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Results in Mathematics

New Subclass of Analytic Functions in Conical Domain Associated with Ruscheweyh q-Differential Operator

Shahid Mahmood and Janusz Sokół

Abstract. The core object of this paper is to define and study a new class of analytic functions using the Ruscheweyh q-differential operator. We also investigate a number of useful properties of this class such structural formula and coefficient estimates for functions. We consider also the Fekete–Szegö problem in the class, we give some subordination results, and some other corollaries.

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1. Introduction and Definitions

Let $\mathcal{A}$ be the class of functions having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathfrak{E} = \{z \in \mathbb{C} : |z| < 1\}$. Further, we denote the class $\mathcal{S}$ of all functions in $\mathcal{A}$ which are univalent in $\mathfrak{E}$ (see [1]). Goodman [2] introduced the class $\mathcal{UCV}$ of uniformly convex functions. A function $f(z) \in \mathcal{S}$ is in the class $\mathcal{UCV}$ if for every circular arc $\xi \subset \mathfrak{E}$, with center in $\mathfrak{E}$, the arc $f(\xi)$ is convex. An interesting characterization of class $\mathcal{UCV}$ was given in [3], see also [4] as:

$$f(z) \in \mathcal{UCV} \Leftrightarrow f(z) \in \mathcal{A} \quad \text{and} \quad 1 > \left|\frac{zf''(z)}{f'(z)}\right| - \mathfrak{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \quad (z \in \mathfrak{E}).$$

In [5], see also [6], it was introduced the class k-uniformly convex functions, $k \geq 0$, denoted by $k-\mathcal{UCV}$ and the class $k-\mathcal{ST}$ related to $k-\mathcal{UCV}$ by Alexandar type relation i.e. $f(z) \in k-\mathcal{UCV} \Leftrightarrow zf'(z) \in k-\mathcal{ST}$, where

$$f(z) \in k-\mathcal{UCV} \Leftrightarrow f(z) \in \mathcal{A} \quad \text{and} \quad 1 > k\left|\frac{zf''(z)}{f'(z)}\right| - \mathfrak{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \quad (z \in \mathfrak{E}).$$

In [5,6] the geometric definitions of $k-\mathcal{UCV}$ and $k-\mathcal{ST}$ and connections with the conic domains were also considered. If $k \geq 0$, then the class $k-\mathcal{UCV}$ is defined purely geometrically as a subclass of univalent functions which map the intersection of $\mathfrak{E}$ with any disk centered at $\zeta, |\zeta| \leq k$, onto a convex domain. Therefore, the notion of k-uniform convexity is a generalization of the notion of convexity. Observe that, if $k = 0$ then the center ζ is the origin and the class $k-\mathcal{UCV}$ reduces to the class $\mathcal{C}$ of convex univalent functions, see [1]. Moreover for $k = 1$ it coincides with the class of uniformly convex functions $\mathcal{UCV}$ introduced by Goodman [2] and studied extensively by Rønning [4] and independently by Ma and Minda [3]. We note that the class $k-\mathcal{UCV}$ started much earlier in papers [7,8] with some additional conditions but without the geometric interpretation.

We say that a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^*_{k,\gamma}$, $k \geq 0$, $\gamma \in \mathbb{C}\backslash\{0\}$, if and only if

$$1 > k\left|\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right| - \mathfrak{Re}\left\{\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} \quad (z \in \mathfrak{E}).$$

Many authors investigated the properties of the class $\mathcal{S}^*_{k,\gamma}$ and their generalizations in several directions e.g. see, [4,6,9–13].

If $f(z)$ and $g(z)$ are analytic in $\mathfrak{E}$, we say that $f(z)$ is subordinate to $g(z)$, written as $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which is analytic in $\mathfrak{E}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in $\mathfrak{E}$, then we have the following equivalence, see [1,14].

$$f(z) \prec g(z) \quad (z \in \mathfrak{E}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathfrak{E}) \subset g(\mathfrak{E}).$$

For two analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in \mathfrak{E}),$$

the convolution (Hadamard product) of $f(z)$ and $g(z)$ is defined as

$$f(z) * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

For $t \in \mathbb{R}$ and $q > 0$, $q \neq 1$, the number $[t, q]$ is defined in [15] as

$$[t, q] = \frac{1 - q^t}{1 - q}, \quad [0, q] = 0.$$

For any non-negative integer n the q-number shift factorial is defined by

$$[n,q]! = [1,q]\,[2,q]\,[3,q]\cdots[n,q]\,, \quad ([0,q]! = 1)\,.$$

We have $\lim_{q\to 1}[n,q] = n$. Throughout in this paper we will assume q to be fixed number between 0 and 1.
The q-derivative operator or q-difference operator for $f \in \mathcal{A}$ is defined as

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad z \in \mathfrak{E}.$$

It can easily be seen that for $n \in \mathbb{N} := \{1,2,3,\ldots\}$ and $z \in \mathfrak{E}$

$$\partial_q z^n = [n,q]\,z^{n-1}, \quad \partial_q\left\{\sum_{n=1}^{\infty} a_n z^n\right\} = \sum_{n=1}^{\infty}[n,q]\,a_n z^{n-1}.$$

The q-generalized Pochhammer symbol for $t \in \mathbb{R}$ and $n \in \mathbb{N}$ is defined as

$$[t,q]_n = [t,q]\,[t+1,q]\,[t+2,q]\cdots[t+n-1,q]\,,$$

and for $t > 0$, let q-gamma function is defined as

$$\Gamma_q(t+1) = [t,q]\,\Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1.$$

Definition 1.1 [15]. For a function $f(z) \in \mathcal{A}$, the Ruscheweyh q-differential operator is defined as

$$L_q^{\lambda} f(z) = \phi(q,\lambda+1;z) * f(z) = z + \sum_{n=2}^{\infty}\psi_{n-1}a_n z^n, \quad (z \in \mathfrak{E} \text{ and } \lambda > -1)\,, \tag{1.2}$$

where

$$\phi(q,\lambda+1;z) = z + \sum_{n=2}^{\infty}\psi_{n-1}z^n, \tag{1.3}$$

and

$$\psi_{n-1} = \frac{\Gamma_q(\lambda+n)}{[n-1,q]!\,\Gamma_q(\lambda+1)} = \frac{[\lambda+1,q]_{n-1}}{[n-1,q]!}. \tag{1.4}$$

From (1.2), it can be seen that

$$L_q^0 f(z) = f(z) \quad \text{and} \quad L_q^1 f(z) = z\partial_q f(z),$$

and

$$L_q^m f(z) = \frac{z\partial_q^m\left(z^{m-1}f(z)\right)}{[m,q]!}, \quad (m \in \mathbb{N})\,.$$

$$\lim_{q\to 1^-}\phi(q,\lambda+1;z) = \frac{z}{(1-z)^{\lambda+1}},$$

and

$$\lim_{q\to 1^-} L_q^{\lambda} f(z) = f(z) * \frac{z}{(1-z)^{\lambda+1}}.$$

This shows that in case of $q \to 1^-$, the Ruscheweyh q-differential operator reduces to the Ruscheweyh differential operator $D^\delta(f(z))$ (see [16]). From (1.2) the following identity can easily be derived.

$$z\partial L_q^\lambda f(z) = \left(1 + \frac{[\lambda, q]}{q^\lambda}\right) L_q^{\lambda+1} f(z) - \frac{[\lambda, q]}{q^\lambda} L_q^\lambda f(z). \tag{1.5}$$

If $q \to 1^-$, then

$$z\left(L^\lambda f(z)\right)' = (1+\lambda)\, L^{\lambda+1} f(z) - \lambda L^\lambda f(z).$$

Now using the Ruscheweyh q-differential operator, we define the following class.

Definition 1.2. Let $f(z) \in \mathcal{A}$. Then $f(z)$ is in the class $k - \mathcal{UST}_q^\lambda(\gamma)$, $\gamma \in \mathbb{C}\backslash\{0\}$, if it satisfies the condition

$$\mathfrak{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{z\partial_q L_q^\lambda f(z)}{L_q^\lambda f(z)} - 1\right)\right\} > \left|\frac{1}{\gamma}\left(\frac{z\partial_q L_q^\lambda f(z)}{L_q^\lambda f(z)} - 1\right)\right| \quad (z \in \mathfrak{E}).$$

Geometric Interpretation

A function $f(z) \in \mathcal{A}$ is in the class $k - \mathcal{UST}_q^\lambda(\gamma)$ if and only if $\frac{z\partial_q L_q^\lambda f(z)}{L_q^\lambda f(z)}$ takes all the values in the conic domain $\Omega_{k,\gamma} = p_{k,\gamma}(\mathfrak{E})$ such that

$$\Omega_{k,\gamma} = \gamma\Omega_k + (1-\gamma),$$

where

$$\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\},$$

or equivalently,

$$\frac{z\partial_q L_q^\lambda f(z)}{L_q^\lambda f(z)} \prec p_{k,\gamma}(z), \quad \Omega_{k,\gamma} = p_{k,\gamma}(\mathfrak{E}). \tag{1.6}$$

The boundary $\partial\Omega_{k,\gamma}$ of the above set becomes the imaginary axis when $k = 0$ while a hyperbola when $0 < k < 1$. In this case $0 \le k < 1$, we have

$$p_{k,\gamma}(z) = 1 + \frac{2\gamma}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\ \operatorname{arc\,tanh}\sqrt{z}\right\}, \quad (z \in \mathfrak{E}).$$

For $k = 1$ the boundary $\partial\Omega_{k,\gamma}$ becomes a parabola and

$$p_{1,\gamma}(z) = 1 + \frac{2\gamma}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad (z \in \mathfrak{E}).$$

It is an ellipse when $k > 1$ and in this case

$$p_{k,\gamma}(z) = 1 + \frac{\gamma}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{u(z)/\sqrt{t}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\mathrm{d}x\right) + \frac{\gamma}{1-k^2},$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $z \in \mathfrak{E}$ and $t \in (0,1)$ is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{R(t)}\right)$, $R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$, see [5,6,17]. Moreover, $p_{k,\gamma}(\mathfrak{E})$ is convex univalent in $\mathfrak{E}$, see [5,6]. All of these curves have the vertex at the point $(k+\gamma)/(k+1)$. Therefore the domain $\Omega_{k,\gamma}$ is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic for $k = 1$ and right half plane when $k = 0$; ever symmetric with respect to the real axis. Because $p_{k,\gamma}(\mathfrak{E}) = \Omega_{k,\gamma}$, the functions $p_{k,\gamma}$, play the role of extremal functions for several problems for the class $k-\mathcal{UST}_q^{\lambda}(\gamma)$.

2. Preliminary Results

Lemma 2.1 [18]. *Let $p(z) = \sum_{n=1}^{\infty} p_n z^n \prec F(z) = \sum_{n=1}^{\infty} d_n z^n$ in $\mathfrak{E}$. If $F(z)$ is convex univalent in $\mathfrak{E}$ then*

$$|p_n| \leq |d_1|, \quad n \geq 1.$$

Lemma 2.2 [19]. *Let $k \in [0,\infty)$ be fixed and let $p_{k,\gamma}$ be defined as above. If*

$$p_{k,\gamma}(z) = 1 + Q_1 z + Q_2 z^2 + \cdots,$$

then

$$Q_1 = \begin{cases} \frac{2\gamma A^2}{1-k^2} & 0 \leq k < 1, \\ \frac{8\gamma}{\pi^2} & k = 1, \\ \frac{\pi^2 \gamma}{4\sqrt{t}(k^2-1)R^2(t)(1+t)} & k > 1 \end{cases} \tag{2.1}$$

and

$$Q_2 = \begin{cases} \frac{\left(A^2+2\right)}{3} Q_1 & 0 \leq k < 1, \\ \frac{2}{3} Q_1 & k = 1, \\ \frac{4R^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}R^2(t)(1+t)} Q_1 & k > 1, \end{cases} \tag{2.2}$$

where

$$A = \frac{2\cos^{-1} k}{\pi},$$

and $t \in (0,1)$ is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{R(t)}\right)$, $R(t)$ is the Legendre's complete elliptic integral of the first kind.

Lemma 2.3 [20]. *Let $h(z) = 1 + \sum_{k=1}^{\infty} c_n z^n \in \mathcal{P}$, i.e., let $h(z)$ be analytic in $\mathfrak{E}$ and satisfy $\mathfrak{Re}\{h(z)\} > 0$ for z in $\mathfrak{E}$, then the following sharp estimate holds*

$$\left|c_2 - \nu c_1^2\right| \leq 2\max\{1, |2\nu - 1|\} \quad \textit{for all } \nu \in \mathbb{C}.$$

3. Main Results

Theorem 3.1. *Let* $f(z) \in k - \mathcal{UST}_q^{\lambda}(\gamma)$. *Then*

$$L_q^{\lambda} f(z) \prec z \exp \int_0^z \frac{p_{k,\gamma}(w(\xi)) - 1}{\xi} \mathrm{d}\xi, \tag{3.1}$$

where $w(z)$ *is analytic in* $\mathfrak{E}$ *with* $w(0) = 0$ *and* $|w(z)| < 1$. *Moreover, for* $|z| = \rho$, *we have*

$$\exp\left(\int_0^1 \frac{p_{k,\gamma}(-\rho) - 1}{\rho} \mathrm{d}\rho\right) \leq \left|\frac{L_q^{\lambda} f(z)}{z}\right| \leq \exp\left(\int_0^1 \frac{p_{k,\gamma}(\rho) - 1}{\rho} \mathrm{d}\rho\right),$$

where $p_{k,\gamma}(z)$ *is defined below* (1.6).

Proof. If $f(z) \in k - \mathcal{UL}_q^{\lambda}(\gamma)$, then using the identity (1.6), we obtain

$$\frac{\partial L_q^{\lambda} f(z)}{L_q^{\lambda} f(z)} - \frac{1}{z} = \frac{p_{k,\gamma}(w(z)) - 1}{z}, \tag{3.2}$$

for some function $w(z)$, analytic in $\mathfrak{E}$ with $w(0) = 0$ and $|w(z)| < 1$. Integrating (3.2), we have

$$L_q^{\lambda} f(z) \prec z \exp \int_0^z \frac{p_{k,\gamma}(w(\xi)) - 1}{\xi} \mathrm{d}\xi. \tag{3.3}$$

This proves (3.1). Noting that the univalent function $p_{k,\gamma}(z)$ maps the disk $|z| < \rho$ $(0 < \rho \leq 1)$ onto a region which is convex and symmetric with respect to the real axis, we see

$$\frac{k+\gamma}{k+1} < p_{k,\gamma}(-\rho|z|) \leq \mathfrak{Re}\{p_{k,\gamma}(w(\rho z))\} \leq p_{k,\gamma}(\rho|z|) \quad (0 < \rho \leq 1,\ z \in \mathfrak{E}). \tag{3.4}$$

Using (3.4), gives

$$\int_0^1 \frac{p_{k,\gamma}(-\rho|z|) - 1}{\rho} \mathrm{d}\rho \leq \mathfrak{Re} \int_0^1 \frac{p_{k,\gamma}(w(\rho z)) - 1}{\rho} \mathrm{d}\rho \leq \int_0^1 \frac{p_{k,\gamma}(\rho|z|) - 1}{\rho} \mathrm{d}\rho,$$

for $z \in \mathfrak{E}$. Consequently, the subordination (3.3) leads us to

$$\int_0^1 \frac{p_{k,\gamma}(-\rho|z|) - 1}{\rho} \mathrm{d}\rho \leq \log\left|\frac{L_q^{\lambda} f(z)}{z}\right| \leq \int_0^1 \frac{p_{k,\gamma}(\rho|z|) - 1}{\rho} \mathrm{d}\rho,$$

$p_{k,\gamma}(-\rho) \leq p_{k,\gamma}(-\rho|z|)$, $p_{k,\gamma}(\rho|z|) \leq p_{k,\gamma}(\rho)$ implies that

$$\exp\left(\int_0^1 \frac{p_{k,\gamma}(-\rho) - 1}{\rho} \mathrm{d}\rho\right) \leq \left|\frac{L_q^{\lambda} f(z)}{z}\right| \leq \exp\left(\int_0^1 \frac{p_{k,\gamma}(\rho) - 1}{\rho} \mathrm{d}\rho\right).$$

This completes the proof. □

Theorem 3.2. *If $f(z) \in k - \mathcal{UST}_q^{\lambda}(\gamma)$, then*

$$|a_2| \leq \frac{\sigma}{\psi_1}, \quad |a_n| \leq \frac{\sigma}{[n-1,q]!\psi_{n-1}} \prod_{j=1}^{n-2} \left(1 + \frac{\sigma}{[j,q]}\right), \quad (n = 3,4,\ldots), \tag{3.5}$$

where ψ_{n-1} is defined by (1.4) and $\sigma = |Q_1|/q$ with Q_1 is given by (2.1).

Proof. Let

$$\frac{z\partial_q L_q^{\lambda} f(z)}{L_q^{\lambda} f(z)} = p(z),$$

where $p(z)$ is analytic in $\mathfrak{E}$. This can be written as

$$z\partial_q L_q^{\lambda} f(z) = L_q^{\lambda} f(z) p(z). \tag{3.6}$$

Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and $L_q^{\lambda} f(z)$ is given by (1.2). Then (3.6) becomes

$$z + \sum_{n=2}^{\infty} [n,q]\, \psi_{n-1} a_n z^n = \left(\sum_{n=0}^{\infty} p_n z^n\right)\left(z + \sum_{n=2}^{\infty} \psi_{n-1} z^n\right).$$

Now comparing the coefficients of z^n, we obtain

$$[n,q]\, \psi_{n-1} a_n = \psi_{n-1} a_n + \sum_{j=1}^{n-1} \psi_{j-1} a_j p_{n-j},$$

or

$$a_n = \frac{1}{[n-1,q]\, q\psi_{n-1}} \sum_{j=1}^{n-1} \psi_{j-1} a_j p_{n-j}.$$

Using the result that $|p_n| \leq |Q_1|$ given in [17], we have

$$|a_n| \leq \frac{|Q_1|}{[n-1,q]\, q\psi_{n-1}} \sum_{j=1}^{n-1} \psi_{j-1} |a_j|.$$

Let us take $\sigma = \frac{|Q_1|}{q}$. Then, we have

$$|a_n| \leq \frac{\sigma}{[n-1,q]\, \psi_{n-1}} \sum_{j=1}^{n-1} \psi_{j-1} |a_j|. \tag{3.7}$$

So for $n = 2$, we have from (3.7)

$$|a_2| \leq \frac{\sigma}{\psi_1}, \tag{3.8}$$

which shows that (3.5) holds for $n = 2$. To prove (3.5) we apply mathematical induction. For $n = 3$, we have from (3.7)

$$|a_3| \leq \frac{\sigma}{[2,q]\, \psi_2} \{1 + \psi_1 |a_2|\},$$

using (3.8), we have

$$|a_3| \leq \frac{\sigma}{[2,q]\,\psi_2(a,c)}(1+\sigma) = \frac{\sigma([1,q]+\sigma)}{[2,q]\,\psi_2(a,c)},$$

which shows that (3.5) holds for $n=3$. Let us assume that (3.5) is true for $n \leq t$, that is,

$$|a_t| \leq \frac{\sigma}{[t-1,q]!\psi_{t-1}} \prod_{j=1}^{t-2}\left(1+\frac{\sigma}{[j,q]}\right).$$

Consider

$$\begin{aligned} |a_{t+1}| &\leq \frac{\sigma}{[t,q]\,\psi_t}\left[1+\psi_1|a_2|+\psi_2|a_3|+\cdots+\psi_{t-1}|a_t|,q\right] \\ &\leq \frac{\sigma}{[t,q]\,\psi_t}\left[1+\sigma+\sigma\left(1+\frac{\sigma}{[1,q]}\right)+\sigma\left(1+\frac{\sigma}{[1,q]}\right)\left(1+\frac{\sigma}{[2,q]}\right)\right. \\ &\qquad \left. +\cdots+\sigma\prod_{j=1}^{t-2}\left(1+\frac{\sigma}{[j,q]}\right),q\right] \\ &= \frac{\sigma}{[t,q]\psi_t}\prod_{j=1}^{t-1}\left(1+\frac{\sigma}{[j,q]}\right). \end{aligned}$$

Therefore, the result is true for $n=t+1$. Consequently, using mathematical induction, we have proved that (3.5) holds true for all n, $n \geq 2$. This completes the proof. □

Theorem 3.3. *Let $f(z) \in k-\mathcal{UST}_q^{\lambda}(\gamma)$. Then $f(\mathfrak{E})$ contains an open disk of radius*

$$\frac{q[\lambda+1,q]}{2q[\lambda+1,q]+|Q_1|},$$

where Q_1 is defined by (2.1).

Proof. Let $\omega_0 \neq 0$ be a complex number such that $f(z) \neq \omega_0$ for $z \in \mathfrak{E}$. Then

$$f_1(z) = \frac{\omega_0 f(z)}{\omega_0 - f(z)} = z + \left(a_2 + \frac{1}{\omega_0}\right)z^2 + \cdots.$$

Since $f_1(z)$ is univalent, so

$$\left|a_2 + \frac{1}{\omega_0}\right| \leq 2.$$

Now using Theorem 3.2, we have

$$\left|\frac{1}{\omega_0}\right| \leq 2 + \frac{|Q_1|}{q[\lambda+1,q]},$$

and hence

$$|\omega_0| \geq \frac{q[\lambda+1,q]}{2q[\lambda+1,q]+|Q_1|}.$$

□

Theorem 3.4. *Let* $0 \leq k < \infty$ *be fixed and let* $f(z) \in k - \mathcal{UST}_q^{\lambda}(\gamma)$ *with the form* (1.1). *Then for a complex number* μ

$$\left|a_3 - \mu a_2^2\right| \leq \frac{Q_1}{q\left[\lambda+1,q\right]\left[\lambda+2,q\right]} \max\left\{1, \left|2\nu - 1\right|\right\},$$

where

$$\nu = \frac{1}{2}\left\{1 - \frac{qQ_2}{Q_1} - \frac{Q_1}{2q^2\left[\lambda+1,q\right]} + \frac{\mu\left[\lambda+2,q\right]Q_1}{q\left[\lambda+1,q\right]}\right\}, \tag{3.9}$$

Q_1 *and* Q_2 *are given by* (2.1) *and* (2.2).

Proof. If $f(z) \in k - \mathcal{UST}_q^{\lambda}(\gamma)$, then there exist a Schwarz function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{z\partial L_q^{\lambda} f(z)}{L_q^{\lambda} f(z)} = p_{k,\gamma}(w(z)) \quad (z \in \mathfrak{E}). \tag{3.10}$$

Let $h(z) \in \mathcal{P}$ be a function defined as

$$h(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathfrak{E}).$$

This gives

$$w(z) = \frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots,$$

and

$$p_{k,\gamma}(w(z)) = 1 + \frac{Q_1 c_1}{2}z + \left\{\frac{c_1^2 Q_2}{4} + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)Q_1\right\}z^2 + \cdots. \tag{3.11}$$

Using (3.11) in (3.10) along with (1.2) , we obtain

$$a_2 = \frac{Q_1 c_1}{2q\left[\lambda+1,q\right]},$$

and

$$a_3 = \frac{1}{q\left[\lambda+1,q\right]\left[\lambda+2,q\right]} \times \left\{q\left\{\frac{c_1^2 Q_2}{4} + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)Q_1\right\} + \frac{Q_1^2 c_1^2}{4q^2\left[\lambda+1,q\right]^2}\right\}.$$

For any complex number μ, we have

$$a_3 - \mu a_2^2 = \frac{1}{q\left[\lambda+1,q\right]\left[\lambda+2,q\right]} \times \left\{q\left\{\frac{c_1^2 Q_2}{4} + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)Q_1\right\} + \frac{Q_1^2 c_1^2}{4q^2\left[\lambda+1,q\right]^2}\right\} - \mu\frac{Q_1^2 c_1^2}{4q^2\left[\lambda+1,q\right]^2}. \tag{3.12}$$

Then (3.12) can be written as

$$a_3 - \mu a_2^2 = \frac{Q_1}{2q\,[\lambda+1,q]\,[\lambda+2,q]}\left\{c_2 - \nu c_1^2\right\},$$

where ν is defined by (3.9). Now, taking absolute value on both sides and using Lemma 2.3, we obtain the required result. □

Theorem 3.5. *If a function $f(z) \in \mathcal{A}$ has the form (1.1) and it satisfies*

$$\sum_{n=2}^{\infty}\left\{q\,[n-1,q]\,(k+1) + |\gamma|\right\}|\psi_{n-1}|\,|a_n| \le |\gamma|, \tag{3.13}$$

then $f(z) \in k - \mathcal{UST}_q^{\lambda}(\gamma)$.

Proof. Let we note that

$$\begin{aligned}\left|\frac{\partial L_q^{\lambda} f(z)}{L_q^{\lambda} f(z)} - 1\right| &= \left|\frac{\partial L_q^{\lambda} f(z) - L_q^{\lambda} f(z)}{L_q^{\lambda} f(z)}\right| \\ &= \left|\frac{\sum_{n=2}^{\infty} q\,[n-1,q]\,\psi_{n-1} a_n z^n}{z + \sum_{n=2}^{\infty}\psi_{n-1} a_n z^n}\right| \\ &\le \frac{\sum_{n=2}^{\infty}|q\,[n-1,q]\,\psi_{n-1}(a,c)|\,|a_n|}{1 - \sum_{n=2}^{\infty}|\psi_{n-1}(a,c)|\,|a_n|}\end{aligned} \tag{3.14}$$

because from (3.13) it follows that

$$1 - \sum_{n=2}^{\infty}|\psi_{n-1}(a,c)|\,|a_n| > 0.$$

To show that $f(z) \in k - \mathcal{UST}_q^{\lambda}(\gamma)$ it suffices that

$$\left|\frac{k}{\gamma}\left(\frac{\partial L_q^{\lambda} f(z)}{L_q^{\lambda} f(z)} - 1\right)\right| - \mathfrak{Re}\left\{\frac{1}{\gamma}\left(\frac{\partial L_q^{\lambda} f(z)}{L_q^{\lambda} f(z)} - 1\right)\right\} \le 1.$$

From (3.14), we have

$$\begin{aligned}
&\left|\frac{k}{\gamma}\left(\frac{\partial L_q^\lambda f(z)}{L_q^\lambda f(z)}-1\right)\right| - \mathfrak{Re}\left\{\frac{1}{\gamma}\left(\frac{\partial L_q^\lambda f(z)}{L_q^\lambda f(z)}-1\right)\right\} \\
&\quad\leq \frac{k}{|\gamma|}\left|\frac{\partial L_q^\lambda f(z)}{L_q^\lambda f(z)}-1\right| + \frac{1}{|\gamma|}\left|\frac{\partial L_q^\lambda f(z)}{L_q^\lambda f(z)}-1\right| \\
&\quad\leq \frac{(k+1)}{|\gamma|}\left|\frac{\partial L_q^\lambda f(z)}{L_q^\lambda f(z)}-1\right| \\
&\quad\leq \frac{(k+1)}{|\gamma|}\frac{\sum_{n=2}^{\infty}|q\,[n-1,q]\,\psi_{n-1}|\,|a_n|}{1-\sum_{n=2}^{\infty}|\psi_{n-1}|\,|a_n|} \\
&\quad\leq 1
\end{aligned}$$

because of (3.13). □

Theorem 3.6. *If $L_q^\lambda f(z) \neq 0$ in $\mathfrak{E}$, and if*

$$\left(1+\frac{[\lambda,q]}{q^\lambda}\right)\frac{L^{\lambda+1}f(z)}{L_q^\lambda f(z)} - \frac{[\lambda,q]}{q^\lambda} \prec p_{k,\gamma}(z),$$

then $f \in k-\mathcal{UST}_q^\lambda(\gamma)$.

Proof. Because $L_q^\lambda f(z) \neq 0$ in $\mathfrak{E}$ we can define the function $p(z)$ by

$$\frac{z\partial L_q^\lambda f(z)}{L_q^\lambda f(z)} = p(z) \quad (z \in \mathfrak{E}).$$

From (1.5), we have

$$\left(1+\frac{[\lambda,q]}{q^\lambda}\right)\frac{L^{\lambda+1}f(z)}{L_q^\lambda f(z)} - \frac{[\lambda,q]}{q^\lambda} = p(z).$$

Therefore, $p(z) \prec p_{k,\gamma}(z)$, now (1.6) shows that $f \in k-\mathcal{UST}_q^\lambda(\gamma)$. □

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Shahid Mahmood
Department of Mechanical Engineering
Sarhad University of Science
I. T Landi Akhun Ahmad
Hayatabad Link
Ring Road
Peshawar
Pakistan
e-mail: shahidmahmood757@gmail.com

Janusz Sokół
Department of Mathematics
Institute of Mathematics
University of Rzeszów
ul. Rejtana 16A
35-310, Rzeszow
Poland
e-mail: jsokol@prz.edu.pl

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