits_{\tau(x)}^{t}\Big(sgn(u-\tau(x)) + \frac{\theta-1}{\theta+1}\Big)\mathrm{d}u\bigg)K_n^{\tau,\theta}(x,t)\mathrm{d}t\bigg] \\
&\le \Big|(f\circ\tau^{-1})'(\tau(x+)) - (f\circ\tau^{-1})'(\tau(x-))\Big|\int_0^1 |t-\tau(x)|\, K_n^{\tau,\theta}(x,t)\mathrm{d}t \\
&= \Big|(f\circ\tau^{-1})'(\tau(x+)) - (f\circ\tau^{-1})'(\tau(x-))\Big| D_n^{\tau,\theta}\Big(\,|\tau(t)-\tau(x)|\,;x\Big).
\end{aligned} \tag{4.11}
$$

Combining Eqs. (4.7–4.11), on an application of Cauchy–Schwarz inequality and Remark 1, we obtain

$$
\begin{aligned}
|D_n^{\tau,\theta}(f;x) - f(x)| &\le \frac{1}{\theta+1}\Big|(f\circ\tau^{-1})'(\tau(x+)) + \theta(f\circ\tau^{-1})'(\tau(x-))\Big|\sqrt{\frac{2\theta}{n+2}}\delta_{n,\tau}(x) \\
&+ \Big|(f\circ\tau^{-1})'(\tau(x+)) - (f\circ\tau^{-1})'(\tau(x-))\Big|\sqrt{\frac{2\theta}{n+2}}\delta_{n,\tau}(x) \\
&+ \bigg|\int_0^{\tau(x)}\bigg(\int_{\tau(x)}^t (f\circ\tau^{-1})_x'(u)\mathrm{d}u\bigg)K_n^{\tau,\theta}(x,t)\mathrm{d}t\bigg| \\
&+ \bigg|\int_{\tau(x)}^1\bigg(\int_{\tau(x)}^t (f\circ\tau^{-1})_x'(u)\mathrm{d}u\bigg)K_n^{\tau,\theta}(x,t)\mathrm{d}t\bigg|.
\end{aligned} \tag{4.12}
$$

Now, let

$$
A_n^{\tau,\theta}((f\circ\tau^{-1})_x', x) = \int_0^{\tau(x)}\bigg(\int_{\tau(x)}^t (f\circ\tau^{-1})_x'(u)\mathrm{d}u\bigg)K_n^{\tau,\theta}(x,t)\mathrm{d}t,
$$

and

$$
B_n^{\tau,\theta}((f\circ\tau^{-1})_x', x) = \int_{\tau(x)}^1\bigg(\int_{\tau(x)}^t (f\circ\tau^{-1})_x'(u)\mathrm{d}u\bigg)K_n^{\tau,\theta}(x,t)\mathrm{d}t.
$$

Now to calculate the estimates of the terms $A_n^{\tau,\theta}((f\circ\tau^{-1})'_x, x)$, using the definition of $\xi_n^{\tau,\theta}$ given in Lemma 4, we can write

$$A_n^{\tau,\theta}((f\circ\tau^{-1})'_x, x) = \int_0^{\tau(x)} \left(\int_{\tau(x)}^{t} (f\circ\tau^{-1})'_x(u)\mathrm{d}u\right) \frac{\partial}{\partial t}\xi_n^{\tau,\theta}(x,t)\mathrm{d}t.$$

Applying the integration by parts, we get

$$\begin{aligned}\left|A_n^{\tau,\theta}((f\circ\tau^{-1})'_x, x)\right| &\leq \int_0^{\tau(x)} |(f\circ\tau^{-1})'_x(t)|\xi_n^{\tau,\theta}(x,t)\mathrm{d}t\\ &\leq \int_0^{\tau(x)-\frac{\tau(x)}{\sqrt{n}}} |(f\circ\tau^{-1})'_x(t)|\xi_n^{\tau,\theta}(x,t)\mathrm{d}t\\ &\quad + \int_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)} |(f\circ\tau^{-1})'_x(t)|\xi_n^{\tau,\theta}(x,t)\mathrm{d}t := I_1 + I_2.\end{aligned}$$

Since $(f\circ\tau^{-1})'_x(\tau(x)) = 0$ and $\xi_n^{\tau,\theta}(x,t) \leq 1$, we have

$$\begin{aligned} I_2 &:= \int_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)} \left|(f\circ\tau^{-1})'_x(t) - (f\circ\tau^{-1})'_x(\tau(x))\right|\xi_n^{\tau,\theta}(x,t)\mathrm{d}t\\ &\leq \int_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)} \left(\bigvee_t^{\tau(x)} (f\circ\tau^{-1})'_x\right)\mathrm{d}t\\ &\leq \left(\bigvee_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)} (f\circ\tau^{-1})'_x\right)\int_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)} \mathrm{d}t = \frac{\tau(x)}{\sqrt{n}}\left(\bigvee_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)} f'_x\right).\end{aligned}$$

By applying Lemma 4 and considering $t = \tau(x) - \dfrac{\tau(x)}{u}$, we get

$$\begin{aligned} I_1 &\leq \frac{2\theta}{n+2}\delta_{n,\tau}^2(x)\int_0^{\tau(x)-\frac{\tau(x)}{\sqrt{n}}} \left|(f\circ\tau^{-1})'_x(t) - (f\circ\tau^{-1})'_{\tau(x)}(x)\right|\frac{\mathrm{d}t}{(\tau(x)-t)^2}\\ &\leq \frac{2\theta}{n+2}\delta_{n,\tau}^2(x)\int_0^{\tau(x)-\frac{\tau(x)}{\sqrt{n}}} \left(\bigvee_t^{\tau(x)} (f\circ\tau^{-1})'_x\right)\frac{\mathrm{d}t}{(\tau(x)-t)^2}\\ &= \frac{2\theta}{n+2}\frac{\delta_{n,\tau}^2(x)}{\tau(x)}\int_1^{\sqrt{n}} \left(\bigvee_{\tau(x)-\frac{\tau(x)}{u}}^{\tau(x)} (f\circ\tau^{-1})'_x\right)\mathrm{d}u\\ &\leq \frac{2\theta}{n+2}\frac{\delta_{n,\tau}^2(x)}{\tau(x)}\sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{\tau(x)-\frac{\tau(x)}{k}}^{\tau(x)} (f\circ\tau^{-1})'_x\right).\end{aligned}$$

Therefore,

$$|A_n^{\tau,\theta}((f\circ\tau^{-1})'_x,x)| \le \frac{2\theta}{n+2}\frac{\delta_{n,\tau}^2(x)}{\tau(x)}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{\tau(x)-\frac{\tau(x)}{k}}^{\tau(x)}(f\circ\tau^{-1})'_x\right) + \frac{\tau(x)}{\sqrt{n}}\left(\bigvee_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)}(f\circ\tau^{-1})'_x\right). \tag{4.13}$$

Also, using integration by parts in $B_n^{\tau,\theta}(f'_x,x)$ and applying Lemma 4 with $z=\tau(x)+(1-\tau(x))/\sqrt{n}$, we have

$$\begin{aligned}
&|B_n^{\tau,\theta}((f\circ\tau^{-1})'_x,x)| = \left|\int_{\tau(x)}^{1}\left(\int_{\tau(x)}^{t}(f\circ\tau^{-1})'_x(u)\mathrm{d}u\right)K_n^{\tau,\theta}(x,t)\mathrm{d}t\right|\\
&= \left|\int_{\tau(x)}^{z}\left(\int_{\tau(x)}^{t}(f\circ\tau^{-1})'_x(u)\mathrm{d}u\right)\frac{\partial}{\partial t}(1-\xi_n^{\tau,\theta}(x,t))\mathrm{d}t\right.\\
&\quad\left.+\int_{z}^{1}\left(\int_{\tau(x)}^{t}(f\circ\tau^{-1})'_x(u)\mathrm{d}u\right)\frac{\partial}{\partial t}(1-\xi_n^{\tau,\theta}(x,t))\mathrm{d}t\right|\\
&= \left|\left[\int_{\tau(x)}^{t}((f\circ\tau^{-1})'_x(u)\mathrm{d}u)(1-\xi_n^{\tau,\theta}(x,t))\right]_{\tau(x)}^{z}\right.\\
&\quad-\int_{\tau(x)}^{z}(f\circ\tau^{-1})'_x(t)(1-\xi_n^{\tau,\theta}(x,t))\mathrm{d}t\\
&\quad+\left[\int_{\tau(x)}^{t}((f\circ\tau^{-1})'_x(u)\mathrm{d}u)(1-\xi_n^{\tau,\theta}(x,t))\right]_{z}^{1}\\
&\quad\left.-\int_{z}^{1}(f\circ\tau^{-1})'_x(t)(1-\xi_n^{\tau,\theta}(x,t))\mathrm{d}t\right|\\
&= \left|\int_{\tau(x)}^{z}(f\circ\tau^{-1})'_x(t)(1-\xi_n^{\tau,\theta}(x,t))\mathrm{d}t\right.\\
&\quad\left.+\int_{z}^{1}(f\circ\tau^{-1})'_x(t)(1-\xi_n^{\tau,\theta}(x,t))\mathrm{d}t\right|\\
&\le \frac{2\theta}{n+2}\delta_{n,\tau}^2(x)\int_{z}^{1}\left(\bigvee_{\tau(x)}^{t}(f\circ\tau^{-1})'_x\right)(t-\tau(x))^{-2}\mathrm{d}t\\
&\quad+\int_{\tau(x)}^{z}\bigvee_{\tau(x)}^{t}(f\circ\tau^{-1})'_x\mathrm{d}t
\end{aligned}$$

$$\leq \frac{2\theta}{n+2}\delta_{n,\tau}^2(x)\int_{\tau(x)+\frac{(1-\tau(x))}{\sqrt{n}}}^{1}\left(\bigvee_{\tau(x)}^{t}(f\circ\tau^{-1})'_x\right)(t-\tau(x))^{-2}\mathrm{d}t$$
$$+\frac{1-\tau(x)}{\sqrt{n}}\left(\bigvee_{\tau(x)}^{\tau(x)+\frac{(1-\tau(x))}{\sqrt{n}}}(f\circ\tau^{-1})'_x\right).$$

By substituting $u=(1-\tau(x))/(t-\tau(x))$, we get

$$|B_n^{\tau,\theta}f'_x,x)|\leq \frac{2\theta}{n+2}\delta_{n,\tau}^2(x)\int_1^{\sqrt{n}}\left(\bigvee_{\tau(x)}^{\tau(x)+\frac{(1-\tau(x))}{u}}(f\circ\tau^{-1})'_x\right)(1-\tau(x))^{-1}\mathrm{d}u$$
$$+\frac{1-\tau(x)}{\sqrt{n}}\left(\bigvee_{\tau(x)}^{\tau(x)+\frac{(1-\tau(x))}{\sqrt{n}}}(f\circ\tau^{-1})'_x\right)$$
$$\leq \frac{2\theta}{n+2}\frac{\delta_{n,\tau}^2(x)}{1-\tau(x)}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{\tau(x)}^{\tau(x)+\frac{(1-\tau(x))}{k}}(f\circ\tau^{-1})'_x\right)$$
$$+\frac{1-\tau(x)}{\sqrt{n}}\left(\bigvee_{\tau(x)}^{\tau(x)+\frac{(1-\tau(x))}{k}}(f\circ\tau^{-1})'_x\right). \tag{4.14}$$

Collecting the estimates (4.12–4.14), we get the required result. This completes the proof. □

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