Results Math 71 (2017), 1491–1498

1422-6383/17/031491-8
published online October 6, 2016
DOI 10.1007/s00025-016-0609-9

Results in Mathematics

On New Sequences Converging Towards the Ioachimescu's Constant

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Abstract. The purpose of this paper is to give some sequences that converge quickly to Ioachimescu's constant related to Ramanujan formula by multiple-correction method.

Mathematics Subject Classification. 11Y60, 11A55, 41A25.

Keywords. Ioachimescu's constant, multiple-correction method, approximation, rate of convergence.

1. Introduction

In 1895, Ioachimescu (see [1]) introduced a constant ℓ, which today bears his names, as the limit of the sequence defined by

$$I_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - 2(\sqrt{n} - 1), \quad n \in \mathbb{N}.$$

The sequence $I(n)_{n\geq 1}$ has attracted much attention lately and several generalizations have been geiven (see, e.g., [2,3]). Recently, Chen, Li and Xu [4] have obtained the complete asymptotic expansion of the Ioachimescu's sequence,

$$I_n \sim \ell + \frac{1}{2\sqrt{n}} - \sum_{k=1}^{\infty} \frac{\mathbf{b}_{2k}}{(2k)!} \frac{(4k-3)!!}{2^{2k-1} n^{2k-1/2}}, \quad n \in \mathbb{N},$$

where $\mathbf{b}_n$ denotes the n-th Bernoulli number.

We are grateful to the editor and anonymous reviewers for their valuable comments and references that helped improve the original version of this paper. The author is supported by the National Natural Science Foundation of China under Grant 61403034.

One easily obtains the following representations of the Ioachimescu's constant:

$$\ell = \int_0^\infty \frac{1 - x + \lfloor x \rfloor}{2(1+x)^{3/2}} dx$$

and

$$\ell = 2 - \sum_{k=1}^{\infty} \frac{1}{(\sqrt{k} + \sqrt{k-1})^2 \sqrt{k}}.$$

A representation of the Ioachimescu's constant has also been given by Ramanujan (1915) [27]

$$\ell = 2 - (\sqrt{2} + 1) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}.$$

From it one easily obtains a representation of the Ioachimescu's constant in terms of the extended ζ function

$$\ell = \zeta\left(\frac{1}{2}\right) + 2.$$

As comes out from [2], we have $\ell = 0.539645491\ldots$.

Let $a \in (0, +\infty)$ and $s \in (0, 1)$, The sequence

$$y_n(a,s) = \frac{1}{a^s} + \frac{1}{(a+1)^s} + \cdots \frac{1}{(a+n-1)^s} \\ - \frac{1}{1-s}[(a+n-1)^{1-s} - a^{1-s}], n \in \mathbb{N},$$

is convergent [3] and its limit is a generalized Euler constant denoted by $\ell(a, s)$. Clearly, $\ell(1, 1/2) = \ell$. Furthermore, Sîntămărian has proved that

$$\lim_{n\to\infty} n^s(y_n(a,s) - \ell(a,s)) = \frac{1}{2}.$$

Also in [3], considering the sequence

$$u_n(a,s) = y_n(a,s) - \frac{1}{2(a+n-1)^s},$$

she has proved that

$$\lim_{n\to\infty} n^{s+1}(\ell(a,s) - u_n(a,s)) = \frac{s}{12}$$

and, for the sequence

$$\alpha_n(a,s) = \frac{1}{a^s} + \frac{1}{(a+1)^s} + \cdots \frac{1}{(a+n-1)^s} \\ - \frac{1}{1-s}\left(\left(a + n - \frac{1}{2}\right)^{1-s} - a^{1-s}\right), n \in \mathbb{N},$$

she has proved that

$$\lim_{n\to\infty} n^{s+1}(\alpha_n(a,s)-\ell(a,s))=\frac{s}{24}.$$

In [9,10], Sîntămărian has obtained some new sequences that convergence to $\ell(a,s)$ with the rate of convergence n^{-s-15}. Other results regarding $\ell(a,s)$ can be found in [6–8] and some of the references therein. Recently, Mortici also proposed some sequences involving the (generalized) harmonic sum in [11–17]. Motivated by these works, in this paper we will give some sequences that converge quickly to Ioachimescu's constant ℓ by multiple-correction method [18–20], basing the sequence

$$I(n)=1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}-2(\sqrt{n}-1),\quad n\in\mathbb{N}.$$

This method could be used to solve other problems, such as Euler–Mascheroni constant, Glaisher–Kinkelin's and Bendersky–Adamchik's constants, Somos' quadratic recuurence constant, and so on [21–24].

2. Main Result

The following lemma gives a method for measuring the rate of convergence, for its proof see Mortici [25,26].

Lemma 2.1. *If the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to zero and there exists the limit*

$$\lim_{n\to+\infty} n^s(x_n-x_{n+1})=l\in[-\infty,+\infty], \tag{2.1}$$

with $s>1$, then

$$\lim_{n\to+\infty} n^{s-1}x_n=\frac{l}{s-1}. \tag{2.2}$$

Now we apply *multiple-correction method* to study faster convergence sequences for Ioachimescu's constant.

Theorem 2.2. *For Ioachimescu's constant, we have the following convergent sequence*

$$I_i(n)=\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-2\left(\sqrt{n}-1\right)+\frac{a}{\sqrt[4]{n^2+b_1n+b_0+\frac{u_1}{n+v_1+\frac{u_2}{n+v_2+\frac{u_3}{n+v_3+\frac{u_4}{n+v_4+\cdots}}}}}}, \tag{2.3}$$

where

$$a = -\frac{1}{2}, b_1 = \frac{1}{3}, b_0 = \frac{5}{72}; u_1 = -\frac{1}{108}, v_1 = -\frac{145}{192}; u_2 = \frac{14795}{12288}, v_2 = \frac{7428401}{8521920};$$
$$u_3 = \frac{4028691184}{17730254025}, v_3 = -\frac{1881480291468257}{2145761498422080};$$
$$u_4 = \frac{50269960572954497463539}{19476423187247186227200},$$
$$v_4 = \frac{3876460949905197978250864085009}{4106566008387077514657180075840}; \ldots.$$

Proof. **(Step 1) The initial-correction.** We choose $\eta_0(n) = 0$, and let

$$I_0(n) := I(n) + \eta_0(n) = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\left(\sqrt{n} - 1\right) + \eta_0(n). \tag{2.4}$$

Developing the expression (2.4) into power series expansion in $1/n$, we easily obtain

$$I_0(n) - I_0(n+1) = \frac{1}{4}\frac{1}{n^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right). \tag{2.5}$$

By Lemma 2.1, we get the rate of convergence of the $(I_0(n) - \ell)_{n\in\mathbb{N}}$ is $n^{-\frac{1}{2}}$, since

$$\lim_{n\to\infty} n^{\frac{1}{2}} \left(I_0(n) - \ell\right) = \frac{1}{2}.$$

(Step 2) The first-correction. Ramanujan [27] made the claim (without proof) for the gamma function

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{\theta_x}{30}\right)^{\frac{1}{6}},$$

where $\theta_x \to 1$ as $x \to +\infty$ and $\frac{3}{10} < \theta_x < 1$. This open problem was solved by Karatsuba [28]. This formula provides a more accurate estimation for the factorial function. Motivated by his idea, we let

$$\eta_1(n) = \frac{a}{\sqrt[4]{n^2 + b_1 n + b_0}} \tag{2.6}$$

and define

$$I_1(n) := \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\left(\sqrt{n} - 1\right) + \eta_1(n). \tag{2.7}$$

Developing (2.7) into power series expansion in $1/n$, we have

$$\begin{aligned}I_1(n)-I_1(n+1)=&\frac{1}{4}(2a+1)\frac{1}{n^{\frac{3}{2}}}-\frac{1}{8}(2+3a(1+b_1))\frac{1}{n^{\frac{5}{2}}}\\&+\frac{5}{64}(3+a(4-8b_0+6b_1+5b_1^2))\frac{1}{n^{\frac{7}{2}}}\\&-\frac{7}{256}(8+5a(1+b_1)(2-8b_0+2b_1+3b_1^2))\frac{1}{n^{\frac{9}{2}}}+O\left(\frac{1}{n^{\frac{11}{2}}}\right).\end{aligned}\tag{2.8}$$

(i) If $a\neq-\frac{1}{2}$, then the rate of convergence of the $(I_1(n)-\ell)_{n\in\mathbb{N}}$ is $n^{-\frac{1}{2}}$, since

$$\lim_{n\to\infty}n^{\frac{1}{2}}\left(I_1(n)-\ell\right)=\frac{1}{2}(2a+1)\neq 0.$$

(ii) If $a_1=-\frac{1}{2}$, $b_1=\frac{1}{3}$ and $b_0=\frac{5}{72}$, from (2.8) we have

$$I_1(n)-I_1(n+1)=\frac{7}{1728}\frac{1}{n^{\frac{9}{2}}}+O\left(\frac{1}{n^{\frac{11}{2}}}\right).$$

Hence the rate of convergence of the $(I_1(n)-\ell)_{n\in\mathbb{N}}$ is $n^{-\frac{7}{2}}$, since

$$\lim_{n\to\infty}n^{\frac{7}{2}}\left(I_1(n)-\ell\right)=\frac{1}{864}.$$

(Step 3) The second-correction. We set the second-correction function in the form of

$$\eta_2(n)=\frac{a}{\sqrt[4]{n^2+b_1n+b_0+\frac{u_1}{n+v_1}}}\tag{2.9}$$

and define

$$I_2(n):=\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-2\left(\sqrt{n}-1\right)+\eta_2(n).\tag{2.10}$$

Developing (2.10) into power series expansion in $1/n$, we have

$$\begin{aligned}I_2(n)-I_2(n+1)=&\left(\frac{7}{1728}+\frac{7u_1}{16}\right)\frac{1}{n^{\frac{9}{2}}}-\left(\frac{271}{36864}+\frac{3}{32}u_1(13+6v_1)\right)\frac{1}{n^{\frac{11}{2}}}\\&+\frac{11}{36864}(21+16u_1(523+384v_1+144v_1^2))\frac{1}{n^{\frac{13}{2}}}+O\left(\frac{1}{n^{\frac{15}{2}}}\right).\end{aligned}\tag{2.11}$$

By the same method as above, we find $u_1=-\frac{1}{108}$, $v_1=-\frac{145}{192}$.

Applying Lemma 2.1 again, one has

$$\lim_{n\to\infty}n^{\frac{13}{2}}\left(I_2(n)-I_2(n+1)\right)=-\frac{162745}{21233664},\tag{2.12}$$

$$\lim_{n\to\infty}n^{\frac{11}{2}}\left(I_2(n)-\ell\right)=-\frac{14795}{10616832}.\tag{2.13}$$

(Step 4) The third-correction. Similarly, we set the third-correction function in the form of

$$\eta_3(n) = \frac{a}{\sqrt[4]{n^2 + b_1 n + b_0 + \frac{u_1}{n+v_1+\frac{u_2}{n+v_2}}}} \tag{2.14}$$

and define

$$I_3(n) := \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\left(\sqrt{n} - 1\right) + \eta_3(n). \tag{2.15}$$

By the same method as above, we find $u_2 = \frac{14795}{12288}$, $v_2 = \frac{7428401}{8521920}$.

Applying Lemma 2.1 again, one has

$$\lim_{n\to\infty} n^{\frac{17}{2}} \left(I_3(n) - I_3(n+1)\right) = \frac{251793199}{106026319872}, \tag{2.16}$$

$$\lim_{n\to\infty} n^{\frac{15}{2}} \left(I_3(n) - \ell\right) = \frac{251793199}{795197399040}. \tag{2.17}$$

Similarly, repeat the above approach for Ioachimescu's constant (the details omitted here), we can prove Theorem 2.2.

Remark 2.3. It is worth to pointing out that Theorem 2.2 provides some sequences with faster rate of convergence for Ioachimescu's constant related to Ramanujan formula by multiply-correction method. Similarly, repeat the above approach step by step, we can get some sequences with more and more faster rate of convergence for Ioachimescu's constant. Meanwhile, parameters that need to be calculated are also greatly increased, this maybe bring some computations increase, the details omitted here.

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Received: April 25, 2016.
Accepted: September 28, 2016.