



Further Generalizations, Refinements, and Reverses of the Young and Heinz Inequalities

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Abstract. In this paper, we give a new inequality for convex functions of real variables, and we apply this inequality to obtain considerable generalizations, refinements, and reverses of the Young and Heinz inequalities for positive scalars. Applications to unitarily invariant norm inequalities involving positive semidefinite matrices are also given.

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1. Introduction

The numerical Young's inequality for positive real numbers says that

$$a^\nu b^\nu \leq \nu a + (1 - \nu)b, \quad (1.1)$$

where $a, b > 0$ and $0 \leq \nu \leq 1$. Equivalently,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Kittaneh and Manasrah in [6] and [7], respectively, refined the inequality (1.1) and gave a reverse of it in the following forms:

$$a^\nu b^{1-\nu} + r_0 \left(\sqrt{a} - \sqrt{b} \right)^2 \leq \nu a + (1 - \nu)b \quad (1.2)$$

and

$$\nu a + (1 - \nu)b \leq a^\nu b^{1-\nu} + R_0 \left(\sqrt{a} - \sqrt{b} \right)^2, \tag{1.3}$$

where $r_0 = \min \{ \nu, 1 - \nu \}$ and $R_0 = \max \{ \nu, 1 - \nu \}$.

For our purpose in this paper, the inequalities (1.2) and (1.3) are combined and expressed so that

$$r_0 \left((a + b) - 2\sqrt{ab} \right) \leq \nu a + (1 - \nu)b - a^\nu b^{1-\nu} \leq R_0 \left((a + b) - 2\sqrt{ab} \right). \tag{1.4}$$

In [4], Hirzallah and Kittaneh proved that if $a, b > 0$ and $0 \leq \nu \leq 1$, then

$$\left(a^\nu b^{1-\nu} \right)^2 + r_0^2 (a - b)^2 \leq (\nu a + (1 - \nu)b)^2, \tag{1.5}$$

where $r_0 = \min \{ \nu, 1 - \nu \}$.

In [7], Kittaneh and Manasrah gave a reverse of the inequality (1.5) in the following form:

$$(\nu a + (1 - \nu)b)^2 \leq \left(a^\nu b^{1-\nu} \right)^2 + R_0^2 (a - b)^2, \tag{1.6}$$

where $R_0 = \max \{ \nu, 1 - \nu \}$.

Also, for our purpose in this paper, the inequalities (1.5) and (1.6) are combined and expressed so that

$$r_0^2 \left((a + b)^2 - 4ab \right) \leq (\nu a + (1 - \nu)b)^2 - \left(a^\nu b^{1-\nu} \right)^2 \leq R_0^2 \left((a + b)^2 - 4ab \right). \tag{1.7}$$

Recently, the authors [1] proved the following theorem.

Theorem 1. *If $a, b > 0$ and $0 \leq \nu \leq 1$, then for $m = 1, 2, 3, \dots$, we have*

$$\left(a^\nu b^{1-\nu} \right)^m + r_0^m \left(a^{\frac{m}{2}} - b^{\frac{m}{2}} \right)^2 \leq (\nu a + (1 - \nu)b)^m, \tag{1.8}$$

where $r_0 = \min \{ \nu, 1 - \nu \}$.

In fact, this is a generalization of the inequalities (1.2) and (1.5), which correspond to the cases $m = 1$ and $m = 2$, respectively.

The Heinz means are defined as

$$H_\nu(a, b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}$$

for $a, b > 0$ and $0 \leq \nu \leq 1$. These interesting means interpolate between the geometric and arithmetic means. In fact, the Heinz inequalities assert that

$$\sqrt{ab} \leq H_\nu(a, b) \leq \frac{a + b}{2}.$$

Interchanging a and b in the inequalities (1.4), and adding the resulting inequalities to the inequalities (1.4), we have

$$2r_0 \left(a + b - 2\sqrt{ab} \right) \leq a + b - \left(a^\nu b^{1-\nu} + a^{1-\nu} b^\nu \right) \leq 2R_0 \left(a + b - 2\sqrt{ab} \right). \tag{1.9}$$

Equivalently,

$$2r_0 \left(\frac{a+b}{2} - \sqrt{ab} \right) \leq \frac{a+b}{2} - H_\nu(a,b) \leq 2R_0 \left(\frac{a+b}{2} - \sqrt{ab} \right), \tag{1.10}$$

where $r_0 = \min \{ \nu, 1 - \nu \}$ and $R_0 = \max \{ \nu, 1 - \nu \}$.

In Sect. 2, we present a new inequality for convex functions of real variables. We apply this inequality to obtain considerable generalizations, refinements, and reverses of the Young and Heinz inequalities (1.2)–(1.10). Applications to unitarily invariant norm inequalities involving positive semidefinite matrices are given in Sect. 3.

2. Main Results

Let f be a convex function defined on an interval I . If x, y, z and w are points in I such that $w < z < y < x$, then it follows by a slop argument that

$$\frac{f(z) - f(w)}{z - w} \leq \frac{f(y) - f(w)}{y - w} \leq \frac{f(y) - f(z)}{y - z} \leq \frac{f(x) - f(y)}{x - y} \tag{2.1}$$

(see, e.g., [8, p. 21]).

Our first result is a consequence of the inequalities (2.1) for convex functions of real variables.

Theorem 2. *Let ϕ be a strictly increasing convex function defined on an interval I . If x, y, z , and w are points in I such that*

$$z - w \leq x - y, \tag{2.2}$$

where $w \leq z \leq x$ and $y \leq x$, then

$$(0 \leq) \quad \phi(z) - \phi(w) \leq \phi(x) - \phi(y). \tag{2.3}$$

Proof. First of all, observe that if $x = y$, then $z = w$, and so the inequality (2.3) becomes an equality. If $y = w$ or $z = w$, then the inequality (2.3) holds. Also, if $x = z$, then according to the inequality (2.2), we have $w \geq y$, so $\phi(w) \geq \phi(y)$, and hence $\phi(z) - \phi(w) \leq \phi(x) - \phi(y)$.

Assume that $x \neq y, y \neq w, z \neq w$, and $x \neq z$. Then we have three cases for ordering the points x, y, z, w as follows:

Case 1: $w < y \leq z < x$

Case 2: $w < z < y < x$

Case 3: $y < w < z < x$.

Now, if $y = z$, then the case 1 becomes $w < y = z < x$, and so by the inequalities (2.1), we have $\frac{\phi(z) - \phi(w)}{z - w} \leq \frac{\phi(x) - \phi(y)}{x - y}$, which implies that $\phi(z) - \phi(w) \leq \phi(x) - \phi(y)$. Suppose $y \neq z$. Then apply the inequalities (2.1) to the cases 1 and 2 to get the inequality (2.3).

To discuss the third case, we apply the strictly increasing property of the function ϕ to the sequence of points $y < w < z < x$, so we have $\phi(y) < \phi(w) <$

$\phi(z) < \phi(x)$ and this implies that $\phi(y) - \phi(w) < 0 < \phi(x) - \phi(z)$, and hence $\phi(z) - \phi(w) < \phi(x) - \phi(y)$.

Thus, from the discussion above we have

$$\phi(z) - \phi(w) \leq \phi(x) - \phi(y).$$

This completes the proof. □

As a direct consequence of Theorem 2, we have the following generalizations of the inequalities (1.4).

Corollary 1. *Let $\phi : [0, \infty) \rightarrow \mathbb{R}$ be a strictly increasing convex function. If $a, b > 0$ and $0 \leq \nu \leq 1$, then we have*

$$\begin{aligned} \phi(r_0(a+b)) - \phi(2r_0\sqrt{ab}) &\leq \phi(\nu a + (1-\nu)b) - \phi(a^\nu b^{1-\nu}) \\ &\leq \phi(R_0(a+b)) - \phi(2R_0\sqrt{ab}), \end{aligned} \tag{2.4}$$

where $r_0 = \min\{\nu, 1-\nu\}$ and $R_0 = \max\{\nu, 1-\nu\}$.

Proof. Let $x = \nu a + (1-\nu)b$, $y = a^\nu b^{1-\nu}$, $z = r_0(a+b)$, $w = 2r_0\sqrt{ab}$, $z' = R_0(a+b)$, and $w' = 2R_0\sqrt{ab}$. Then based on the inequalities (1.4) and the arithmetic–geometric mean inequality, we have

$$z - w \leq x - y \leq z' - w'.$$

The first and the second inequalities in (2.4) follow directly by applying Theorem 2 to the inequalities $z - w \leq x - y$, with $w \leq z \leq x$, $y \leq x$ and $x - y \leq z' - w'$ with $y \leq x \leq z'$, $w' \leq z'$, respectively. This completes the proof. □

A particular case of Corollary 1, which is obtained by taking $\phi(x) = x^p$ ($p \in \mathbb{R}$, $p \geq 1$) can be stated as follows.

Corollary 2. *If $a, b > 0$ and $0 \leq \nu \leq 1$, then for $p \in \mathbb{R}$, $p \geq 1$, we have*

$$\begin{aligned} r_0^p \left((a+b)^p - (2\sqrt{ab})^p \right) &\leq (\nu a + (1-\nu)b)^p - (a^\nu b^{1-\nu})^p \\ &\leq R_0^p \left((a+b)^p - (2\sqrt{ab})^p \right), \end{aligned} \tag{2.5}$$

where $r_0 = \min\{\nu, 1-\nu\}$ and $R_0 = \max\{\nu, 1-\nu\}$.

One can observe that the inequalities (2.5) are reduced to the inequalities (1.4) and (1.7) when $p = 1$ and $p = 2$, respectively.

The next theorem demonstrates the relationship between the inequalities (2.5) and (1.8). In fact, it confirms that the first inequality (2.5) is uniformly better than the inequality (1.8), and the second inequality in (2.5) provides a reverse of the same inequality. It should be noted here that the variable p in the inequalities (2.5) is continuous ($p \in \mathbb{R}$, $p \geq 1$), while the variable m in the inequality (1.8) is discrete ($m = 1, 2, 3, \dots$).

Theorem 3. *If $a, b > 0$ and $0 \leq \nu \leq 1$, then we have*

$$\begin{aligned}
 r_0^m \left(a^{\frac{m}{2}} - b^{\frac{m}{2}} \right)^2 &\leq r_0^m \left((a+b)^m - \left(2\sqrt{ab} \right)^m \right) \\
 &\leq (\nu a + (1-\nu)b)^m - (a^\nu b^{1-\nu})^m \\
 &\leq R_0^m \left((a+b)^m - \left(2\sqrt{ab} \right)^m \right)
 \end{aligned}
 \tag{2.6}$$

for $m = 1, 2, 3, \dots$, where $r_0 = \min \{ \nu, 1 - \nu \}$.

Proof. It is clear that the second and third inequalities in (2.6) are special cases of the inequalities (2.5). So, it is enough to prove the first inequality in (2.6).

To do this, observe that the cases $m = 1$ and $m = 2$ degenerate to an equality. For $m = 3, 4, 5, \dots$, we discuss two cases.

Case 1: If m is even, we have

$$\begin{aligned}
 (a+b)^m &= a^m + \sum_{i=1}^{\frac{m}{2}-1} \binom{m}{i} (a^i b^{m-i} + a^{m-i} b^i) + \binom{m}{\frac{m}{2}} b^{\frac{m}{2}} + b^m \\
 &\geq a^m + 2a^{\frac{m}{2}} b^{\frac{m}{2}} \sum_{i=1}^{\frac{m}{2}-1} \binom{m}{i} + \binom{m}{\frac{m}{2}} a^{\frac{m}{2}} b^{\frac{m}{2}} + b^m \\
 &\hspace{10em} \text{(by the arithmetic-geometric mean inequality)} \\
 &= a^m + a^{\frac{m}{2}} b^{\frac{m}{2}} \left(2 \sum_{i=1}^{\frac{m}{2}-1} \binom{m}{i} + \binom{m}{\frac{m}{2}} \right) + b^m \\
 &= a^m + b^m + a^{\frac{m}{2}} b^{\frac{m}{2}} \sum_{i=1}^{m-1} \binom{m}{i} \\
 &= a^m + b^m + (2^m - 2) a^{\frac{m}{2}} b^{\frac{m}{2}}.
 \end{aligned}$$

Case 2: If m is odd, we have

$$\begin{aligned}
 (a+b)^m &= a^m + \sum_{i=1}^{\frac{m-1}{2}} \binom{m}{i} (a^i b^{m-i} + a^{m-i} b^i) + b^m \\
 &\geq a^m + b^m + 2a^{\frac{m}{2}} b^{\frac{m}{2}} \sum_{i=1}^{\frac{m-1}{2}} \binom{m}{i} \\
 &\hspace{10em} \text{(by the arithmetic-geometric mean inequality)} \\
 &= a^m + b^m + a^{\frac{m}{2}} b^{\frac{m}{2}} \sum_{i=1}^{m-1} \binom{m}{i} \\
 &= a^m + b^m + (2^m - 2) a^{\frac{m}{2}} b^{\frac{m}{2}}.
 \end{aligned}$$

Now,

for $m = 3, 4, 5, \dots$, we have

$$\begin{aligned} (a + b)^m - (2\sqrt{ab})^m &= (a + b)^m - 2^m a^{\frac{m}{2}} b^{\frac{m}{2}} \\ &\geq a^m + b^m - 2a^{\frac{m}{2}} b^{\frac{m}{2}} \\ &= \left(a^{\frac{m}{2}} - b^{\frac{m}{2}}\right)^2. \end{aligned}$$

Thus,

$$r_0^m \left(a^{\frac{m}{2}} - b^{\frac{m}{2}}\right)^2 \leq r_0^m \left((a + b)^m - (2\sqrt{ab})^m\right) \leq (\nu a + (1 - \nu)b)^m - (a^\nu b^{1-\nu})^m$$

for $m = 1, 2, 3, \dots$. This completes the proof. \square

Applying Theorem 2 again, we have the following generalizations of the inequalities (1.9) and (1.10), respectively:

$$\begin{aligned} \phi(2r_0(a + b)) - \phi(4r_0\sqrt{ab}) &\leq \phi(a + b) - \phi(a^\nu b^{1-\nu} + a^{1-\nu} b^\nu) \\ &\leq \phi(2R_0(a + b)) - \phi(4R_0\sqrt{ab}) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \phi\left(2r_0\left(\frac{a + b}{2}\right)\right) - \phi(2r_0\sqrt{ab}) &\leq \phi\left(\frac{a + b}{2}\right) - \phi(H_\nu(a, b)) \\ &\leq \phi\left(2R_0\left(\frac{a + b}{2}\right)\right) - \phi(2R_0\sqrt{ab}). \end{aligned}$$

In particular, if $\phi(x) = x^p$ ($p \in \mathbb{R}, p \geq 1$), we have

$$\begin{aligned} (2r_0)^p \left((a + b)^p - (2\sqrt{ab})^p\right) &\leq (a + b)^p - (a^\nu b^{1-\nu} + a^{1-\nu} b^\nu)^p \\ &\leq (2R_0)^p \left((a + b)^p - (2\sqrt{ab})^p\right). \end{aligned} \tag{2.8}$$

3. Some Inequalities for Unitarily Invariant Norms

In this section, we obtain matrix versions of the scalar inequalities presented in Sects. 1 and 2.

Let $M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices. A norm $\|\cdot\|$ on $M_n(\mathbb{C})$ is called unitarily invariant if $\|UAV\| = \|A\|$ for all $A \in M_n(\mathbb{C})$ and for all unitary matrices $U, V \in M_n(\mathbb{C})$. An example of unitarily invariant norms is the Schatten p -norm, denoted by $\|\cdot\|_p$ and defined, for $1 \leq p < \infty$, by

$$\|A\|_p = \left(\sum_{i=1}^n s_i^p(A)\right)^{\frac{1}{p}},$$

where $s_1(A) \geq s_2(A) \geq \dots \geq s_n(A)$ are the singular values of $A \in M_n(\mathbb{C})$. The Schatten 1-norm of A is the trace norm, which can be expressed as $\|A\|_1 = \text{tr} |A|$. The Schatten 2-norm of $A = [a_{ij}]$ is known as the Hilbert-Schmidt (or the Frobenius) norm, which can be expressed as

$$\|A\|_2 = \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}}.$$

Another important example of unitarily invariant norms on $M_n(\mathbb{C})$ is the spectral (or the usual operator) norm $\|\cdot\|$, given by

$$\|A\| = s_1(A).$$

Based on the refined and reversed Young inequalities (1.5) and (1.6), Hirzallah and Kittaneh [4], and Kittaneh and Manasrah [7], respectively, proved that if $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite, then

$$r_0^2 \|AX - XB\|_2^2 \leq \|\nu AX + (1 - \nu)XB\|_2^2 - \|A^\nu XB^{1-\nu}\|_2^2 \tag{3.1}$$

and

$$\|\nu AX + (1 - \nu)XB\|_2^2 - \|A^\nu XB^{1-\nu}\|_2^2 \leq R_0^2 \|AX - XB\|_2^2, \tag{3.2}$$

where $0 \leq \nu \leq 1$, $r_0 = \min \{\nu, 1 - \nu\}$ and $R_0 = \max \{\nu, 1 - \nu\}$.

It can be easily shown that

$$\|AX - XB\|_2^2 = \|AX + XB\|_2^2 - 4 \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^2.$$

Thus, the inequalities (3.1) and (3.2) can be combined and expressed so that

$$\begin{aligned} r_0^2 \left(\|AX + XB\|_2^2 - 4 \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^2 \right) &\leq \|\nu AX + (1 - \nu)XB\|_2^2 - \|A^\nu XB^\nu\|_2^2 \\ &\leq R_0^2 \left(\|AX + XB\|_2^2 - 4 \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^2 \right). \end{aligned} \tag{3.3}$$

Applying Theorem 2 to the inequalities (3.3), the following general result holds.

Theorem 4. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite. If $\phi : [0, \infty) \rightarrow \mathbb{R}$ is a strictly increasing convex function, then we have*

$$\begin{aligned} &\phi \left(r_0^2 \|AX + XB\|_2^2 \right) - \phi \left(4r_0^2 \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^2 \right) \\ &\leq \phi \left(\|\nu AX + (1 - \nu)XB\|_2^2 \right) - \phi \left(\|A^\nu XB^\nu\|_2^2 \right) \\ &\leq \phi \left(R_0^2 \|AX + XB\|_2^2 \right) - \phi \left(4R_0^2 \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^2 \right), \end{aligned}$$

where $r_0 = \min \{\nu, 1 - \nu\}$ and $R_0 = \max \{\nu, 1 - \nu\}$.

In particular, when $\phi(x) = x^{\frac{p}{2}}$ ($p \in \mathbb{R}$ and $p \geq 2$), we have

$$\begin{aligned} r_0^p \left(\|AX + XB\|_2^p - 2^p \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^p \right) &\leq \|\nu AX + (1 - \nu)XB\|_2^p - \|A^\nu XB^\nu\|_2^p \\ &\leq R_0^p \left(\|AX + XB\|_2^p - 2^p \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^p \right). \end{aligned}$$

If $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite, then it is known that for any unitarily invariant norm, the function $f(\nu) = \left\| \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\| \right\|$ is convex on $[0, 1]$ and attains its minimum at $\nu = \frac{1}{2}$ (see, e.g., [2, p. 265]).

Bhatia and Davis [3] proved that if $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite, then

$$2 \left\| \|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \right\| \leq \left\| \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\| \right\| \leq \| \|AX + XB\| \|,$$

where $0 \leq \nu \leq 1$. These inequalities are known as Heinz norm inequalities.

Kittaneh [5] proved that if $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite, then

$$\begin{aligned} & 2r_0 \left(\| \|AX + XB\| \| - 2 \left\| \|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \right\| \right) \\ & \leq \| \|AX + XB\| \| - \left\| \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\| \right\|, \end{aligned} \tag{3.4}$$

where $0 \leq \nu \leq 1$ and $r_0 = \min\{\nu, 1 - \nu\}$.

In the next theorem, we obtain a reverse of the inequality (3.4) as follows.

Theorem 5. *If $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite, then*

$$\begin{aligned} & \| \|AX + XB\| \| - \left\| \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\| \right\| \\ & \leq 2R_0 \left(\| \|AX + XB\| \| - 2 \left\| \|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \right\| \right), \end{aligned} \tag{3.5}$$

where $0 \leq \nu \leq 1$ and $R_0 = \max\{\nu, 1 - \nu\}$.

Proof. If $\nu = 0$, $\nu = \frac{1}{2}$, or $\nu = 1$, then the inequality (3.5) is obviously true. Suppose that $0 < \nu < 1$, $\nu \neq \frac{1}{2}$.

If $0 < \nu < \frac{1}{2} < 1$, then based on the convexity of the function $f(\nu) = \left\| \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\| \right\|$, we have

$$\frac{f\left(\frac{1}{2}\right) - f(\nu)}{\frac{1}{2} - \nu} \leq \frac{f(1) - f(\nu)}{1 - \nu},$$

and so

$$(1 - \nu) \left(f\left(\frac{1}{2}\right) - f(\nu) \right) \leq \left(\frac{1}{2} - \nu\right) (f(1) - f(\nu)).$$

Adding $f(1)$ to both sides, gives

$$f(1) - f(\nu) \leq 2(1 - \nu) \left(f(1) - f\left(\frac{1}{2}\right) \right).$$

i.e.,

$$\begin{aligned} & \| \|AX + XB\| \| - \left\| \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\| \right\| \\ & \leq 2(1 - \nu) \left(\| \|AX + XB\| \| - 2 \left\| \|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \right\| \right). \end{aligned}$$

If $0 < \frac{1}{2} < \nu < 1$, then

$$\frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0} \leq \frac{f(\nu) - f(0)}{\nu - 0},$$

and so

$$f(0) - f(\nu) \leq 2\nu \left(f(0) - f\left(\frac{1}{2}\right) \right),$$

i.e.,

$$\begin{aligned} & \left| \|AX + XB\| - \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\| \right| \\ & \leq 2\nu \left(\|AX + XB\| - 2 \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} \right\| \right). \end{aligned}$$

Thus, from the above two norm inequalities, we get the inequality (3.5). □

The inequalities (3.4) and (3.5) can be combined so that

$$\begin{aligned} & 2r_0 \left(\|AX + XB\| - 2 \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} \right\| \right) \\ & \leq \left| \|AX + XB\| - \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\| \right| \\ & \leq 2R_0 \left(\|AX + XB\| - 2 \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} \right\| \right). \end{aligned} \tag{3.6}$$

Applying Theorem 2 to the inequalities (3.6), we have the following general result.

Corollary 3. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite. If $\phi : [0, \infty) \rightarrow \mathbb{R}$ is a strictly increasing convex function, then we have*

$$\begin{aligned} & \phi(2r_0 \|AX + XB\|) - \phi\left(4r_0 \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} \right\|\right) \\ & \leq \phi(\|AX + XB\|) - \phi\left(\|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\|\right) \\ & \leq \phi(2R_0 \|AX + XB\|) - \phi\left(4R_0 \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} \right\|\right), \end{aligned}$$

where $0 \leq \nu \leq 1$, $r_0 = \min\{\nu, 1 - \nu\}$ and $R_0 = \max\{\nu, 1 - \nu\}$.

In particular, if $\phi(x) = x^q$ ($q \in \mathbb{R}, q \geq 1$) and $\|\cdot\| = \|\cdot\|_p$ (the Schatten p -norm $p \in \mathbb{R}, p \geq 1$), we have

$$\begin{aligned} & (2r_0)^q \left(\|AX + XB\|_p^q - 2^q \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} \right\|_p^q \right) \\ & \leq \|AX + XB\|_p^q - \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\|_p^q \\ & \leq (2R_0)^q \left(\|AX + XB\|_p^q - 2^q \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} \right\|_p^q \right). \end{aligned}$$

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