



One-Dimensional Phase Retrieval with Additional Interference Intensity Measurements

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Abstract. The one-dimensional phase retrieval problem consists in the recovery of a complex-valued signal from its Fourier intensity. Due to the well-known ambiguousness of this problem, the determination of the original signal within the extensive solution set is challenging and can only be done under suitable a priori assumptions or additional information about the unknown signal. Depending on the application, one has sometimes access to further interference intensity measurements between the unknown signal and a reference signal. Beginning with the reconstruction in the discrete-time setting, we show that each signal can be uniquely recovered from its Fourier intensity and two further interference intensity measurements between the unknown signal and a modulation of the signal itself. Afterwards, we consider the continuous-time problem, where we obtain an equivalent result. Moreover, the unique recovery of a continuous-time signal can also be ensured by using interference intensity measurements with a known or an unknown reference which is unrelated to the unknown signal.

Mathematics Subject Classification. 42A05, 94A08, 94A12.

Keywords. Phase retrieval, One-dimensional signals, Compact support, Interference intensity measurements.

1. Introduction

Phase retrieval problems occur in a wide range of applications in physics and engineering such as crystallography [15, 20, 25], astronomy [11, 14], and laser optics [33, 34]. All of these applications have in common that one needs to recover an unknown signal from the intensity of its Fourier transform. Because

of the well-known ambiguousness of this problem, the recovery of an analytic or a numerical solution is generally challenging. To determine a meaningful solution within the solution set, one therefore needs further a priori information or additional data.

In this paper, we consider the one-dimensional phase retrieval problem for discrete-time and continuous-time signals. In both cases, we restrict ourselves to the recovery of an unknown signal with finite or compact support. The ambiguities of these problems can be explicitly specified by an appropriate factorization of the corresponding autocorrelation signal, see [7, 11, 28] for the discrete-time and [1, 2, 17, 35] for the continuous-time setting.

Depending on the application, one can sometimes superpose the unknown signal x with an appropriate reference signal h . In the discrete-time setting, the additional Fourier intensity of the interference $x + h$ can be used to reduce the solution set to merely two different signals or even to ensure uniqueness for particularly known references [7, 18, 19]. Under some further assumptions, it is also possible to use unknown references to recover the wanted signal [7, 21, 31, 32]. Besides considering reference signals being unrelated to the unknown signal, one may also employ a modulation of the unknown signal itself as reference in order to guarantee uniqueness [13]. Here we extend the results in [13] from the interference intensities $|\widehat{x}(2\pi k/M) + \widehat{x}(2\pi(k-\ell)/M)|$ and $|\widehat{x}(2\pi k/M) - i\widehat{x}(2\pi(k-\ell)/M)|$ with $k = 0, \dots, M-1$, where M is the length of the unknown signal and $\ell \in \mathbb{Z}$, to interference intensities of the form $|\widehat{x}(\omega_k) + e^{i\alpha}\widehat{x}(\omega_k - \mu)|$ with $\alpha \in [-\pi, \pi)$ and $\mu \in \mathbb{R}$ for finitely many sample points $\omega_k \in \mathbb{R}$. At the same time, we show that neither the support of x has to be known beforehand nor that the samples of the interference intensity have to be non-zero as required in [13].

A different approach to ensure uniqueness by additional data in the frequency domain is presented in [26, 27]. Here the Fourier transform is replaced by the so-called short-time Fourier transform, where the unknown signal is overlapped with a small analysis window at different positions.

If we normalize the support of the unknown signal x to $\{0, \dots, M-1\}$, we can interpret x as an M -dimensional vector. Further, the Fourier intensities $|\widehat{x}(\omega_k)|$ at different points $\omega_k \in [-\pi, \pi)$ can now be written as the intensity measurement $|\langle x, v_k \rangle|$ with $v_k := (e^{i\omega_k m})_{m=0}^{M-1}$. Generalizing this idea, here the question arises how the vectors v_k have to be chosen, and how many vectors v_k are required to ensure the recovery of x from the intensities $|\langle x, v_k \rangle|$, see for instance [4–6, 10] and references therein. To reconstruct the signal x explicitly, one can again exploit suitable interference intensity measurements of the form $|\langle x, v_k + v_\ell \rangle|$, see [3].

Besides employing interference intensity measurements, the ambiguities can also be avoided by further information about the unknown signal in the time domain. In [8], it has been shown that almost every complex-valued discrete-time signal x is uniquely determined by its Fourier intensity $|\widehat{x}|$ and

one absolute value $|x[n]|$ for a fixed n within the support of x . A similar statement holds if two phases $\arg x[n]$ and $\arg x[m]$ for appropriate n and m are known beforehand.

The continuous-time phase retrieval problem, however, has a completely different behavior than the discrete-time equivalent. Nonetheless, also here the ambiguousness is a challenging problem. If the additional Fourier intensity of an appropriate modulation of the unknown signal is available, one can here solve the corresponding phase retrieval problem uniquely by comparing the zeros of the analytic continuation of the given intensities [36]. Using a combination of oversampling and modulations, Pohl et al. show that the unknown signal can be recovered up to a global phase [29]. Similarly to the discrete-time setting, the unknown signal can also be uniquely recovered by exploiting interference intensity measurements. For this purpose, Burge et al. have constructed suitable reference signals depending on the given Fourier intensity [12].

Besides these approaches, we show that the well-known uniqueness results using interference intensity measurements with known and unknown reference signals [7, 18, 19, 21, 31, 32] can be generalized from the discrete-time to the continuous-time phase retrieval problem. More precisely, we observe that the additional Fourier intensity of the signal $f + h$, where f is the true signal and h a known reference signal, reduces the set of usually infinitely many non-trivial solutions to merely two signals. Furthermore, the second solution can be explicitly constructed from f and h . In the case that the reference signal is also unknown, we can ensure the uniqueness under some slight assumptions, which are comparable to the discrete-time setting. More precisely, we assume that the Laplace transformed signals $\mathcal{L}[f]$ and $\mathcal{L}[h]$ have no common zeros. Finally, we transfer our uniqueness results based on the interference with a modulated version of the unknown signal itself to the continuous-time phase retrieval problem.

The paper is organized as follows. In Sect. 2, we introduce the discrete-time phase retrieval problem. Here we investigate interference intensity measurements of the unknown signal with a modulation of the signal itself. Using Prony's method, we adapt and extend the uniqueness results in [3, 13] to our specific problem formulation and show that each discrete-time signal with finite support can always be recovered uniquely. Moreover, we can drop the assumption that the support of the unknown signal must be known beforehand as supposed in [3, 13], since we can recover the support from the given Fourier intensities.

In Sect. 3, we transfer our findings from the discrete-time to the continuous-time setting. Moreover, we consider interference intensity measurements where the reference signal is known beforehand. Similar to the discrete-time setting, these measurements reduce the solution set to two different signals. Giving a novel proof of the corresponding results in [22], we can here explicitly represent the second possible solution. Furthermore, we show that

the uniqueness results for an unknown reference signal [7, 21, 31] can also be transferred. For this purpose, we will employ the characterization of the arising ambiguities in [17].

2. Discrete-Time Phase Retrieval

2.1. Trivial and Non-Trivial Ambiguities

We begin with the *one-dimensional discrete-time phase retrieval problem*, where we wish to recover a complex-valued signal $x := (x[n])_{n \in \mathbb{Z}}$ from its Fourier intensity $|\widehat{x}|$, where \widehat{x} denotes the *discrete-time Fourier transform* given by

$$\widehat{x}(\omega) := \mathcal{F}[x](\omega) := \sum_{n \in \mathbb{Z}} x[n] e^{-i\omega n} \quad (\omega \in \mathbb{R}).$$

In the following, we assume that the unknown signal x has a finite support of length N . In other words, we always find an $n_0 \in \mathbb{Z}$ such that $x[n] = 0$ for $n < n_0$ and $n \geq n_0 + N$. As a consequence of this a priori condition, the corresponding *autocorrelation signal*

$$a[n] := \sum_{k \in \mathbb{Z}} \overline{x[k]} x[k+n] \quad (n \in \mathbb{Z})$$

possesses the finite support $\{-N+1, \dots, N-1\}$. Moreover, the *autocorrelation function*

$$\widehat{a}(\omega) = \sum_{n \in \mathbb{Z}} a[n] e^{-i\omega n} = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} x[n] \overline{x[k]} e^{-i\omega(n-k)} = |\widehat{x}(\omega)|^2$$

is here always a non-negative trigonometric polynomial of degree $N-1$, and the support length N of the unknown signal x is hence explicitly encoded in the given Fourier intensity $|\widehat{x}|$.

On the other side, since the autocorrelation function $\widehat{a} = |\widehat{x}|^2$ is trigonometric polynomial of degree $N-1$ and is thus completely determined by $2N-1$ samples at different points in $[-\pi, \pi)$, our phase retrieval problem is equivalent to the task to recover the unknown signal x from the $2N-1$ values

$$\left| \widehat{x}\left(\frac{2\pi k}{N}\right) \right| \quad (k = -N+1, \dots, N-1)$$

if the support length N or an upper bound of the support length of the unknown signal is known beforehand.

It is well known that the (discrete-time) phase retrieval problem is not uniquely solvable in general. The simplest occurring ambiguities are caused by rotation (multiplication with a unimodular constant), time shift, and conjugation and reflection [7].

Proposition 1. *Let x be a complex-valued signal with finite support. Then*

- (i) *the rotated signal $(e^{i\alpha} x[n])_{n \in \mathbb{Z}}$ for $\alpha \in \mathbb{R}$*
- (ii) *the time shifted signal $(x[n - n_0])_{n \in \mathbb{Z}}$ for $n_0 \in \mathbb{Z}$*

(iii) *the conjugated and reflected signal $(\overline{x[-n]})_{n \in \mathbb{Z}}$ have the same Fourier intensity $|\widehat{x}|$.*

Although the ambiguities in Proposition 1 always occur, they are of minor interest since they are closely related to the original signal. Usually, besides the rotations, shifts, and conjugation and reflection, the discrete-time phase retrieval problem has an extensive set of further solutions, which can completely differ from the original signal. Similarly to [7], we distinguish between trivial and non-trivial ambiguities.

Definition 1. A *trivial ambiguity* of the discrete-time phase retrieval problem is caused by a rotation, time shift, conjugation and reflection, or by a combinations of these. All other occurring ambiguities are called *non-trivial*.

Unlike the trivial ambiguities, where we have infinitely many possibilities to choose the rotation parameter $\alpha \in \mathbb{R}$ and the shift parameter $n_0 \in \mathbb{Z}$, the phase retrieval problem to recover a signal with finite support can only possess finitely many non-trivial solutions, see [7, Corollary 2.6]. More precisely, depending on the support length N of the unknown signal, the solution set can consists of at most 2^{N-2} non-trivially different signals.

2.2. Interference Intensity Measurements in the Discrete-Time Setting

Considering the large number of non-trivial solutions, how can we determine the original signal within the set of ambiguities? One possibility to reduce the solution set significantly is the exploitation of additional interference intensity measurements of the form $|\mathcal{F}[x + h]|$, where h is a suitable reference signal with finite support. If h is a known reference signal unrelated to x , then the corresponding phase retrieval problem has at most two non-trivially different solutions, see [7, 19]. If h is also an unknown signal, under some additional assumption, both signals x and h can be uniquely recovered from $|\widehat{x}|$, $|\widehat{h}|$, and $|\widehat{x} + \widehat{h}|$, see [7, 21, 31], where in [31] the additional measurements $|\widehat{x} + i\widehat{h}|$ are employed.

In this section, we examine interference intensity measurements of a slightly different kind. More precisely, we replace the reference signal h with a modulated version of the unknown signal x itself. This idea goes back to [3, 13], where the phase retrieval of a finite-dimensional vector from the intensities of the discrete Fourier transform or the intensities of a suitably constructed frame is considered.

Transferring this approach to the discrete-time phase retrieval problem, we try to recover the complex-valued signal x with finite support from its Fourier intensity $|\widehat{x}|$ and a set of interference intensity measurements of the form

$$|\mathcal{F}[x + e^{i\alpha} e^{i\mu \cdot} x]|, \tag{1}$$

where the modulations and rotations are described by $\mu \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. Since the Fourier transform of the modulated signal is given by

$$\mathcal{F}[e^{i\alpha} e^{i\mu \cdot} x](\omega) = e^{i\alpha} \sum_{n \in \mathbb{Z}} x[n] e^{-i(\omega - \mu)n} = e^{i\alpha} \widehat{x}(\omega - \mu),$$

we can also interpret the considered interference intensity measurements as interferences with certain shifts of the Fourier transform \widehat{x} in the frequency domain.

2.2.1. Phase Reconstruction by Using a Polarization Identity. Our first approach to recover x is to apply a suitable polarization identity, which allows us to determine the unknown phase of \widehat{x} . For this purpose, we generalize the Mercedes-Benz polarization identity introduced by Alexeev et al. [3]. In the following, the primitive K th root of unity is denoted by $\zeta_K := e^{2\pi i/K}$.

Lemma 1. *Let z_1 and z_2 be two complex numbers. Then the polarization identity*

$$\bar{z}_1 z_2 = \frac{1}{K} \sum_{k=0}^{K-1} \zeta_K^k |z_1 + \zeta_K^{-k} z_2|^2 \quad (2)$$

holds for every integer $K > 2$.

Proof. The assertion can be proven by generalizing the ideas in [3, p. 38]. We expand the right-hand side of (2) and obtain

$$\begin{aligned} \frac{1}{K} \sum_{k=0}^{K-1} \zeta_K^k |z_1 + \zeta_K^{-k} z_2|^2 &= \frac{1}{K} \sum_{k=0}^{K-1} \zeta_K^k (|z_1|^2 + 2\Re[\zeta_K^{-k} \bar{z}_1 z_2] + |z_2|^2) \\ &= \frac{2}{K} \sum_{k=0}^{K-1} \zeta_K^k \Re[\zeta_K^{-k} \bar{z}_1 z_2] \end{aligned}$$

since the sum over all roots ζ_K^k is zero. Writing the real part of the product $\zeta_K^{-k} \bar{z}_1 z_2$ as

$$\Re[\zeta_K^{-k} \bar{z}_1 z_2] = \Re[\zeta_K^{-k}] \Re[\bar{z}_1 z_2] - \Im[\zeta_K^{-k}] \Im[\bar{z}_1 z_2],$$

and using the identities $\Re \zeta_K^{-1} = \Re \zeta_K^k$ and $\Im \zeta_K^{-k} = -\Im \zeta_K^k$, we further have

$$\frac{1}{K} \sum_{k=0}^{K-1} \zeta_K^k |z_1 + \zeta_K^{-k} z_2|^2 = \frac{2}{K} \sum_{k=0}^{K-1} \zeta_K^k (\Re[\zeta_K^k] \Re[\bar{z}_1 z_2] + \Im[\zeta_K^k] \Im[\bar{z}_1 z_2]).$$

We consider the real and imaginary part of this equation separately. Here the substitution $\zeta_K^k := \Re \zeta_K^k + i \Im \zeta_K^k$ yields

$$\Re \left[\frac{1}{K} \sum_{k=0}^{K-1} \zeta_K^k |z_1 + \zeta_K^{-k} z_2|^2 \right] = \frac{2}{K} \left(\Re [\bar{z}_1 z_2] \sum_{k=0}^{K-1} [\Re \zeta_K^k]^2 + \Im [\bar{z}_1 z_2] \sum_{k=0}^{K-1} \Re \zeta_K^k \Im \zeta_K^k \right)$$

and

$$\Im \left[\frac{1}{K} \sum_{k=0}^{K-1} \zeta_K^k |z_1 + \zeta_K^{-k} z_2|^2 \right] = \frac{2}{K} \left(\Re [\bar{z}_1 z_2] \sum_{k=0}^{K-1} \Im \zeta_K^k \Re \zeta_K^k + \Im [\bar{z}_1 z_2] \sum_{k=0}^{K-1} [\Im \zeta_K^k]^2 \right).$$

Rewriting the real and imaginary parts as $\Re \zeta_K^k = 1/2(\zeta_K^{-k} + \zeta_K^k)$ and $\Im \zeta_K^k = i/2(\zeta_K^{-k} - \zeta_K^k)$, we can compute the occurring sums by

$$\begin{aligned} \sum_{k=0}^{K-1} [\Re \zeta_K^k]^2 &= \frac{1}{4} \sum_{k=0}^{K-1} (\zeta_K^{-2k} + 2 + \zeta_K^{2k}) = \frac{K}{2}, \\ \sum_{k=0}^{K-1} [\Im \zeta_K^k]^2 &= -\frac{1}{4} \sum_{k=0}^{K-1} (\zeta_K^{-2k} - 2 + \zeta_K^{2k}) = \frac{K}{2}, \end{aligned}$$

and

$$\sum_{k=0}^{K-1} \Re \zeta_K^k \Im \zeta_K^k = \frac{i}{4} \sum_{k=0}^{K-1} (\zeta_K^{-2k} - \zeta_K^{2k}) = 0,$$

which completes the proof. □

Remark 1. Obviously, Lemma 1 cannot be valid if $K < 3$ because the right-hand side would be always a real number. For the special case $K = 3$, Lemma 1 coincides with the original Mercedes-Benz polarization identity introduced in [3, Lemma 2.1].

The polarization identity in Lemma 1 can now be used to reveal the relation between the values $\hat{x}(\omega)$ and $\hat{x}(\omega - \mu)$ hidden in the interference intensity measurements

$$|\mathcal{F} [x + \zeta_K^{-k} e^{i\mu \cdot} x]| \quad (k = 0, \dots, K - 1)$$

for some $K \geq 3$ and $\mu \in \mathbb{R}$. The following theorem shows that the knowledge of this relationship is sufficient to recover each discrete-time signal up to trivial rotations.

Theorem 1. *Let x be a discrete-time signal with finite support of length N . If $\mu \neq 2\pi p/q$ for all $p \in \mathbb{Z}$ and $q \in \{1, \dots, N - 1\}$, then the signal x can be uniquely recovered up to rotation from its Fourier intensity $|\hat{x}|$ and the interference intensity measurements*

$$|\mathcal{F} [x + \zeta_K^{-k} e^{i\mu \cdot} x]| \quad (k = 0, \dots, K - 1)$$

for every $K \geq 3$.

Proof. Firstly, we apply the polarization identity in Lemma 1 to the given interference intensity measurements $|\hat{x}(\cdot) + \zeta_K^{-k} \hat{x}(\cdot - \mu)|$ and obtain

$$\frac{1}{K} \sum_{k=0}^{K-1} \zeta_K^k |\hat{x}(\omega) + \zeta_K^{-k} \hat{x}(\omega - \mu)|^2 = |\hat{x}(\omega)| |\hat{x}(\omega - \mu)| e^{i(\phi(\omega - \mu) - \phi(\omega))},$$

where ϕ denotes the phase of $\hat{x} = |\hat{x}| e^{i\phi}$. If $\hat{x}(\omega)$ and $\hat{x}(\omega - \mu)$ are non-zero, we can hence determine the phase difference $\phi(\omega - \mu) - \phi(\omega)$.

Since the unknown signal can only be recovered up to rotations, we can arbitrarily choose the phase $\phi(\omega_0)$ of $\hat{x}(\omega_0)$ for one ω_0 without loss of generality. Starting from this point, we can now use the phase differences to compute a series of relative phases

$$\phi(\omega_0 + \mu k) \quad (k = 0, \dots, 2N - 1)$$

with respect to $\phi(\omega_0)$. To ensure that $\hat{x}(\omega_0 + \mu k) \neq 0$ for $k = 0, \dots, 2N - 1$, we notice that \hat{x} can be written as

$$\hat{x}(\omega) = e^{-i\omega n_0} \sum_{n=0}^{N-1} c_n e^{-i\omega n}, \quad (3)$$

with $c_n := x[n+n_0]$ because of the finite support. Since the occurring algebraic polynomial in $e^{-i\omega}$ of degree $N - 1$ can only have finitely many zeros on the unit circle, we can always find a suitable ω_0 that enables us to recover $2N$ relative phases and hence $2N$ values of \hat{x} itself.

Next, we consider the exponential sum $Q := \hat{x}(\omega_0 + \mu \cdot)$ given by

$$Q(\omega) := \sum_{n=0}^{N-1} \gamma_n e^{-i\omega \tau_n} = \sum_{n=0}^{N-1} \left[c_n e^{-i\omega_0(n+n_0)} \right] e^{-i\omega \mu(n+n_0)} \quad (4)$$

with the complex coefficients $\gamma_n := c_n e^{-i\omega_0(n+n_0)}$ and the real frequencies $\tau_n := \mu(n+n_0)$. If we can recover the unknown parameters γ_n and τ_n from the function values $Q(k) = \hat{x}(\omega_0 + \mu k)$ with $k = 0, \dots, 2N - 1$, then we can compute the values c_n and n_0 in (3) and thus the true signal x . For this concern, we apply Prony's method to the recovered values $\hat{x}(\omega_0 + \mu k)$ in the frequency domain.

Following the lines in [30] or [16, Sect. 9.4], we define the Prony polynomial A by

$$A(z) := \prod_{n=0}^{N-1} (z - e^{-i\tau_n}) = \sum_{k=0}^N \lambda_k z^k$$

with $\lambda_N = 1$. Using the definition of the Prony polynomial, we immediately obtain

$$\sum_{k=0}^N \lambda_k Q(k+m) = \sum_{k=0}^N \sum_{n=0}^{N-1} \lambda_k \gamma_n e^{-i(k+m)\tau_n} = \sum_{n=0}^{N-1} \gamma_n e^{-im\tau_n} A(e^{-i\tau_n}) = 0$$

for $m = 0, \dots, N - 1$. Bringing $\lambda_N Q(N+m)$ with $\lambda_N = 1$ to the right-hand side, we can determine the remaining coefficients $\boldsymbol{\lambda} := (\lambda_0, \dots, \lambda_{N-1})^T$ by solving the linear equation system

$$\mathbf{H}\boldsymbol{\lambda} = \mathbf{h} \quad (5)$$

with the Hankel matrix $\mathbf{H} := (Q(k+m))_{m,k=0}^{N-1}$ and the right-hand side $\mathbf{h} := (Q(N+m))_{m=0}^{N-1}$. Knowing the coefficients λ_k of A , we can compute

the unknown frequencies τ_n via the roots of the Prony polynomial and the unknown coefficients γ_n as solution of the over-determined equation system

$$\sum_{n=0}^{N-1} \gamma_n e^{-ik\tau_n} = Q(k) \quad (k = 0, \dots, 2N - 1). \tag{6}$$

In order to ensure the solvability and uniqueness of the equation systems (5) and (6), we notice that \mathbf{H} can be written as

$$\mathbf{H} = \mathbf{V}^T \text{diag}(\gamma_0, \dots, \gamma_{N-1}) \mathbf{V}$$

with the Vandermonde matrix $\mathbf{V} = (e^{-ik\tau_n})_{n,k=0}^{N-1}$. If the coefficients γ_n are non-zero, both equation systems are uniquely solvable whenever the values $e^{-i\tau_n}$ differ pairwise. By the definition of τ_n , this is equivalent to

$$\mu(n_1 + n_0) \neq \mu(n_2 + n_0) + 2\pi\ell \quad \text{or} \quad \mu \neq 2\pi\ell / (n_1 - n_2)$$

for all $\ell \in \mathbb{Z}$ and $n_1 \neq n_2$ with $n_1, n_2 \in \{0, \dots, N - 1\}$, which leads us to the assumptions of the theorem. If some of the coefficients γ_n are zero and the related exponentials vanish in Q , then we can compute the number of non-vanishing exponentials by considering the singular value decomposition of \mathbf{H} , and the solvability of the smaller equation systems is also covered by the assumptions.

All in all, we can always apply Prony’s method to recover the unknown coefficients γ_n and frequencies τ_n of the exponential sum Q . Consequently, we can always determine \hat{x} and hence x up to rotation. \square

Remark 2. The main reason for the application of Prony’s method in the proof of Theorem 1 is the lack of information about the integer n_0 in the frequency representation (3). Considering the influence of the modulation $e^{-i\omega n_0}$ in the time domain, we only know the support length N of the unknown signal x but not the exact position of the support itself. If we additionally assume that x has the support $\{0, \dots, N - 1\}$, we can recover the trigonometric polynomial \hat{x} directly from the constructed function values by solving a linear equation system because all occurring frequencies are known beforehand. Unfortunately, we cannot neglect the restrictions on the parameter μ since these are needed to ensure the invertibility of the arising Vandermonde matrix.

Remark 3. Considering the assumptions of Theorem 1, we have to choose μ in a way that μ is not of the form $2\pi p/q$ with $p \in \mathbb{Z}$ and $q \in \{1, \dots, N - 1\}$. Hence, if we know the support length N or an upper bound of N , we can easily find an approximate modulation parameter μ . Furthermore, if we choose μ as an irrational multiple of 2π , we can theoretically recover every signal from the given interference intensity measurements without any information about the actual support length N .

Remark 4. If we assume that the support of the unknown discrete-time signal x with support length N is contained in the set $\{0, \dots, M - 1\}$ with

$M \geq N$ such that $(x[n])_{n \in \mathbb{Z}}$ can be identified with an M -dimensional vector, then we can interpret the Fourier intensity $|\widehat{x}(\omega_n)|$ for a certain point ω_n in the frequency domain as intensity measurement $|\langle x, v_n \rangle|$ with the frame vector $v_n := (e^{i\omega_n k})_{k=0}^{M-1}$. Choosing at least M pairwise different points ω_n in $[-\pi, \pi)$ beforehand, we can consequently transfer the whole theory developed by Alexeev et al. [3] to recover the unknown vector x from the given intensity measurements $|\widehat{x}(\omega_n)|$ and the given interference intensity measurements

$$|\widehat{x}(\omega_n) + \zeta_3^{-k} \widehat{x}(\omega_m)| \quad (k = 0, 1, 2)$$

for a larger number of randomly chosen index pairs (n, m) . In [3] it has been shown that the unknown vector can be recovered with high probability from approximately $240M$ measurements for an arbitrary frame, see [3, p. 41].

In contrast to the findings of Alexeev et al. for arbitrary frames, we exploit that the Fourier transform of a complex-valued discrete-time signal with finite support of length N is a trigonometric polynomial. For the special case $K = 3$ considered by Alexeev et al., this enables us to recover the unknown signal x always from merely $8N - 4$ measurements or, more precisely, from $2N - 1$ measurements for each of the four Fourier intensities

$$|\widehat{x}(\cdot)| \quad \text{and} \quad |\widehat{x}(\cdot) + \zeta_3^{-k} \widehat{x}(\cdot - \mu)| \quad (k = 0, 1, 2),$$

cf. Sect. 2.1. Moreover, the procedure in the proof of Theorem 1 allows us to determine the position of the current support from the given Fourier intensities. In other words, we can recover the unknown signal x without the assumption that the support of x is contained in some specific set $\{0, \dots, M - 1\}$.

2.2.2. Reducing the Number of Required Interference Intensity Measurements. Looking back at Theorem 1, we observe that the actual number of interference intensity measurements depends on the chosen root of unity ζ_K or, more precisely, on the chosen integer K . Consequently, it seems that the given interference intensity measurements are highly redundant. This impression is confirmed by a result in [13], where Candès et al. employ only two of the interference intensity measurements like in Theorem 1 to recover a finite-dimensional vector.

In this section, we adapt the approach of Candès et al. to the discrete-time phase retrieval problem and simultaneously generalize the result to show that each complex-valued discrete-time signal can be recovered from its Fourier intensity and two further interference intensity measurements of the form (1), where the two rotations can be chosen almost arbitrarily. In particular, these rotations do not have to arise from the K th roots of unity for a certain integer K .

Theorem 2. *Let x be a discrete-time signal with finite support of length N . If $\mu \neq 2\pi p/q$ for all $p \in \mathbb{Z}$ and $q \in \{1, \dots, N - 1\}$, then the signal x can be uniquely recovered up to a rotation from its Fourier intensity $|\widehat{x}|$ and the two*

interference intensity measurements

$$|\mathcal{F}[x + e^{i\alpha_1} e^{i\mu} x]| \quad \text{and} \quad |\mathcal{F}[x + e^{i\alpha_2} e^{i\mu} x]|,$$

where α_1 and α_2 are two real numbers satisfying $\alpha_1 - \alpha_2 \neq \pi k$ for all integer k .

Proof. We follow the lines of the proof of Theorem 1. Again, the crucial point is the extraction of the required relative phases from the given interference intensity measurements. Writing $\widehat{x}(\omega) = |\widehat{x}(\omega)| e^{i\phi(\omega)}$, we can rearrange the first interference intensity measurement to

$$\begin{aligned} & |\widehat{x}(\omega) + e^{i\alpha_1} \widehat{x}(\omega - \mu)|^2 \\ &= |\widehat{x}(\omega)|^2 + |\widehat{x}(\omega - \mu)|^2 + 2|\widehat{x}(\omega)||\widehat{x}(\omega - \mu)|\Re\left[e^{i(\phi(\omega - \mu) - \phi(\omega) + \alpha_1)}\right]. \end{aligned}$$

Replacing α_1 by α_2 , we obtain a similar representation for the second interference intensity measurement. Consequently, if both moduli $|\widehat{x}(\omega)|$ and $|\widehat{x}(\omega - \mu)|$ of the Fourier transform \widehat{x} are non-zero, we can determine

$$\Re\left[e^{i(\phi(\omega - \mu) - \phi(\omega) + \alpha_1)}\right] \quad \text{and} \quad \Re\left[e^{i(\phi(\omega - \mu) - \phi(\omega) + \alpha_2)}\right].$$

In order to extract the phase difference $\phi(\omega - \mu) - \phi(\omega)$ from these two values, we apply Euler’s formula and afterwards the addition theorem for cosine. In this manner, we obtain

$$\begin{aligned} \Re\left[e^{i(\phi(\omega - \mu) - \phi(\omega) + \alpha_1)}\right] &= \cos(\alpha_1) \cos(\phi(\omega - \mu) - \phi(\omega)) \\ &\quad - \sin(\alpha_1) \sin(\phi(\omega - \mu) - \phi(\omega)) \end{aligned}$$

and

$$\begin{aligned} \Re\left[e^{i(\phi(\omega - \mu) - \phi(\omega) + \alpha_2)}\right] &= \cos(\alpha_2) \cos(\phi(\omega - \mu) - \phi(\omega)) \\ &\quad - \sin(\alpha_2) \sin(\phi(\omega - \mu) - \phi(\omega)). \end{aligned}$$

Since the values on the left-hand side are known, we can consequently determine the sine and cosine of the wanted phase difference $\phi(\omega - \mu) - \phi(\omega)$ by solving a simple linear equation system. Here, the determinant of the occurring matrix is given by

$$\det\begin{pmatrix} \cos \alpha_1 & -\sin \alpha_1 \\ \cos \alpha_2 & -\sin \alpha_2 \end{pmatrix} = \sin \alpha_1 \cos \alpha_2 - \cos \alpha_1 \sin \alpha_2 = \sin(\alpha_1 - \alpha_2),$$

which ensures a unique solution whenever $\alpha_1 - \alpha_2$ does not coincide with a multiple of π . Consequently, we can always determine the required phase difference $\phi(\omega - \mu) - \phi(\omega)$ for a certain ω . With the extracted phase difference, the theorem can now be justified as discussed in the proof of Theorem 1. \square

Remark 5. As mentioned before, Candès et al. consider a slightly different phase retrieval problem in [13, Theorem 3.1]. More precisely, they deal with the problem to recover a finite-dimensional vector from the intensities of its discrete Fourier transform. Using our notation, we can state this problem as

follows: recover the signal x whose support of length N is contained in the interval from 0 to $M - 1$ from the (discrete-time) Fourier intensities

$$|\widehat{x}\left(\frac{2\pi k}{M}\right)|, \quad \left|\widehat{x}\left(\frac{2\pi k}{M}\right) + \widehat{x}\left(\frac{2\pi(k-\ell)}{M}\right)\right|, \quad \text{and} \quad \left|\widehat{x}\left(\frac{2\pi k}{M}\right) - i\widehat{x}\left(\frac{2\pi(k-\ell)}{M}\right)\right|$$

for all integers $k = 0, \dots, M-1$ and for a certain integer ℓ . Under the additional assumption that ℓ and M are co-prime, and that the given Fourier samples $|\widehat{x}(2\pi k/M)|$ are non-zero, Candès et al. show that the unknown signal x can be uniquely recovered.

Recalling that the Fourier intensity of a discrete-time signal with support length N is entirely determined by $2N - 1$ arbitrary samples, we can directly compare Theorem 2 for $\alpha_1 := 0$, $\alpha_2 := -\pi/2$, and $\mu := 2\pi\ell/M$ with the results of Candès et al. and see that both statements are almost identical. In the special case that N and M coincide, the main difference between both statements is that we need twice as many measurements. Anyway, this enables us to neglect the assumption that the given samples of the Fourier intensity have to be non-zero. Further, we can determine the unknown position of the current support completely from the given Fourier intensities.

Finally, the integers N and M have a slightly different meaning. With the dimension M , we determine the interval $\{0, \dots, M - 1\}$ that contains the non-zero components of the considered signal x . The current support length N of this signal can however be much smaller than the assumed dimension M . Consequently, if M is only a rough estimation, then Theorem 2 allows us to recover the wanted signal from a much smaller set of measurements.

3. Continuous-Time Phase Retrieval

3.1. Characterization of the Occurring Ambiguities

Different from the previous section, we now consider the continuous-time phase retrieval problem, which has a completely different behavior as the discrete-time equivalent. Here we are faced with the recovery of a continuous-time signal or a function $f: \mathbb{R} \rightarrow \mathbb{C}$ in L^2 with compact support from its Fourier intensity $|\mathcal{F}[f]|$, where the corresponding Fourier transform is given by

$$\widehat{f}(\omega) := \mathcal{F}[f](\omega) := \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt.$$

The ambiguousness of this problem has been studied by Akutowicz [1, 2], Walther [35], and Hofstetter [17], for instance. To reduce the set of possible ambiguities, we will again employ interference intensity measurements with different kinds of reference signals. For this purpose, we require a suitable characterization of the solution set. Here we follow the approach by Hofstetter [17], where the solutions are presented as an infinite product with respect to the zero set of the analytic continuations of their Fourier transforms.

In our case, the continuation of $\mathcal{F}[f]$ is given through the theorem of Paley-Wiener, see for instance [9, Theorem 6.8.1]. Using the (two-sided) Laplace transform, we can write this well-known theorem in the following form.

Theorem 3 (Paley-Wiener). *The function $F: \mathbb{C} \rightarrow \mathbb{C}$ is an entire function of exponential type and belongs to L^2 on the imaginary axis if and only if F is the Laplace transform*

$$F(\zeta) := \mathcal{L}[f](\zeta) := \int_{-\infty}^{\infty} f(t) e^{-\zeta t} dt,$$

of a function f in L^2 with compact support.

In this context, a function $F: \mathbb{C} \rightarrow \mathbb{C}$ is called *entire* if the function is analytic over the whole complex plane. If F moreover grows no faster than an exponential, which means that F can be bounded by

$$|F(\zeta)| \leq A e^{B|\zeta|}$$

with appropriate real constants $A \geq 0$ and B , then the entire function F is of *exponential type*, see for instance [37, p. 53]. As a consequence of the theorem of Paley-Wiener, the Laplace transform F is now the unique analytic continuation of the Fourier transform $\mathcal{F}[f]$ from the imaginary axis to the complex plane because the restriction $F(i \cdot)$ of the holomorphic Laplace transform F to the imaginary axis obviously coincides with $\mathcal{F}[f]$.

The next key instruments in the characterization of the occurring ambiguities are the *autocorrelation signal*

$$a(t) := \int_{-\infty}^{\infty} \overline{f(s)} f(s+t) ds$$

and the *autocorrelation function* $A := \mathcal{L}[a]$. Since the autocorrelation signal of a compactly supported function f in L^2 is again a compactly supported function in L^2 , the autocorrelation function A can be interpreted as the analytic continuation of the Fourier transform \hat{a} . Further, the autocorrelation is closely related to the given Fourier intensity, see [17].

Proposition 2. *Let f be a continuous-time signal in L^2 with compact support. Then the autocorrelation function A is the analytic continuation of the squared Fourier intensity $|\mathcal{F}[f]|^2$ from the imaginary axis to the complex plane.*

Proof. Using the definition of the autocorrelation signal, we can write the autocorrelation function A of the considered signal f as

$$\begin{aligned} A(\zeta) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{f(s)} f(s+t) e^{-\zeta t} ds dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{f(s)} f(t) e^{-\zeta t} e^{\zeta s} dt ds = F(\zeta) \overline{F(-\bar{\zeta})}. \end{aligned} \quad (7)$$

If we now consider the restriction of this identity to the imaginary axis, we have

$$A(i\omega) = F(i\omega) \overline{F(i\omega)} = \widehat{f}(\omega) \overline{\widehat{f}(\omega)} = |\widehat{f}(\omega)|^2,$$

which implies that the restriction of the autocorrelation function A coincides with the squared Fourier intensity $|\mathcal{F}[f]|^2$ as claimed. Since A is a holomorphic function by the theorem of Paley-Wiener, the assertion follows. \square

After this preliminaries, we recall the characterization of all occurring ambiguities in the continuous-time phase retrieval problem, [17, Theorem I].

Theorem 4 (Hofstetter). *Let f be a continuous-time signal in L^2 with compact support. Then the Laplace transform of each continuous-time signal g in L^2 with compact support and the same Fourier intensity $|\mathcal{F}[g]| = |\mathcal{F}[f]|$ can be written in the form*

$$G(\zeta) = C \zeta^m e^{\zeta\gamma} \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{\eta_j}\right) e^{\frac{\zeta}{\eta_j}}$$

where the absolute value $|C|$ and the imaginary part $\Im\gamma$ of the complex constants C and γ coincide for all signals g , and where η_j is chosen from the zero pair $(\xi_j, -\bar{\xi}_j)$ of the autocorrelation function A of the signal f .

Proof. For the sake of convenience for the reader, we give a short proof following the lines of [17]. Let f and g be signals with compact support in L^2 , and let F and G be their Laplace transforms being entire functions of exponential type. By Hadamard's factorization theorem [9], we can represent the entire functions F and G by

$$F(\zeta) = C_1 \zeta^{m_1} e^{\zeta\gamma_1} \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{\xi_j}\right) e^{\frac{\zeta}{\xi_j}} \quad \text{and} \quad G(\zeta) = C_2 \zeta^{m_2} e^{\zeta\gamma_2} \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{\eta_j}\right) e^{\frac{\zeta}{\eta_j}}$$

with respect to their non-vanishing zeros ξ_j and η_j .

Assuming that $|\mathcal{F}[g]| = |\mathcal{F}[f]|$, we can conclude that the autocorrelation functions of f and g coincide. Using (7), we can therefore represent the common autocorrelation function A in terms of F or G by

$$A(\zeta) = F(\zeta) \overline{F(-\bar{\zeta})} = G(\zeta) \overline{G(-\bar{\zeta})}. \quad (8)$$

Due to this factorization, all zeros of the autocorrelation function A obviously occur in pairs of the form $(\xi_j, -\bar{\xi}_j)$, where ξ_j is a zero of F . Since an analogous observation follows from the factorization of G , we can resort the zeros η_j so that $\eta_j = \xi_j$ or $\eta_j = -\bar{\xi}_j$. Further, the multiplicities m_1 and m_2 of the zero at the origin must also be equal. Consequently, the possibly infinite products in the factorizations of A coincide, and we can reduce (8) to

$$|C_1|^2 e^{\zeta(\gamma_1 - \bar{\gamma}_1)} = |C_2|^2 e^{\zeta(\gamma_2 - \bar{\gamma}_2)} \quad \text{or} \quad |C_1|^2 e^{2\zeta \Im[\gamma_1]} = |C_2|^2 e^{2\zeta \Im[\gamma_2]},$$

which shows that the absolute values $|C_1|$ and $|C_2|$ and also the imaginary parts $\Im\gamma_1$ and $\Im\gamma_2$ coincide. \square

Similarly to the discrete-time setting, we divide the occurring ambiguities characterized by Theorem 4 in two different classes. Since the rotation, the time shift by a real number, and the reflection and conjugation of a solution always result in a further solution of the considered phase retrieval problem, cf. Proposition 1, we call these ambiguities trivial. The remaining ambiguities are non-trivial. Using the characterization in Theorem 4, we can also represent the ambiguities in the following manner.

Proposition 3. *Let f and g be two continuous-time signals in L^2 with compact support and the same Fourier intensity $|\mathcal{F}[f]| = |\mathcal{F}[g]|$. Then there exist two entire functions F_1 and F_2 of exponential type such that*

$$F(\zeta) = F_1(\zeta) F_2(\zeta)$$

and

$$G(\zeta) = e^{i\alpha} e^{-\zeta t_0} \overline{F_1(-\bar{\zeta})} F_2(\zeta),$$

where α and t_0 are suitable real numbers.

Proof. Applying Theorem 4, we can represent the Laplace transforms F and G by

$$F(\zeta) = C_1 \zeta^m e^{\zeta\gamma_1} \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{\xi_j}\right) e^{\frac{\zeta}{\xi_j}} \quad \text{and} \quad G(\zeta) = C_2 \zeta^m e^{\zeta\gamma_2} \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{\eta_j}\right) e^{\frac{\zeta}{\eta_j}}$$

with $|C_1| = |C_2|$, $\Im\gamma_1 = \Im\gamma_2$, and $\eta_j \in (\xi_j, -\bar{\xi}_j)$. Now, we can resort the zeros of G such that

$$\eta_j = \begin{cases} -\bar{\xi}_j & \xi_j \in A, \\ \xi_j & \text{else} \end{cases}$$

for an appropriate subset A of the zero set $\Xi := \{\xi_j : j \in \mathbb{N}\}$ of F .

Based on this subset, we define the two possibly infinite products F_1 and F_2 by

$$F_1(\zeta) := \prod_{\xi_j \in A} \left(1 - \frac{\zeta}{\xi_j}\right) e^{\frac{\zeta}{\xi_j}} \quad \text{and} \quad F_2(\zeta) := C_1 \zeta^m e^{\zeta\gamma_1} \prod_{\xi_j \in \Xi \setminus A} \left(1 - \frac{\zeta}{\xi_j}\right) e^{\frac{\zeta}{\xi_j}}.$$

Due to the fact that the convergence exponent—the infimum of the positive numbers α for which the series $\sum_{j=1}^{\infty} |\xi_j|^{-\alpha}$ converges—of the zeros of an entire

function is always less than or equal to the order of the entire function, see for instance [9, Theorem 2.5.18], the zeros ξ_j can at most have the convergence exponent one. Since the zeros of F_1 and F_2 are merely subsets of the zeros of F , the corresponding convergence exponents thus also have to be less than or equal to one. Borel's theorem now implies that the possibly infinite products F_1 and F_2 are again entire functions of exponential type, see [24].

Obviously, we have the factorization $F = F_1 F_2$. In order to achieve the factorization of G , we consider the reflection of the first factor given by

$$\overline{F_1(-\bar{\cdot})} = \prod_{\xi_j \in \Lambda} \left(1 - \frac{\zeta}{-\bar{\xi}_j}\right) e^{-\frac{\zeta}{\bar{\xi}_j}}.$$

Hence, the reflection $\overline{F_1(-\bar{\cdot})}$ possesses the zeros $\eta_j = -\bar{\xi}_j$ for all ξ_j in Λ , which implies that the zeros of the product $\overline{F_1(-\bar{\cdot})} F_2$ and G coincide.

Finally, since the absolute values $|C_1|$ and $|C_2|$ and the imaginary parts $\Im\gamma_1$ and $\Im\gamma_2$ have to be equal by Theorem 4, the entire functions $\overline{F_1(-\bar{\cdot})} F_2$ and G can only differ by a rotation $e^{i\alpha}$ and a time shift $e^{-\zeta t_0}$. Choosing the real numbers α and t_0 suitably, we obtain the wanted factorization in the assertion. \square

Remark 6. We can interpret Proposition 3 in a slightly different way: if the restrictions of the entire functions F and G of exponential type to the imaginary axis coincide, then we can always find two entire functions F_1 and F_2 to factorize F and G in the manner of Proposition 3. This observation can now be generalized to entire functions of arbitrary order, see [23, Lemma 1].

Remark 7. For the discrete-time setting, an equivalent statement of Proposition 3 holds where the Laplace transforms F and G are replaced by the discrete-time Fourier transforms of the two signals. Moreover, for two discrete-time signals x and y with $|\hat{x}| = |\hat{y}|$, there always exists two discrete-time signals x_1 and x_2 with finite support such that

$$x = x_1 * x_2 := \left(\sum_{k \in \mathbb{Z}} x_1[n-k] x_2[k] \right)_{n \in \mathbb{Z}} \quad \text{and} \quad y = e^{i\alpha} x_1[\cdot - n_0] * \overline{x_2[-\cdot]}$$

with suitable $\alpha \in \mathbb{R}$ and $n_0 \in \mathbb{Z}$, see [7, Theorem 2.3]. In other words, each ambiguity of the discrete-time phase retrieval problem can be represented by an appropriate convolution and a suitable rotation, shift, and reflection and conjugation of the occurring factors.

3.2. Ensuring Uniqueness in the Continuous-Time Phase Retrieval

Differently from the discrete-time phase retrieval problem, the continuous-time version to recover a certain signal usually possesses infinitely many non-trivial ambiguities. Hence, we are once more faced with the question: how can we ensure the unique recovery of the unknown signal, or how can we at least reduce the occurring ambiguities to a sufficiently small set.

Like in the discrete-time setting, we will employ different kinds of interference intensity measurements. Here Klivanov et al. show that the additional interference intensity measurement with a known reference can almost ensure the unique recovery of a distribution with compact support. Adapting this observation [22, Proposition 6.5] to the continuous-time phase retrieval problem, we obtain the following statement.

Proposition 4 (Klivanov et al.). *Let f and h be two continuous-time signals in L^2 with compact support, where the non-vanishing reference signal h is known beforehand. Then the signal f can be recovered from the Fourier intensities*

$$|\mathcal{F}[f]| \quad \text{and} \quad |\mathcal{F}[f + h]|$$

except for at most one ambiguity.

Proof. We give a new proof of the statement by adapting the proof of the discrete-time equivalent in [7, Theorem 4.3]. Writing the Fourier transforms of the signals f and h in their polar representations

$$\mathcal{F}[f](\omega) = |\mathcal{F}[f](\omega)|e^{i\phi(\omega)} \quad \text{and} \quad \mathcal{F}[h](\omega) = |\mathcal{F}[h](\omega)|e^{i\psi(\omega)},$$

where ϕ and ψ denote the corresponding phase functions, we can represent the given interference intensity measurement by

$$\begin{aligned} & |\mathcal{F}[f + h](\omega)|^2 \\ &= |\mathcal{F}[f](\omega)|^2 + |\mathcal{F}[h](\omega)|^2 + 2|\mathcal{F}[f](\omega)||\mathcal{F}[h](\omega)| \cos(\phi(\omega) - \psi(\omega)). \end{aligned}$$

In other words, the phase difference $\phi - \psi$ is given by

$$\phi(\omega) - \psi(\omega) = \pm \arccos \left(\frac{|\mathcal{F}[f + h](\omega)|^2 - |\mathcal{F}[f](\omega)|^2 - |\mathcal{F}[h](\omega)|^2}{2|\mathcal{F}[f](\omega)||\mathcal{F}[h](\omega)|} \right) + 2\pi k \tag{9}$$

with an appropriate integer k whenever $\mathcal{F}[f](\omega)$ and $\mathcal{F}[h](\omega)$ are non-zero.

Since $\mathcal{F}[f]$ and $\mathcal{F}[h]$ are continuous, we can find a small interval for ω where the sign in (9) does not change and the integer k is fixed. Further, since $\phi - \psi$ is the phase function of the product $\mathcal{F}[f]\overline{\mathcal{F}[h]}$, which is the restriction of the entire function $F(\cdot)\overline{H(-\cdot)}$ to the imaginary axis, we can extend $\phi - \psi$ uniquely from the interval to the complete frequency domain. Consequently, there exist at most two distinct phase differences $\phi - \psi$. Here the phase $\check{\phi}$ of the second solution \check{f} and the phase ϕ are related by

$$\phi(\omega) - \psi(\omega) = -\check{\phi}(\omega) + \psi(\omega) + 2\pi k.$$

Hence, the Fourier transform of the second solution has to be of the form

$$\mathcal{F}[\check{f}] = |\mathcal{F}[f]|e^{-i\phi+2i\psi}, \tag{10}$$

which completes the proof. □

Remark 8. The main benefit of the proof of Proposition 4 given above is that we obtain an explicit representation of the second possible solution in dependence of the phase of the reference signal. Considering the Fourier transform (10), we can have doubts whether the corresponding continuous-time signal \check{f} is really a signal with compact support and hence a valid solution of the problem. Indeed, the Fourier transform (10) does not have to be the restriction of an entire function and does not even have to be a continuous function at all because the phase ψ of the continuous function $\mathcal{F}[h]$ itself can possess discontinuities. Further, the theorem of Paley-Wiener here implies that the second solution \check{f} does not have to have a compact support and may thus be an invalid solution of the considered continuous-time phase retrieval problem. In this case, the original signal f is the only non-trivial solution.

Next, we replace the known reference signal h within the interference $f + h$ by an unknown reference. Inspired by the discrete-time equivalent in [7, Theorem 4.4], we show that the continuous-time signal f and the unknown reference h are uniquely determined by the Fourier intensities of f , h , and $f + h$ up to common trivial ambiguities. This means that we can recover f and h up to common rotations or time shifts or up to the reflection and conjugation of the two signals.

Theorem 5. *Let f and h be two continuous-time signals in L^2 with compact support. If the non-vanishing zeros of the Laplace transformed signal F and H form disjoint sets, then both signals f and h can be recovered from the Fourier intensities*

$$|\mathcal{F}[f]|, \quad |\mathcal{F}[h]|, \quad \text{and}, \quad |\mathcal{F}[f + h]|$$

uniquely up to common trivial ambiguities.

Proof. Let \check{f} and \check{h} be a further solution pair of the considered problem with

$$|\mathcal{F}[f]| = |\mathcal{F}[\check{f}]|, \quad |\mathcal{F}[h]| = |\mathcal{F}[\check{h}]|, \quad \text{and} \quad |\mathcal{F}[f + h]| = |\mathcal{F}[\check{f} + \check{h}]|.$$

Applying Proposition 3, we can represent the two solution pairs in the frequency domain by an appropriate factorization of the Laplace transform F and H of the original signals. In this manner, we obtain the factorizations

$$F(\zeta) = F_1(\zeta) F_2(\zeta) \quad \text{and} \quad \check{F}(\zeta) = e^{i\alpha_1} e^{-\zeta t_1} \overline{F_1(-\bar{\zeta})} F_2(\zeta)$$

and further

$$H(\zeta) = H_1(\zeta) H_2(\zeta) \quad \text{and} \quad \check{H}(\zeta) = e^{i\alpha_2} e^{-\zeta t_2} \overline{H_1(-\bar{\zeta})} H_2(\zeta)$$

for some real numbers $\alpha_1, \alpha_2, t_1, t_2$ and entire functions F_1, F_2, H_1, H_2 .

In the next step, we consider the analytic continuation of the squared interference intensity measurement or the corresponding autocorrelation function, see Proposition 2. With the representation in (7), we can write the given interference intensity measurement as

$$(F(\zeta) + H(\zeta)) \left(\overline{F(-\bar{\zeta})} + \overline{H(-\bar{\zeta})} \right) = \left(\check{F}(\zeta) + \check{H}(\zeta) \right) \left(\overline{\check{F}(-\bar{\zeta})} + \overline{\check{H}(-\bar{\zeta})} \right)$$

or in the simplified form

$$F(\zeta) \overline{H(-\bar{\zeta})} + \overline{F(-\bar{\zeta})} H(\zeta) = \check{F}(\zeta) \overline{\check{H}(-\bar{\zeta})} + \overline{\check{F}(-\bar{\zeta})} \check{H}(\zeta).$$

Incorporating the found factorizations of F and H , we obtain

$$\begin{aligned} F_1(\zeta) F_2(\zeta) \overline{H_1(-\bar{\zeta})} \overline{H_2(-\bar{\zeta})} + \overline{F_1(-\bar{\zeta})} \overline{F_2(-\bar{\zeta})} H_1(\zeta) H_2(\zeta) \\ = e^{i(\alpha_1 - \alpha_2)} e^{-\zeta(t_1 - t_2)} \overline{F_1(-\bar{\zeta})} F_2(\zeta) H_1(\zeta) \overline{H_2(-\bar{\zeta})} \\ + e^{i(\alpha_2 - \alpha_1)} e^{-\zeta(t_2 - t_1)} F_1(\zeta) \overline{F_2(-\bar{\zeta})} \overline{H_1(-\bar{\zeta})} H_2(\zeta) \end{aligned}$$

and thus

$$\begin{aligned} \left[e^{-i\alpha_1} e^{\zeta t_1} F_1(\zeta) \overline{H_1(-\bar{\zeta})} - e^{-i\alpha_2} e^{\zeta t_2} \overline{F_1(-\bar{\zeta})} H_1(\zeta) \right] \\ \cdot \left[e^{i\alpha_1} e^{-\zeta t_1} F_2(\zeta) \overline{H_2(-\bar{\zeta})} - e^{i\alpha_2} e^{-\zeta t_2} \overline{F_2(-\bar{\zeta})} H_2(\zeta) \right] = 0. \end{aligned} \tag{11}$$

Remembering that $F_1, F_2, H_1,$ and H_2 are entire functions, we observe that both factors in (11) are entire functions too, and that at least one of both factors thus has to be constantly zero. In order to investigate the two different cases more precisely, we employ the explicit construction of the entire functions F_1 and F_2 in the proof of Proposition 3. Using a similar procedure for H_1 and H_2 , and denoting the sets of all non-vanishing zeros of F and H by Ξ_1 and Ξ_2 respectively, we can represent the four functions by

$$F_1(\zeta) = \prod_{\xi_j \in A_1} \left(1 - \frac{\zeta}{\xi_j}\right) e^{\frac{\zeta}{\xi_j}} \quad \text{and} \quad F_2(\zeta) = C_1 \zeta^{m_1} e^{\zeta \gamma_1} \prod_{\xi_j \in \Xi_1 \setminus A_1} \left(1 - \frac{\zeta}{\xi_j}\right) e^{\frac{\zeta}{\xi_j}}$$

and further

$$H_1(\zeta) = \prod_{\eta_j \in A_2} \left(1 - \frac{\zeta}{\eta_j}\right) e^{\frac{\zeta}{\eta_j}} \quad \text{and} \quad H_2(\zeta) = C_2 \zeta^{m_2} e^{\zeta \gamma_2} \prod_{\eta_j \in \Xi_2 \setminus A_2} \left(1 - \frac{\zeta}{\eta_j}\right) e^{\frac{\zeta}{\eta_j}},$$

where A_1 and A_2 are appropriate subsets of Ξ_1 and Ξ_2 .

In the following, we firstly assume that the second factor of (11) is zero, which directly implies that the equation

$$\begin{aligned} (-1)^{m_2} C_1 \overline{C_2} e^{i\alpha_1} \zeta^{m_1+m_2} e^{-\zeta(t_1 - \gamma_1 + \bar{\gamma}_2)} \prod_{\xi_j \in \Xi_1 \setminus A_1} \left(1 - \frac{\zeta}{\xi_j}\right) e^{\frac{\zeta}{\xi_j}} \prod_{\eta_j \in \Xi_2 \setminus A_2} \left(1 - \frac{\zeta}{-\eta_j}\right) e^{-\frac{\zeta}{\eta_j}} \\ = (-1)^{m_1} \overline{C_1} C_2 e^{i\alpha_2} \zeta^{m_1+m_2} e^{-\zeta(t_2 + \bar{\gamma}_1 - \gamma_2)} \prod_{\xi_j \in \Xi_1 \setminus A_1} \left(1 - \frac{\zeta}{-\xi_j}\right) e^{-\frac{\zeta}{\xi_j}} \prod_{\eta_j \in \Xi_2 \setminus A_2} \left(1 - \frac{\zeta}{\eta_j}\right) e^{\frac{\zeta}{\eta_j}} \end{aligned} \tag{12}$$

holds for every ζ in the complex plane. Since the possibly infinite products above are again entire functions by Borel's theorem [24], the zeros on both sides of the equality have to coincide. However, due to the assumption that the zeros ξ_j and η_j of the Laplace transforms F and H are pairwise distinct, the

zero sets $\Xi_1 \setminus \Lambda_1$ and $\Xi_2 \setminus \Lambda_2$ of F_2 and H_2 have to be invariant under reflection at the imaginary axis.

Based on this observation, we can immediately conclude that the entire functions F_2 and H_2 are invariant under reflection and conjugation up to an additional rotation and modulation. More precisely, we obtain the identities

$$\begin{aligned} \overline{F_2(-\bar{\zeta})} &= (-1)^{m_1} \bar{C}_1 \zeta^{m_1} e^{-\zeta \bar{\gamma}_1} \prod_{\xi_j \in \Xi_1 \setminus \Lambda_1} \left(1 - \frac{\xi}{-\bar{\xi}_j}\right) e^{-\frac{\zeta}{-\bar{\xi}_j}} \\ &= (-1)^{m_1} e^{-2i \arg C_1} e^{-2\zeta \Re[\gamma_1]} F_2(\zeta) \end{aligned}$$

and similarly

$$\overline{H_2(-\bar{\zeta})} = (-1)^{m_2} e^{-2i \arg C_2} e^{-2\zeta \Re[\gamma_2]} H_2(\zeta).$$

Incorporating these identities in the representation of \check{F} and \check{H} , we can describe the second solution pair in the frequency domain by

$$\check{F}(\zeta) = (-1)^{m_1} e^{i(\alpha_1 + 2 \arg C_1)} e^{-\zeta(t_1 - 2\Re[\gamma_1])} \overline{F(-\bar{\zeta})}$$

and

$$\check{H}(\zeta) = (-1)^{m_2} e^{i(\alpha_2 + 2 \arg C_2)} e^{-\zeta(t_2 - 2\Re[\gamma_2])} \overline{H(-\bar{\zeta})}.$$

Hence, the continuous-time signals \check{f} and \check{h} are merely rotations and shifts of the reflected and conjugated signals f and h .

It remains to prove that the occurring rotations and shifts coincide. For this purpose, we revisit equation (12). Considering that the zeros and hence the possibly infinite products on both sides are equal, we can reduce (12) to

$$(-1)^{m_2} C_1 \bar{C}_2 e^{i\alpha_1} e^{-\zeta(t_1 - \gamma_1 + \bar{\gamma}_2)} = (-1)^{m_1} \bar{C}_1 C_2 e^{i\alpha_2} e^{-\zeta(t_2 + \bar{\gamma}_1 - \gamma_2)}$$

or, by rearranging and combining the individual factors, to

$$(-1)^{m_1} e^{i\alpha_1 + 2 \arg C_1} e^{-\zeta(t_1 - 2\Re[\gamma_1])} = (-1)^{m_2} e^{i\alpha_2 + 2 \arg C_2} e^{-\zeta(t_2 - 2\Re[\gamma_2])},$$

which verifies our conjecture that the second solution pair \check{f} and \check{h} coincides with the first solution pair f and h up to common trivial ambiguities.

For the second case, where the first factor of (11) is constantly zero, an analogous and slightly simpler argumentation yields the representations

$$\check{F}(\zeta) = e^{i\alpha_1} e^{-\zeta t_1} F(\zeta) \quad \text{and} \quad \check{H}(\zeta) = e^{i\alpha_2} e^{-\zeta t_2} H(\zeta),$$

where the occurring rotations and time shifts again coincide. \square

Our last approach to achieve the uniqueness of the continuous-time phase retrieval problem considered in this section is again the idea of using interference intensity measurements of the unknown signal with a modulated version of the signal itself. Generalizing the main results of Sect. 2.2, we will establish two different theorems, which show that each continuous-time signal in L_2 with compact support can be uniquely recovered from an appropriate set of interference intensity measurements.

Theorem 6. *Let f be a continuous-time signal in L^2 with compact support. Then the signal f can be uniquely recovered up to rotation from its Fourier intensity $|\mathcal{F}[f]|$ and the interference intensity measurements*

$$|\mathcal{F}[f + \zeta_K^{-k} e^{i\mu \cdot} f]| \quad (k = 0, \dots, K - 1; \mu \in M)$$

for an integer $K > 2$ and every open neighbourhood M around zero.

Proof. Due to the assumption that the unknown signal f is a square-integrable function with compact support, the theorem of Paley-Wiener implies that the Fourier transform $\mathcal{F}[f]$ is the restriction of an entire function and thus has to be continuous. Consequently, if the signal f does not vanish everywhere, we can find a point ω_0 together with an open neighbourhood where the Fourier transform $\mathcal{F}[f]$ does not vanish.

Similarly to the discrete-time version in Theorem 1, the key element of the proof is to exploit the additional interference intensity measurements $|\widehat{f}(\cdot) + \zeta_K^{-k} \widehat{f}(\cdot - \mu)|$ by using the polarization identity in Lemma 1. We obtain

$$\frac{1}{K} \sum_{k=0}^{K-1} \zeta_K^k |\widehat{f}(\omega_0) + \zeta_K^{-k} \widehat{f}(\omega_0 - \mu)|^2 = \overline{\widehat{f}(\omega_0)} \widehat{f}(\omega_0 - \mu)$$

for every μ in the open set M . Writing the Fourier transform $\mathcal{F}[f]$ in its polar representation $|\mathcal{F}[f]| e^{i\phi}$, we can now extract the relative phases $\phi(\omega_0 - \mu) - \phi(\omega_0)$ from

$$\frac{1}{K} \sum_{k=0}^{K-1} \zeta_K^k |\widehat{f}(\omega_0) + \zeta_K^{-k} \widehat{f}(\omega_0 - \mu)|^2 = |\widehat{f}(\omega_0)| |\widehat{f}(\omega_0 - \mu)| e^{i(\phi(\omega_0 - \mu) - \phi(\omega_0))}.$$

Like for the discrete-time counterpart, the considered phase retrieval problem can merely be solved up to rotations. This enables us to define one phase $\phi(\omega_0)$ in the frequency domain arbitrarily. Beginning from this initial phase, we can further determine the complete phase function ϕ and hence the Fourier transform $\mathcal{F}[f]$ in a small open interval around ω_0 by using the extracted relative phases. Since the unknown Fourier transform $\mathcal{F}[f]$ is the restriction of an entire function as discussed above, the unknown function $\mathcal{F}[f]$ can be uniquely extended from the small interval to the complete frequency domain. Using the inverse Fourier transform, we finally obtain the desired signal f . \square

Theorem 7. *Let f be a continuous-time signal in L^2 with compact support. Then the signal f can be uniquely recovered up to a rotation from its Fourier intensity $|\mathcal{F}[f]|$ and the interference intensity measurements*

$$|\mathcal{F}[f + e^{i\alpha_1} e^{i\mu \cdot} f]| \quad \text{and} \quad |\mathcal{F}[f + e^{i\alpha_2} e^{i\mu \cdot} f]| \quad (\mu \in M)$$

where α_1 and α_2 are two real numbers satisfying $\alpha_1 - \alpha_2 \neq \pi k$ for all integers k , and where M is an open neighbourhood around zero.

Proof. Again, the crucial point to verify the assertion is the extraction of the relative phase from the given interference intensity measurements. Letting ϕ

be the phase function of the unknown Fourier transform $\mathcal{F}[f]$, and following the lines in the proof of the discrete-time counterpart (Theorem 2), we can determine the values

$$\Re\left[e^{i(\phi(\omega-\mu)-\phi(\omega)+\alpha_1)}\right] \quad \text{and} \quad \Re\left[e^{i(\phi(\omega-\mu)-\phi(\omega)+\alpha_2)}\right]$$

and further the relative phase $\phi(\omega-\mu)-\phi(\omega)$ whenever $\mathcal{F}[f](\omega)$ and $\mathcal{F}[f](\omega-\mu)$ are non-zero by solving a linear equation system. Based on the extracted relative phases $\phi(\omega-\mu)-\phi(\omega)$, we now can recover the unknown Fourier transform $\mathcal{F}[f]$ as discussed in the previous proof of Theorem 6. \square

Remark 9. Since a holomorphic function is completely determined by the function values on an arbitrary set that has an accumulation point, we can relax the requirements on M in Theorems 6 and 7. In fact, we can replace the condition that M is an open neighbourhood around zero by the assumption that M possesses at least an accumulation point at zero. Moreover, since the true signal f has a compact support and thus the Fourier transform $\mathcal{F}[f]$ is non-zero almost everywhere, we can easily adapt the proof for arbitrary open sets M or arbitrary sets M with at least one accumulation point.

Acknowledgements

I gratefully acknowledge the funding of this work by the DFG in the framework of the SFB 755 ‘Nanoscale photonic imaging’ and of the GRK 2088 ‘Discovering structure in complex data: Statistics meets Optimization and Inverse Problems’.

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Received: April 28, 2016.

Accepted: November 29, 2016.