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Results in Mathematics

Supercongruences on Truncated Hypergeometric Series

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Abstract. In this paper, we use p-adic Gamma function and certain formulas on hypergeometric series to establish several new supercongruences. In particular, we give a generalization of a p-adic supercongruence conjecture due to van Hamme and Swisher.

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1. Introduction

Following Andrews et al. [3], we define the hypergeometric series by

$$ {}_{r+1}F_s\left(\begin{matrix} a_0, a_1, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{k!(b_1)_k \cdots (b_s)_k} z^k, $$

where $(z)_n$ is the Pochhammer symbol given by

$$ (z)_0 = 1, \quad (z)_n = z(z+1)\cdots(z+n-1) \quad \text{for } n \geq 1. $$

The truncated hypergeometric series, defined by

$$ {}_{r+1}F_s\left(\begin{matrix} a_0, a_1, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; z\right)_n := \sum_{k=0}^{n} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{k!(b_1)_k \cdots (b_s)_k} z^k, $$

are of importance in many fields including algebraic varieties, differential equations, Fuchsian groups, elliptic functions, modular forms and special functions, see, for example, [3,4,8].

Birkhäuser

Recall that the function $\Gamma(x)$ is defined by the formula [3]: for $Re\ x > 0$,

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

One of the most important properties of $\Gamma(z)$ is the following Euler's reflection formula:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Let p be an odd prime and n a positive integer. We define the p-adic Gamma function as

$$\Gamma_p(n) := (-1)^n \prod_{\substack{j<n \\ p\nmid j}} j.$$

Then we extend this to all $x \in \mathbb{Z}_p$ by setting

$$\Gamma_p(x) = \lim_{n\to x} \Gamma_p(n),$$

where n runs through any sequence of positive integers p-adically approaching x and $\Gamma_p(0) = 1$.

Congruences which happen to hold modulo some higher power of a prime p are called supercongruences. Various supercongruences were obtained and conjectured by many mathematicians including Beukers [5,6], van Hamme [25], Zudilin [26], and Sun [21–23].

In [6], Beukers made the following conjecture: for all odd primes p, there holds

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{\frac{p-1}{2}}{k}\binom{2k}{k}^2 \frac{1}{16^k} \equiv \begin{cases} 4x^2-2p & \text{if } p\equiv 1 \mod 4, p=x^2+y^2 \text{ with } x \text{ odd} \\ 0, & \text{if } p\equiv 3 \mod 4 \end{cases} \mod p^2. \quad (1.1)$$

Here and below, we use the notation $A \equiv B \pmod{p^l}$ if $(A-B)/p^l$ is a p-integer for $A, B \in \mathbb{Q}$. Beukers only proved this congruence in the modulus p case, which is equivalent to

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \frac{1}{64^k} \equiv \begin{cases} 4x^2 & \text{if } p\equiv 1 \mod 4, p=x^2+y^2 \text{ with } x \text{ odd} \\ 0 & \text{if } p\equiv 3 \mod 4 \end{cases} \mod p.$$

Congruence (1.1) was proved completely by Ahlgren [1], Ishikawa [10] and Mortenson [18]. It should be pointed out that van Hamme [25] established the following company congruence of (1.1):

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \frac{1}{64^k} \equiv \begin{cases} 4x^2-2p & \text{if } p\equiv 1 \mod 4, p=x^2+y^2 \text{ with } x \text{ odd} \\ 0 & \text{if } p\equiv 3 \mod 4 \end{cases} \mod p^2. \quad (1.2)$$

We shall give a new proof of congruence (1.2) by using a ${}_3F_2$ summation formula.

13 congruences linking the partial sums of certain hypergeometric series to the values of the p-adic Gamma function were presented by van Hamme [25]. He proved 3 congruences and gave a weaker result of another one. Based on the result of Ahlgren and Ono [2], Kilbourn [12] proved the conjecture [25, (M.2)]. In [15], McCarthy and Osburn settled van Hamme's conjecture [25, (A.2)]. Mortenson [16] confirmed the following conjecture [25, (B.2)] by placing it in the context of the Beukers-like supercongruences of Rodriguez-Villegas. Later, Long [13] proved the following conjecture [25, (J.2)] for each $p > 3$ and gave a weaker form of [25, (L.2)]. In 2014, Long and Ramakrishna [14] established a stronger result of [25, (D.2)]. The author in [9, Theorem 1.3] set up a congruence that includes as special cases the conjectures [25, (B.2), (E.2) and (F.2)]. Recently, Osburn and Zudilin [19] proved the conjecture [25, (K.2)]. Swisher in [24] handled the cases [25, (A.2), (C.2), (E.2), (F.2), (G.2) and (L.2)] and in particular proved the cases [25, (A.2), (C.2), and (G.2)] to higher powers of p than those conjectured by van Hamme.

In this article, we shall establish the following supercongruence which includes as special cases [25, (C.2) and (G.2)] as well as some new results.

Theorem 1.1. *Let $l \geq 2$ be an integer and $p \geq 5$ a prime with $p \equiv \pm 1 \mod l$. Then*

$$\sum_{k=0}^{\frac{\varepsilon p-1}{l}} (2lk+1)\binom{-\frac{1}{l}}{k}^4 \equiv \varepsilon p \frac{\Gamma_p\left(\frac{1}{l}\right)\Gamma_p\left(1-\frac{2}{l}\right)}{\Gamma_p\left(1-\frac{1}{l}\right)} \mod p^4, \tag{1.3}$$

where $\varepsilon = \begin{cases} 1 & \text{if } p \equiv 1 \mod l \\ l-1 & \text{if } p \equiv -1 \mod l \end{cases}.$

Letting $l = 2$ in (1.3) and noting that $\Gamma_p(0) = 1$, we get the following result on supercongruence.

Corollary 1.2. *Let $p \geq 5$ be a prime. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} (4k+1)\binom{-1/2}{k}^4 \equiv p \mod p^4. \tag{1.4}$$

Congruence (1.4) is a special case of [13, Theorem 1.1]. (1.4) in the modulus p^3 case was confirmed by L. van Hamme (see [25, (C.2)]). In that paper [25], he used a sequence of orthogonal polynomials.

Taking $l = 3$ in (1.3) and using (2.5), we are led to the following congruences which appear to be new.

Corollary 1.3. *Let $p \geq 5$ be a prime. If $p \equiv 1 \mod 3$, then*

$$\sum_{k=0}^{\frac{p-1}{3}} (6k+1)\binom{-1/3}{k}^4 \equiv -p\Gamma_p\left(\frac{1}{3}\right)^3 \mod p^4;$$

if $p \equiv 2 \mod 3$, *then*

$$\sum_{k=0}^{\frac{2p-1}{3}} (6k+1)\binom{-1/3}{k}^4 \equiv 2p\Gamma_p\left(\frac{1}{3}\right)^3 \mod p^4.$$

Setting $l = 4$ in (1.3), we obtain the following interesting results.

Corollary 1.4. *Let* $p \geq 5$ *be a prime. If* $p \equiv 1 \mod 4$, *then*

$$\sum_{k=0}^{\frac{p-1}{4}} (8k+1)\binom{-1/4}{k}^4 \equiv p\frac{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})}{\Gamma_p(\frac{3}{4})} \mod p^4; \tag{1.5}$$

if $p \equiv 3 \mod 4$, *then*

$$\sum_{k=0}^{\frac{3p-1}{4}} (8k+1)\binom{-1/4}{k}^4 \equiv 3p\frac{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})}{\Gamma_p(\frac{3}{4})} \mod p^4.$$

Congruence (1.5) in the modulus p^3 case was conjectured by van Hamme [25, (G.2)]. The result given here is a refinement of that of van Hamme.

Rodriguez-Villegas [20] proposed 22 conjectured supercongruences which are related to the truncated hypergeometric function associated to a Calabi-Yau manifold at a prime p and the number of its $\mathbb{F}_p$-points. We mention one of them below.

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 16^{-k} \equiv \left(\frac{-4}{p}\right) \mod p^2. \tag{1.6}$$

The above congruence was first confirmed by Mortenson [17].

We give another supercongruence which includes (1.6) as a special case.

Theorem 1.5. *Let* $l \geq 2$ *be an integer and* $p \geq 5$ *a prime with* $p \equiv \pm 1 \mod l$. *Then*

$$\sum_{k=0}^{\frac{\varepsilon p-1}{l}} \binom{-1/l}{k}^2 \equiv -\Gamma_p\left(\frac{1}{l}\right)^2 \Gamma_p\left(1-\frac{2}{l}\right) \mod p^2, \tag{1.7}$$

where $\varepsilon = \begin{cases} 1 & \text{if } p \equiv 1 \mod l \\ l-1 & \text{if } p \equiv -1 \mod l \end{cases}$.

Letting $l = 2$ in (1.7) and noting that $\Gamma_p(0) = 1$ and $\binom{-1/2}{k} = \frac{\binom{2k}{k}}{(-4)^k}$, we get the congruence (1.6).

Taking $l = 3$, $l = 4$ and $l = 6$ in (1.7), we are led to the following three congruences which seem to be new.

Corollary 1.6. *Let $p \geq 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \binom{-1/3}{k}^2 \equiv -\Gamma_p\left(\frac{1}{3}\right)^3 \mod p^2,$$

$$\sum_{k=0}^{p-1} \binom{-1/4}{k}^2 \equiv -\Gamma_p\left(\frac{1}{4}\right)^2 \Gamma_p\left(\frac{1}{2}\right) \mod p^2,$$

$$\sum_{k=0}^{p-1} \binom{-1/6}{k}^2 \equiv -\Gamma_p\left(\frac{1}{6}\right)^2 \Gamma_p\left(\frac{2}{3}\right) \mod p^2.$$

Theorem 1.7. *Let $p \equiv 1 \mod 4$ be a prime and $p = x^2 + 4y^2$ with $x, y \in \mathbb{Z}$. Then*

$${}_3F_2\left(\begin{matrix}\frac{1}{4}, \frac{1}{4}, \frac{1}{2} \\ 1, 1\end{matrix}; 1\right)_{p-1} \equiv 4x^2 - 2p \mod p^2.$$

In the next section, we shall provide some lemmas which will be used in the derivation of Theorems 1.1 and 1.5. A new proof of (1.2) will be given in the third section. Section 4 is devoted to our proof of Theorems 1.1, 1.5 and 1.7.

2. Preliminaries

Let $\zeta = e^{\frac{2\pi i}{3}}$. Then from the fact that

$$\begin{aligned}(a + b\zeta^j p)_k &= (a + b\zeta^j p)(a + b\zeta^j p + 1) \cdots (a + b\zeta^j p + k - 1) \\ &= (a)_k \left(1 + b\zeta^j pA(k) + b^2\zeta^{2j} p^2 B(k)\right) \mod p^3,\end{aligned}$$

where

$$A(k) = \sum_{l=1}^{k} \frac{1}{a + l - 1}$$

and

$$B(k) = \sum_{1 \leq l < m \leq k} \frac{1}{(a + l - 1)(a + m - 1)},$$

for $a \neq 0$ and $j \in \{0, 1, 2\}$, we have

$$(a + bp)_k(a + b\zeta p)_k(a + b\zeta^2 p)_k = (a)_k^3 \mod p^3. \tag{2.1}$$

Another important result we need is the following congruence which is similar to (2.1), namely

$$(a + bp)_k(a - bp)_k \equiv (a)_k^2 \mod p^2. \tag{2.2}$$

Now we recall some basic properties of the Morita p-adic Gamma function.

Lemma 2.1 (See [7, §11.6]). *Let p be an odd prime and $x \in \mathbb{Z}_p$. Then*

(i)
$$\Gamma_p(1) = -1. \tag{2.3}$$

(ii)
$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & \text{if } v_p(x) = 0; \\ -1, & \text{if } v_p(x) > 0; \end{cases} \tag{2.4}$$

where $v_p(x)$ denotes the p-adic evaluation of x.

(iii)
$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{a_0(x)}, \tag{2.5}$$

where $a_0(x) \in \{1, 2, \ldots, p\}$ such that $a_0(x) \equiv x \mod p$.

We need the following important result, which follows readily from the definition of $\Gamma_p(x)$ and Long and Ramakrishna [14] used but did not state explicitly (see [14, Lemma 17]).

Lemma 2.2. *Let p be an odd prime, $m \geq 3$ an integer and ζ a m-th primitive root of unity. Suppose $a \in \mathbb{Z}_p[\zeta]$ and $n \in \mathbb{N}$ such that $a + k \notin p\mathbb{Z}_p[\zeta]$ for all $k \in \{0, 1, \ldots, n-1\}$. Then*

$$(a)_n = (-1)^n \frac{\Gamma_p(a+n)}{\Gamma_p(a)}.$$

The following result on the (p-adic) expansion of p-adic Gamma function is very important in the proof of Theorems 1.1 and 1.5.

Lemma 2.3 (See [14, Theorem 14]). *For $p \geq 5, r \in \mathbb{N}, a \in \mathbb{Z}_p, m \in \mathbb{C}_p$ satisfying $v_p(m) \geq 0$ and $t \in \{0, 1, 2\}$ we have*

$$\frac{\Gamma_p(a+mp^r)}{\Gamma_p(a)} \equiv \sum_{k=0}^{t} \frac{G_k(a)}{k!}(mp^r)^k \mod p^{(t+1)r},$$

where $G_k(a) = \frac{\Gamma_p^{(k)}(a)}{\Gamma_p(a)} \in \mathbb{Z}_p$ and $\Gamma_p^{(k)}(x)$ is the k-th derivative of $\Gamma_p(x)$.

We conclude this section with the following result which is similar to [13, Lemma 2.8].

Consider a $(t+1)$-variable formal power series $F(x_1, x_2, \ldots, x_t; z)$. For example, maybe it is a scalar multiple of a terminating hypergeometric series

$$C \cdot {}_{r+1}F_r \left(\begin{matrix} a_1, a_2, \ldots, a_r & -n \\ b_1, \ldots, b_{r-1} & b_r \end{matrix} ; z \right).$$

We assume that by setting $x_i = a_i$ for $i = 1, \ldots, t$ and $z = z_0$, we have $F(a_1, a_2, \ldots, a_t; z_0) \in \mathbb{Z}_p$. Fix z_0 and deform the parameters a_i into polynomials $a_i(x) \in \mathbb{Z}_p[x]$ such that $a_i(0) = a_i$ for each $i \in \{1, 2, \ldots, t\}$, and assume that the function $F(a_1(x), a_2(x), \ldots, a_t(x); z_0)$ is a formal power series in x^3 with coefficients in $\mathbb{Z}_p$, namely,

$$F(a_1(x), a_2(x), \ldots, a_t(x); z_0) = A_0 + A_3x^3 + A_6x^6 + \cdots$$

for $A_i \in \mathbb{Z}_p$, where $A_0 = F(a_1, a_2, \ldots, a_t; z_0)$. Setting $x = p$ in the above expansion of F, we obtain

Lemma 2.4. *Under the setting above, if* $p^s | A_3$ *for* $s \in \{1, 2, 3\}$, *then*

$$F(a_1(p), a_2(p), \ldots, a_t(p); z_0) \equiv A_0 \mod p^{3+s}.$$

3. A New Proof of (1.2)

Recall from [3, Theorem 3.4.1] the following summation formula:

$$\begin{aligned}&{}_3F_2\left(\begin{matrix} a, -b, -c \\ 1+a+b, 1+a+c \end{matrix}; 1\right)\\ &\quad = \frac{\Gamma(a/2+1)\Gamma(a+b+1)\Gamma(a+c+1)\Gamma(a/2+b+c+1)}{\Gamma(a+1)\Gamma(a/2+b+1)\Gamma(a/2+c+1)\Gamma(a+b+c+1)}.\end{aligned}$$

Letting $a = \frac{1}{2}, b = \frac{p-1}{2}, c = \frac{-1-p}{2}$ yields

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1-p}{2}, \frac{1+p}{2} \\ 1+\frac{p}{2}, 1-\frac{p}{2} \end{matrix}; 1\right) = \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{4}\right)\Gamma\left(1+\frac{p}{2}\right)\Gamma\left(1-\frac{p}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{p}{2}\right)\Gamma\left(\frac{3}{4}-\frac{p}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}. \tag{3.1}$$

By (2.2) and the fact $\frac{(\frac{1}{2})_k}{k!} = \frac{\binom{2k}{k}}{4^k}$,

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1-p}{2}, \frac{1+p}{2} \\ 1+\frac{p}{2}, 1-\frac{p}{2} \end{matrix}; 1\right) \equiv {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; 1\right)_{\frac{p-1}{2}} = \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \frac{1}{64^k} \mod p^2. \tag{3.2}$$

We now consider $\frac{\Gamma(\frac{5}{4})\Gamma(\frac{1}{4})\Gamma(1+\frac{p}{2})\Gamma(1-\frac{p}{2})}{\Gamma(\frac{3}{4}+\frac{p}{2})\Gamma(\frac{3}{4}-\frac{p}{2})\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})} \mod p^2$. It is easily seen from the fact $\Gamma(x+1) = x\Gamma(x)$ and the Euler's reflection formula that

$$\frac{\Gamma(1+\frac{p}{2})\Gamma(1-\frac{p}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})} = (-1)^{\frac{p-1}{2}} p. \tag{3.3}$$

When $p \equiv 1 \mod 4$, it is easy to see from Lemma 2.2 that

$$\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}-\frac{p}{2}\right)} = \left(\frac{3}{4}-\frac{p}{2}\right)_{\frac{p-1}{2}} = \frac{\Gamma_p\left(\frac{1}{4}\right)}{\Gamma_p\left(\frac{3}{4}-\frac{p}{2}\right)}. \tag{3.4}$$

But note that $(\frac{1}{4})_{\frac{p+1}{2}}$ has exactly a multiple of p which is $\frac{p}{4}$. So by Lemma 2.2,

$$\frac{\Gamma\left(\frac{3}{4}+\frac{p}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} = \frac{4\Gamma\left(\frac{3}{4}+\frac{p}{2}\right)}{\Gamma\left(\frac{1}{4}\right)} = 4\left(\frac{1}{4}\right)_{\frac{p+1}{2}} = -p\frac{\Gamma_p\left(\frac{3}{4}+\frac{p}{2}\right)}{\Gamma_p\left(\frac{1}{4}\right)}. \tag{3.5}$$

It follows from (3.3)–(3.5) that

$$\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{4}\right)\Gamma\left(1+\frac{p}{2}\right)\Gamma\left(1-\frac{p}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{p}{2}\right)\Gamma\left(\frac{3}{4}-\frac{p}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)} = \frac{\Gamma_p\left(\frac{1}{4}\right)^2}{\Gamma_p\left(\frac{3}{4}+\frac{p}{2}\right)\Gamma_p\left(\frac{3}{4}-\frac{p}{2}\right)} \tag{3.6}$$

By Lemma 2.3,

$$\Gamma_p\left(\frac{3}{4}+\frac{p}{2}\right) \equiv \Gamma_p\left(\frac{3}{4}\right)\left(1+\frac{G_1\left(\frac{3}{4}\right)p}{2}\right) \mod p^2,$$

$$\Gamma_p\left(\frac{3}{4}-\frac{p}{2}\right) \equiv \Gamma_p\left(\frac{3}{4}\right)\left(1-\frac{G_1(\frac{3}{4})p}{2}\right) \mod p^2.$$

Then

$$\Gamma_p\left(\frac{3}{4}+\frac{p}{2}\right)\Gamma_p\left(\frac{3}{4}-\frac{p}{2}\right) \equiv \Gamma_p\left(\frac{3}{4}\right)^2 \mod p^2. \tag{3.7}$$

Using (3.7) and (2.5) in (3.6), we get

$$\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{4}\right)\Gamma\left(1+\frac{p}{2}\right)\Gamma\left(1-\frac{p}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{p}{2}\right)\Gamma\left(\frac{3}{4}-\frac{p}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)} \equiv -\Gamma_p\left(\frac{1}{4}\right)^4 \mod p^2. \tag{3.8}$$

Hence, by (3.1), (3.2) and (3.8),

$$\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^3\frac{1}{64^k} \equiv -\Gamma_p\left(\frac{1}{4}\right)^4 \mod p^2. \tag{3.9}$$

According to a result in [11, p. 200], we have if $p \equiv 1 \mod 4$ is a prime and $p = x^2+y^2$ with x odd, then

$$\lambda_p = 4x^2 - 2p, \tag{3.10}$$

where λ_p is defined by

$$\sum_{n=1}^{\infty}\lambda_n q^n = q\prod_{k=1}^{\infty}(1-q^{4k})^6.$$

In view of (3.9), (3.10) and [25, Proposition 1], we obtain

$$\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^3\frac{1}{64^k} \equiv 4x^2-2p \mod p^2.$$

When $p \equiv 3 \mod 4$, noticing that $(\frac{3}{4}-\frac{p}{2})_{\frac{p-1}{2}}$ has a multiple of p which is $-\frac{p}{4}$ and using Lemma 2.2, we find $\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4}-\frac{p}{2})} \equiv 0 \mod p$ and $\frac{\Gamma(\frac{3}{4}+\frac{p}{2})}{\Gamma(\frac{5}{4})} \not\equiv 0 \mod p$. Then, by (3.3),

$$\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{4}\right)\Gamma\left(1+\frac{p}{2}\right)\Gamma\left(1-\frac{p}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{p}{2}\right)\Gamma\left(\frac{3}{4}-\frac{p}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)} \equiv 0 \mod p^2. \tag{3.11}$$

In view of (3.1), (3.2) and (3.11), we deduce that

$$\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^3\frac{1}{64^k} \equiv 0 \mod p^2.$$

This completes the proof. □

4. Proof of Theorems 1.1, 1.5 and 1.7

Proof of Theorem 1.1. We first consider the case $p \equiv 1 \pmod{l}$. Recall the following important identity on hypergeometric series (see [3, Corollary 3.4.3]):

$$ {}_5F_4\left(\begin{matrix} a, \frac{a}{2}+1, c, d, -m \\ \frac{a}{2}, a-c+1, a-d+1, a+m+1 \end{matrix}; 1\right) = \frac{(a+1)_m(a-c-d+1)_m}{(a-c+1)_m(a-d+1)_m}, \tag{4.1} $$

where m is a positive integer.

Let $\zeta = e^{\frac{2\pi i}{3}}$. Applying (4.1) with $a = \frac{1}{l}, c = \frac{1-\zeta p}{l}, d = \frac{1-\zeta^2 p}{l}, m = \frac{p-1}{l}$ gives

$$ {}_5F_4\left(\begin{matrix} \frac{1}{l}, \frac{1}{2l}+1, \frac{1-p}{l}, \frac{1-\zeta p}{l}, \frac{1-\zeta^2 p}{l} \\ \frac{1}{2l}, \frac{p}{l}+1, \frac{\zeta p}{l}+1, \frac{\zeta^2 p}{l}+1 \end{matrix}; 1\right) = \frac{\left(\frac{1}{l}+1\right)_{\frac{p-1}{l}}\left(1-\frac{1+p}{l}\right)_{\frac{p-1}{l}}}{\left(1+\frac{\zeta p}{l}\right)_{\frac{p-1}{l}}\left(1+\frac{\zeta^2 p}{l}\right)_{\frac{p-1}{l}}}. \tag{4.2} $$

With the help of (2.1) and the identity $\binom{x}{k} = (-1)^k \frac{(-x)_k}{k!}$, we get

$$ {}_5F_4\left(\begin{matrix} \frac{1}{l}, \frac{1}{2l}+1, \frac{1-p}{l}, \frac{1-\zeta p}{l}, \frac{1-\zeta^2 p}{l} \\ \frac{1}{2l}, \frac{p}{l}+1, \frac{\zeta p}{l}+1, \frac{\zeta^2 p}{l}+1 \end{matrix}; 1\right) \equiv \sum_{k=0}^{\frac{p-1}{l}} (2lk+1)\binom{-\frac{1}{l}}{k}^4 \mod p^3. \tag{4.3} $$

By Lemma 2.2 and (2.4), we have

$$ \left(1-\frac{1+p}{l}\right)_{\frac{p-1}{l}} = (-1)^{\frac{p-1}{l}} \frac{\Gamma_p\left(1-\frac{2}{l}\right)}{\Gamma_p\left(1-\frac{1+p}{l}\right)}, $$

$$ \left(1+\frac{\zeta p}{l}\right)_{\frac{p-1}{l}} = (-1)^{\frac{p-1}{l}} \frac{\Gamma_p\left(1-\frac{1+\zeta^2 p}{l}\right)}{\Gamma_p\left(1+\frac{\zeta p}{l}\right)}, $$

$$ \left(1+\frac{\zeta^2 p}{l}\right)_{\frac{p-1}{l}} = (-1)^{\frac{p-1}{l}} \frac{\Gamma_p\left(1-\frac{1+\zeta p}{l}\right)}{\Gamma_p\left(1+\frac{\zeta^2 p}{l}\right)}, $$

and

$$ \left(\frac{1}{l}+1\right)_{\frac{p-1}{l}} = p\left(\frac{1}{l}\right)_{\frac{p-1}{l}} = (-1)^{\frac{p-1}{l}} p \frac{\Gamma_p\left(\frac{p}{l}\right)}{\Gamma_p\left(\frac{1}{l}\right)} = (-1)^{1+\frac{p-1}{l}} p \frac{\Gamma_p\left(1+\frac{p}{l}\right)}{\Gamma_p\left(\frac{1}{l}\right)}. $$

Then

$$ \frac{\left(\frac{1}{l}+1\right)_{\frac{p-1}{l}}\left(1-\frac{1+p}{l}\right)_{\frac{p-1}{l}}}{\left(1+\frac{\zeta p}{l}\right)_{\frac{p-1}{l}}\left(1+\frac{\zeta^2 p}{l}\right)_{\frac{p-1}{l}}} = -p\frac{\Gamma_p\left(1-\frac{2}{l}\right)\Gamma_p\left(1+\frac{p}{l}\right)\Gamma_p\left(1+\frac{\zeta p}{l}\right)\Gamma_p\left(1+\frac{\zeta^2 p}{l}\right)}{\Gamma_p\left(\frac{1}{l}\right)\Gamma_p\left(1-\frac{1+p}{l}\right)\Gamma_p\left(1-\frac{1+\zeta p}{l}\right)\Gamma_p\left(1-\frac{1+\zeta^2 p}{l}\right)}. $$

By Lemma 2.3,

$$\Gamma_p\left(1+\frac{\zeta^j p}{l}\right) \equiv \Gamma_p(1)\left(1+\frac{G_1(1)\zeta^j}{l}p+\frac{G_2(1)\zeta^{2j}}{2l^2}p^2\right) \mod p^3,$$

$$\Gamma_p\left(1-\frac{1+\zeta^j p}{l}\right) \equiv \Gamma_p\left(1-\frac{1}{l}\right)\left(1+\frac{G_1\left(1-\frac{1}{l}\right)\zeta^j}{l}p+\frac{G_2\left(1-\frac{1}{l}\right)\zeta^{2j}}{2l^2}p^2\right) \mod p^3,$$

for $j \in \{0,1,2\}$. It follows that

$$\Gamma_p\left(1+\frac{p}{l}\right)\Gamma_p\left(1+\frac{\zeta p}{l}\right)\Gamma_p\left(1+\frac{\zeta^2 p}{l}\right) \equiv \Gamma_p(1)^3 \mod p^3,$$

$$\Gamma_p\left(1-\frac{1+p}{l}\right)\Gamma_p\left(1-\frac{1+\zeta p}{l}\right)\Gamma_p\left(1-\frac{1+\zeta^2 p}{l}\right) \equiv \Gamma_p\left(1-\frac{1}{l}\right)^3 \mod p^3.$$

Hence, by (2.3) and (2.5), we arrive at

$$\begin{aligned}\frac{\left(\frac{1}{l}+1\right)_{\frac{p-1}{l}}\left(1-\frac{1+p}{l}\right)_{\frac{p-1}{l}}}{\left(1+\frac{\zeta p}{l}\right)_{\frac{p-1}{l}}\left(1+\frac{\zeta^2 p}{l}\right)_{\frac{p-1}{l}}} &\equiv -p\frac{\Gamma_p\left(1-\frac{2}{l}\right)\Gamma_p(1)^3}{\Gamma_p\left(\frac{1}{l}\right)\Gamma_p\left(1-\frac{1}{l}\right)^3} \\ &= p\frac{\Gamma_p\left(1-\frac{2}{l}\right)\Gamma_p\left(\frac{1}{l}\right)}{\Gamma_p\left(1-\frac{1}{l}\right)} \mod p^4. \end{aligned} \tag{4.4}$$

In view of (4.2)–(4.4), we obtain

$$\sum_{k=0}^{\frac{p-1}{l}}(2lk+1)\binom{-\frac{1}{l}}{k}^4 \equiv p\frac{\Gamma_p\left(1-\frac{2}{l}\right)\Gamma_p\left(\frac{1}{l}\right)}{\Gamma_p\left(1-\frac{1}{l}\right)} \mod p^3.$$

Below we consider the following truncated hypergeometric series:

$${}_5F_4\left(\begin{matrix}\frac{1}{l}, \frac{1}{2l}+1, \frac{1-x}{l}, \frac{1-\zeta x}{l}, \frac{1-\zeta^2 x}{l} \\ \frac{1}{2l}, \frac{x}{l}+1, \frac{\zeta x}{l}+1, \frac{\zeta^2 x}{l}+1\end{matrix}; 1\right)_{\frac{p-1}{l}}.$$

By symmetry,

$$\begin{aligned}&{}_5F_4\left(\begin{matrix}\frac{1}{l}, \frac{1}{2l}+1, \frac{1-x}{l}, \frac{1-\zeta x}{l}, \frac{1-\zeta^2 x}{l} \\ \frac{1}{2l}, \frac{x}{l}+1, \frac{\zeta x}{l}+1, \frac{\zeta^2 x}{l}+1\end{matrix}; 1\right)_{\frac{p-1}{l}} \\ &\quad = {}_5F_4\left(\begin{matrix}\frac{1}{2l}+1, \frac{1}{l}, \frac{1}{l}, \frac{1}{l}, \frac{1}{l} \\ \frac{1}{2l}, 1, 1, 1\end{matrix}; 1\right)_{\frac{p-1}{l}} + A_3x^3 + A_6x^6 + \cdots \end{aligned} \tag{4.5}$$

for some $A_3 \in \mathbb{Z}_p$. In addition, we consider the following transformation of hypergeometric series (see [3, Theorem 3.4.5]):

$$
\begin{aligned}
&{}_7F_6\left(\begin{matrix} a, \frac{a}{2}+1, b, c, d, e, f \\ \frac{a}{2}, a-b+1, a-c+1, a-d+1, a-e+1\ a-f+1 \end{matrix}; 1\right) \\
&\quad= \frac{\Gamma(a-d+1)\Gamma(a-e+1)\Gamma(a-f+1)\Gamma(a-d-e-f+1)}{\Gamma(1+a)\Gamma(a-e-f+1)\Gamma(a-d-e+1)\Gamma(a-d-f+1)} \\
&\qquad\times {}_4F_3\left(\begin{matrix} a-b-c+1, d, e, f \\ a-b+1, a-c+1, d+e+f-a \end{matrix}; 1\right), \qquad (4.6)
\end{aligned}
$$

provided the series on the right side is terminating and the one on the left converges.

Taking $a = \frac{1}{l}$, $b = \frac{1-x}{l}$, $c = \frac{1-\zeta x}{l}$, $d = \frac{1-\zeta^2 x}{l}$, $e = \frac{1-p}{l}$, $f = 1$ in (4.6), we find

$$
\begin{aligned}
&{}_7F_6\left(\begin{matrix} \frac{1}{2l}+1, \frac{1}{l}, \frac{1-x}{l}, \frac{1-\zeta x}{l}, \frac{1-\zeta^2 x}{l}, \frac{1-p}{l}, 1 \\ \frac{1}{2l}, 1+\frac{x}{l}, 1+\frac{\zeta x}{l}, 1+\frac{\zeta^2 x}{l}, 1+\frac{p}{l}, \frac{1}{l} \end{matrix}; 1\right) \\
&= \frac{\Gamma\left(1+\frac{\zeta^2 x}{l}\right)\Gamma\left(\frac{p-1+\zeta^2 x}{l}\right)\Gamma\left(1+\frac{p}{l}\right)\Gamma\left(\frac{1}{l}\right)}{\Gamma\left(1+\frac{p-1+\zeta^2 x}{l}\right)\Gamma\left(\frac{\zeta^2 x}{l}\right)\Gamma\left(1+\frac{1}{l}\right)\Gamma\left(\frac{p}{l}\right)} {}_4F_3\left(\begin{matrix} 1-\frac{1+\zeta^2 x}{l}, \frac{1-\zeta^2 x}{l}, \frac{1-p}{l}, 1 \\ 1+\frac{x}{l}, 1+\frac{\zeta x}{l}, 1+\frac{1+\zeta p}{l} \end{matrix}; 1\right).
\end{aligned}
$$

By the fact $\Gamma(x+1) = x\Gamma(x)$,

$$
\frac{\Gamma\left(1+\frac{\zeta^2 x}{l}\right)\Gamma\left(\frac{p-1+\zeta^2 x}{l}\right)\Gamma\left(1+\frac{p}{l}\right)\Gamma\left(\frac{1}{l}\right)}{\Gamma\left(1+\frac{p-1+\zeta^2 x}{l}\right)\Gamma\left(\frac{\zeta^2 x}{l}\right)\Gamma\left(1+\frac{1}{l}\right)\Gamma\left(\frac{p}{l}\right)} = \frac{p\zeta^2 x}{p-1+\zeta^2 x},
$$

which implies that each coefficient of x^l for $l \geq 0$ on the left side is in $p\mathbb{Z}_p$. On the other hand, modulo p, the left side is congruent to that of (4.5). So when we expand the left side of (4.5) in terms of powers of x, the coefficients are all in $p\mathbb{Z}_p$. In particular, $A_3 \in p\mathbb{Z}_p$. Then, setting $x = p$ in (4.5), by Lemma 2.4 we are led to the congruence:

$$
{}_5F_4\left(\begin{matrix} \frac{1}{l}, \frac{1}{2l}+1, \frac{1-p}{l}, \frac{1-\zeta p}{l}, \frac{1-\zeta^2 p}{l} \\ \frac{1}{2l}, \frac{p}{l}+1, \frac{\zeta p}{l}+1, \frac{\zeta^2 p}{l}+1 \end{matrix}; 1\right)_{\frac{p-1}{l}} \equiv {}_5F_4\left(\begin{matrix} \frac{1}{2l}+1, \frac{1}{l}, \frac{1}{l}, \frac{1}{l}, \frac{1}{l} \\ \frac{1}{2l}, 1, 1, 1 \end{matrix}; 1\right)_{\frac{p-1}{l}} \mod p^4
$$

On the other hand, by (4.2) and (4.4),

$$
{}_5F_4\left(\begin{matrix} \frac{1}{l}, \frac{1}{2l}+1, \frac{1-p}{l}, \frac{1-\zeta p}{l}, \frac{1-\zeta^2 p}{l} \\ \frac{1}{2l}, \frac{p}{l}+1, \frac{\zeta p}{l}+1, \frac{\zeta^2 p}{l}+1 \end{matrix}; 1\right)_{\frac{p-1}{l}} \equiv p\frac{\Gamma_p\left(1-\frac{2}{l}\right)\Gamma_p\left(\frac{1}{l}\right)}{\Gamma_p\left(1-\frac{1}{l}\right)} \mod p^4.
$$

Hence,

$$
{}_5F_4\left(\begin{matrix} \frac{1}{2l}+1, \frac{1}{l}, \frac{1}{l}, \frac{1}{l}, \frac{1}{l} \\ \frac{1}{2l}, 1, 1, 1 \end{matrix}; 1\right)_{\frac{p-1}{l}} \equiv p\frac{\Gamma_p\left(1-\frac{2}{l}\right)\Gamma_p\left(\frac{1}{l}\right)}{\Gamma_p\left(1-\frac{1}{l}\right)} \mod p^4,
$$

namely,

$$\sum_{k=0}^{\frac{p-1}{l}}(2lk+1)\binom{-\frac{1}{l}}{k}^4 \equiv p\frac{\Gamma_p\left(1-\frac{2}{l}\right)\Gamma_p\left(\frac{1}{l}\right)}{\Gamma_p\left(1-\frac{1}{l}\right)} \mod p^4.$$

Similarly, Theorem 1.1 in the case $p \equiv -1 \pmod{l}$ follows from (4.1) and (4.6). So we omit the details. □

Proof of Theorem 1.5. Recall the following Chu–Vandermonde identity (see [3, Corollary 2.2.3]):

$$_2F_1\left(\begin{matrix}-n, a\\ c\end{matrix}; 1\right) = \frac{(c-a)_n}{(c)_n}, \tag{4.7}$$

where n is a nonnegative integer.

Applying (4.7) with $a = \frac{1+\varepsilon p}{l}, c = 1, n = \frac{\varepsilon p-1}{l}$, we have

$$_2F_1\left(\begin{matrix}\frac{1-\varepsilon p}{l}, \frac{1+\varepsilon p}{l}\\ 1\end{matrix}; 1\right) = \frac{\left(1-\frac{1+\varepsilon p}{l}\right)_{\frac{\varepsilon p-1}{l}}}{(1)_{\frac{\varepsilon p-1}{l}}}. \tag{4.8}$$

With the help of (2.2) and the identity $\binom{x}{k} = (-1)^k\frac{(-x)_k}{k!}$, we get

$$_2F_1\left(\begin{matrix}\frac{1-\varepsilon p}{l}, \frac{1+\varepsilon p}{l}\\ 1\end{matrix}; 1\right) \equiv \sum_{k=0}^{\frac{\varepsilon p-1}{l}}\binom{-1/l}{k}^2 \mod p^2. \tag{4.9}$$

By Lemma 2.2, we find

$$\left(1-\frac{1+\varepsilon p}{l}\right)_{\frac{\varepsilon p-1}{l}} = (-1)^{\frac{\varepsilon p-1}{l}}\frac{\Gamma_p(1-\frac{2}{l})}{\Gamma_p\left(1-\frac{1+\varepsilon p}{l}\right)},$$

$$(1)_{\frac{\varepsilon p-1}{l}} = (-1)^{\frac{\varepsilon p-1}{l}}\frac{\Gamma_p\left(1+\frac{\varepsilon p-1}{l}\right)}{\Gamma_p(1)}.$$

Then

$$\frac{\left(1-\frac{1+\varepsilon p}{l}\right)_{\frac{\varepsilon p-1}{l}}}{(1)_{\frac{\varepsilon p-1}{l}}} = \frac{\Gamma_p(1)\Gamma_p\left(1-\frac{2}{l}\right)}{\Gamma_p\left(1-\frac{1}{l}-\frac{\varepsilon p}{l}\right)\Gamma_p\left(1-\frac{1}{l}+\frac{\varepsilon p}{l}\right)}$$

By Lemma 2.3,

$$\Gamma_p\left(1-\frac{1}{l}-\frac{\varepsilon p}{l}\right) \equiv \Gamma_p\left(1-\frac{1}{l}\right)\left(1-\frac{G(1-\frac{1}{l})\varepsilon}{l}p\right) \mod p^2,$$

$$\Gamma_p\left(1-\frac{1}{l}+\frac{\varepsilon p}{l}\right) \equiv \Gamma_p\left(1-\frac{1}{l}\right)\left(1+\frac{G(1-\frac{1}{l})\varepsilon}{l}p\right) \mod p^2.$$

It follows that

$$\Gamma_p\left(1-\frac{1}{l}-\frac{\varepsilon p}{l}\right)\Gamma_p\left(1-\frac{1}{l}+\frac{\varepsilon p}{l}\right) \equiv \Gamma_p\left(1-\frac{1}{l}\right)^2 \mod p^2.$$

Hence, by (2.3) and (2.5), we arrive at

$$\frac{\left(1-\frac{1+\varepsilon p}{l}\right)_{\frac{\varepsilon p-1}{l}}}{(1)_{\frac{\varepsilon p-1}{l}}} \equiv -\Gamma_p\left(\frac{1}{l}\right)^2 \Gamma_p\left(1-\frac{2}{l}\right) \mod p^2. \tag{4.10}$$

In view of (4.8)–(4.10), we obtain

$$\sum_{k=0}^{\frac{\varepsilon p-1}{l}} \binom{-1/l}{k}^2 \equiv -\Gamma_p\left(\frac{1}{l}\right)^2 \Gamma_p\left(1-\frac{2}{l}\right) \mod p^2.$$

This completes the proof of Theorem 1.5. □

Proof of Theorem 1.7. Recall the following formula (see [3, p. 177, 5(b)]):

$${}_3F_2\left(\begin{matrix}-2n, 2b, c\\ -n+b+\frac{1}{2}, d\end{matrix}; 1\right) = {}_4F_3\left(\begin{matrix}-n, b, c, d-c\\ -n+b+\frac{1}{2}, d/2, (d+1)/2\end{matrix}; 1\right),$$

where n is a nonnegative integer.

Taking $n=\frac{p-1}{4}$, $b=\frac{p+1}{4}$, $c=1/2, d=1$ in the above identity, we have

$${}_3F_2\left(\begin{matrix}\frac{1-p}{2}, \frac{1+p}{2}, \frac{1}{2}\\ 1, 1\end{matrix}; 1\right) = {}_3F_2\left(\begin{matrix}\frac{1-p}{4}, \frac{1+p}{4}, \frac{1}{2}\\ 1, 1\end{matrix}; 1\right).$$

By (2.2) and the fact $\frac{(\frac{1}{2})_k}{k!} = \frac{\binom{2k}{k}}{4^k}$,

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{64^k} = {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ 1, 1\end{matrix}; 1\right)_{\frac{p-1}{2}} \equiv {}_3F_2\left(\begin{matrix}\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\\ 1, 1\end{matrix}; 1\right)_{\frac{p-1}{4}} \mod p^2. \tag{4.11}$$

In view of (4.11) and (1.2), we obtain

$${}_3F_2\left(\begin{matrix}\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\\ 1, 1\end{matrix}; 1\right)_{\frac{p-1}{4}} \equiv 4x^2-2p \mod p^2,$$

which implies

$${}_3F_2\left(\begin{matrix}\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\\ 1, 1\end{matrix}; 1\right)_{p-1} \equiv 4x^2-2p \mod p^2,$$

since $(\frac{1}{4})_k \equiv 0 \mod p$ for $p/4 < k < p$. This finishes the proof of Theorem 1.7. □

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