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Results in Mathematics

On Recovering the Dirac Operator with an Integral Delay from the Spectrum

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Abstract. An inverse spectral problem for the Dirac operator with an integral delay is studied. We show, that the considered operator can be uniquely recovered from one spectrum, provide a constructive procedure for the solution of the inverse problem, and obtain necessary and sufficient conditions for its solvability.

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1. Introduction

Inverse spectral problems consist in recovering operators from given their spectral characteristics. The greatest success in the inverse spectral theory has been achieved for the Sturm-Liouville and Dirac differential operators (see, e.g., [1–4] and the references therein) and afterwards for higher-order differential operators and differential systems with an arbitrary location of roots of the characteristic polynomial (see [4–7]). For integro-differential, integral and other classes of non-local operators the classical methods (transformation operator method [1–3] and the method of spectral mappings [3–6]) do not work, and for such operators the general inverse spectral theory does not exist, but there are some particular results in this direction (see [8–17] and the references therein). At the same time, nonlocal and, in particular, integro-differential operators are of great interest, because they have many applications in natural sciences and engineering (see, e.g., [18]). In [8,10,13,15,17] various aspects

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were studied of the inverse problem for a Volterra convolution perturbation of the Sturm-Liouville operator. In [12,13,17] the inverse problems were reduced to the main nonlinear convolutional integral equations, which were solved globally. This allowed obtaining the global solution of the inverse problems. In [15] this approach was expanded for the situation involving a more general nonlinear integral equation. In the present paper we develop this approach for an integro-differential Dirac operator.

Consider the boundary value problem $D = D(p, q)$ for the integro-differential Dirac system

$$By' + \int_0^x M(x-t)y(t)\,dy = \lambda y, \quad 0 < x < \pi, \tag{1}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M(x) = \begin{pmatrix} p(x) & q(x) \\ -q(x) & p(x) \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

the functions $p(x)$, $q(x)$ are complex-valued, $(\pi - x)p(x)$, $(\pi - x)q(x) \in L_2(0, \pi)$ and λ is the spectral parameter; along with the boundary conditions

$$y_1(0) = y_1(\pi) = 0. \tag{2}$$

We study the following inverse problem.

Inverse Problem 1. Given the spectrum of D, construct the functions $p(x)$ and $q(x)$.

We note that inverse problems for systems of integro-differential equations have not been studied before. In the next section we reduce Inverse Problem 1 to a system of nonlinear integral equations, and prove the global solvability of this system. In Sect. 3 using this result we prove the uniqueness of the solution of this inverse problem and obtain necessary and sufficient conditions of its solvability (Theorems 3, 4). The proof is constructive and gives an algorithm for solving the inverse problem (Algorithm 1).

2. System of Nonlinear Main Equations

Consider the solution

$$S(x, \lambda) = \begin{pmatrix} S_1(x, \lambda) \\ S_2(x, \lambda) \end{pmatrix}, \quad S(0, \lambda) = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

of Eq. (1). One can show that there exists the transformation operator, i.e.

$$S(x, \lambda) = S_0(x, \lambda) + \int_0^x K(x, \xi) S_0(\xi, \lambda)\,d\xi, \quad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix},$$
$$S_0(x, \lambda) = \begin{pmatrix} \sin \lambda x \\ -\cos \lambda x \end{pmatrix}, \tag{3}$$

where the elements of the kernel $K(x,\xi)$ satisfy the integral equations

$$K_{11}(x,\xi) = -\xi q(x-\xi) - \int_0^{x-\xi} ds \int_{x-\xi}^{x} \Big(q(s)K_{11}(t-s,\xi-x+t) + p(s)K_{12}(t-s,\xi-x+t)\Big)\, dt,$$

$$K_{12}(x,\xi) = \xi p(x-\xi) + \int_0^{x-\xi} ds \int_{x-\xi}^{x} \Big(p(s)K_{11}(t-s,\xi-x+t) - q(s)K_{12}(t-s,\xi-x+t)\Big)\, dt,$$

$$K_{11}(x,\xi) = K_{22}(x,\xi), \quad K_{12}(x,\xi) = -K_{21}(x,\xi).$$

Solving them by the method of successive approximations, we obtain

$$\begin{cases} K_{11}(x,\xi) = -\xi q(x-\xi) + \sum_{n=2}^{\infty}\sum_{j=0}^{n} \dfrac{a_{nj}\xi^n}{n!}\left(p^{*j} * q^{*(n-j)}\right)(x-\xi), \\ K_{12}(x,\xi) = \xi p(x-\xi) + \sum_{n=2}^{\infty}\sum_{j=0}^{n} \dfrac{b_{nj}\xi^n}{n!}\left(p^{*j} * q^{*(n-j)}\right)(x-\xi), \end{cases} \tag{4}$$

where $(f^{*0} * g)(x) = (g * f^{*0})(x) = g(x)$,

$$(f*g)(x) = \int_0^x f(t)g(x-t)\,dt, \quad f^{*1}(x) = f(x), \quad f^{*(\nu+1)}(x) = (f^{*\nu} * f)(x),\ \nu \ge 1,$$

$$a_{nj} = -a_{n-1,j} - b_{n-1,j-1}, \quad b_{nj} = a_{n-1,j-1} - b_{n-1,j}, \quad n \in \mathbb{N}\backslash\{1\},\ j = \overline{0,n},$$

$$a_{11} = b_{10} = 0, \quad a_{10} = -b_{11} = -1, \quad a_{nj} = b_{nj} = 0, \quad j<0 \text{ or } j>n.$$

One can show by induction, that

$$\sum_{j=0}^{n} |a_{nj}| \le 2^{n-1}, \quad \sum_{j=0}^{n} |b_{nj}| \le 2^{n-1}, \quad n \in \mathbb{N}. \tag{5}$$

The eigenvalues of D coincide with the zeros of its characteristic function $\Delta(\lambda) := S_1(\pi,\lambda)$. According to (3) we have

$$\Delta(\lambda) = \sin\lambda\pi + \int_0^{\pi} \Big(w_1(\xi)\sin\lambda\xi + w_2(\xi)\cos\lambda\xi\Big)\, d\xi, \quad w_1(\xi), w_2(\xi) \in L_2(0,\pi). \tag{6}$$

In (6) we have $w_1(\xi) = K_{11}(\pi,\xi)$, $w_2(\xi) = -K_{12}(\pi,\xi)$. It follows from (4), that

$$\begin{cases} -w_1(\pi-\xi) = (\pi-\xi)q(\xi) - \sum_{n=2}^{\infty}\sum_{j=0}^{n} \dfrac{a_{nj}(\pi-\xi)^n}{n!}\left(p^{*j} * q^{*(n-j)}\right)(\xi), \\ -w_2(\pi-\xi) = (\pi-\xi)p(\xi) + \sum_{n=2}^{\infty}\sum_{j=0}^{n} \dfrac{b_{nj}(\pi-\xi)^n}{n!}\left(p^{*j} * q^{*(n-j)}\right)(\xi). \end{cases} \tag{7}$$

We consider the relations (7) as a system of nonlinear equations with respect to p and q, and use them for solving the inverse problem.

Theorem 1. *For any functions $w_1(x)$, $w_2(x) \in L_2(0,\pi)$ the system (7) has a unique solution $p(x)$, $q(x)$, such that $(\pi-x)p(x)$, $(\pi-x)q(x) \in L_2(0,\pi)$.*

The proof of Theorem 1 is based on the following general result.

Theorem 2. *Consider the system of nonlinear equations*

$$w_k(x) = y_k(x) + \sum_{n=2}^{\infty}\sum_{j=0}^{m_n}\left(\psi_{knj}(x)Q_{nj}[y](x) + \int_0^x \Psi_{knj}(x,t)Q_{nj}[y](t)\,dt\right),$$
$$k=\overline{1,N},\ x\in(0,T), \tag{8}$$

where $Q_{nj}[y]$, $j=\overline{0,m_n}$, *are all possible terms of the form*

$$y_1^{*j_{n1}} * y_2^{*j_{n2}} * \cdots * y_N^{*j_{nN}}, \quad \sum_{\nu=1}^{N} j_{n\nu} = n,$$

the functions $\psi_{knj}(x)$ *and* $\Psi_{knj}(x,t)$ *are square integrable, and there exist square integrable functions* $u(x)$, $U(x,t)$, *such that*

$$\sum_{j=0}^{m_n}|\psi_{knj}(x)| \le u(x), \quad \sum_{j=0}^{m_n}|\Psi_{knj}(x,t)| \le U(x,t), \quad k=\overline{1,N}, \quad n\in\mathbb{N}\backslash\{1\}. \tag{9}$$

For any functions $w_k(x)\in L_2(0,T)$, $k=\overline{1,N}$, *the system* (8) *has a unique solution* $y_k(x)\in L_2(0,T)$, $k=\overline{1,N}$.

Theorem 2 can be proved similarly to Theorem 5 from [12].

Proof of Theorem 1. The system (7) on an interval $(0,T)$, $0<T<\pi$, takes the form (8) with

$$N=2, \quad m_n=n, \quad \psi_{1nj}(\xi) = -\frac{a_{nj}(\pi-\xi)^{n-1}}{n!},$$
$$\psi_{2nj}(\xi) = \frac{b_{nj}(\pi-\xi)^{n-1}}{n!}, \quad \Psi_{knj}=0.$$

The estimates (9) follow from (5) and the relation

$$\frac{(2(\pi-\xi))^{n-1}}{n!} \le \exp(2(\pi-\xi)).$$

Thus, by Theorem 2 the system (7) on $(0,\pi)$ has a unique solution $p(x)$, $q(x)\in L_2(0,T)$ for each $T\in(0,\pi)$. Let $p(x)=p_1(x)+p_2(x)$, $q(x)=q_1(x)+q_2(x)$, where p_1, $q_1\in L_2(0,\pi)$ and $p_2(x)=q_2(x)=0$ on $(0,\pi/2)$. Then

$$(p_1+p_2)^{*j} * (q_1+q_2)^{*(n-j)}$$
$$= p_1^{*j} * q_1^{*(n-j)} + jp_1^{*(j-1)} * q_1^{*(n-j)} * p_2 + (n-j)p_1^{*j} * q_1^{*(n-j-1)} * q_2,$$

and for $\pi/2<x<\pi$ the system (7) takes the form

$$\mu(\xi) = y(\xi) + \int_0^{\xi} H(\xi,\tau)y(\tau)\,d\tau, \tag{10}$$

where

$$\mu(\xi) = \begin{pmatrix} \mu_1(\xi) \\ \mu_2(\xi) \end{pmatrix}, \quad y(\xi) = (\pi - \xi) \begin{pmatrix} q_2(\xi) \\ p_2(\xi) \end{pmatrix}, \quad H(\xi, \tau) = \begin{pmatrix} H_{11}(\xi, \tau) & H_{12}(\xi, \tau) \\ H_{21}(\xi, \tau) & H_{22}(\xi, \tau) \end{pmatrix},$$

$$\mu_k(\xi) = -w_k(\pi - \xi) - \sum_{n=2}^{\infty} \sum_{j=0}^{n} \frac{c_{knj}(\pi - \xi)^n}{n!} (p_1^{*j} * q_1^{*(n-j)})(\xi) \in L_2(0, \pi),$$
$$k = 1, 2,$$

$$H_{kl}(\xi, \tau) = \frac{\pi - \xi}{\pi - \tau} \sum_{n=0}^{\infty} \sum_{j=1}^{n} \frac{c_{klnj}(\pi - \xi)^n}{n!} (p_1^{*j} * q_1^{*(n-j)})(\xi - \tau) \in L_2\Big((0, \pi) \times (0, \pi)\Big),$$
$$k, l = 1, 2,$$

and c_{knj}, c_{klnj} are some constant coefficients. The Volterra integral Eq. (10) is uniquely solvable in $L_2(0, \pi) \times L_2(0, \pi)$. Hence $(\pi - \xi)p(\xi)$, $(\pi - \xi)q(\xi) \in L_2(0, \pi)$. □

3. Solution of the Inverse Problem

Using the representation (6) by the standard method involving Rouché's theorem (see, e.g., [2]) one can prove the following theorem.

Theorem 3. *The problem D has countably many eigenvalues λ_k, $k \in \mathbb{Z}$, of the form*

$$\lambda_k = k + \varkappa_k, \quad \{\varkappa_k\} \in l_2. \tag{11}$$

Moreover, analogously to Theorem 1.1.4 in [3] using Hadamard's factorization theorem one can obtain the following assertion.

Lemma 1. *The characteristic function is uniquely determined by its zeros by the formula*

$$\Delta(\lambda) = \pi(\lambda - \lambda_0) \prod_{k \neq 0} \frac{\lambda_k - \lambda}{k} \exp\left(\frac{\lambda}{k}\right). \tag{12}$$

The following theorem establishes unique solvability of Inverse Problem 1 along with the fact that the asymtotics (11), being necessary, is also a sufficient condition of its solvability.

Theorem 4. *For any sequence of complex numbers $\{\lambda_k\}_{k \in \mathbb{Z}}$ of the form (11) there exist unique (up to values on a set of measure zero) functions $p(x)$, $q(x)$ such that $(\pi - x)p(x)$, $(\pi - x)q(x) \in L_2(0, \pi)$ and $\{\lambda_k\}_{k \in \mathbb{Z}}$ is the spectrum of the boundary value problem $D(p, q)$ of the form (1) and (2).*

For the proof of Theorem 4 we need the following lemma.

Lemma 2. *Let arbitrary complex numbers λ_k, $k \in \mathbb{Z}$, of the form (11) be given. Then the function $\Delta(\lambda)$ determined by (12) has the form (6) with certain functions $w_1(x)$, $w_2(x) \in L_2(0, \pi)$.*

Proof. The following representation holds:

$$\sin \lambda\pi = \pi\lambda \prod_{k \neq 0} \frac{k-\lambda}{k} \exp\left(\frac{\lambda}{k}\right),$$

and hence

$$\Delta(\lambda) = \sin \lambda\pi F(\lambda), \quad F(\lambda) = \prod_{k=-\infty}^{\infty} \left(1 - \frac{\varkappa_k}{\lambda - k}\right). \tag{13}$$

Let us prove that $F(\lambda)$ is bounded in the region $G_\delta := \{\lambda \in \mathbb{C} \colon |\lambda - k| \geq \delta,\ k \in \mathbb{Z}\}$, $\delta > 0$. Choose such N, that $|\varkappa_k| < \delta/2$ for $|k| > N$. Represent $F(\lambda)$ in the form

$$F(\lambda) = \prod_{|k| \leq N} \left(1 - \frac{\varkappa_k}{\lambda - k}\right) \exp(H_N(\lambda)),$$

where

$$H_N(\lambda) = \sum_{|k|>N} \ln\left(1 - \frac{\varkappa_k}{\lambda - k}\right) = \sum_{|k|>N} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\varkappa_k^n}{(\lambda - k)^n}.$$

One can easily see that

$$|H_N(\lambda)| \leq 2 \sum_{|k|>N} \frac{|\varkappa_k|}{|\lambda - k|} \leq 2 \left(\sum_{|k|>N} |\varkappa_k|^2\right)^{1/2} \left(\sum_{|k|>N} \frac{1}{|\lambda - k|^2}\right)^{1/2}.$$

Consequently, $|F(\lambda)| < C_\delta$ for $\lambda \in G_\delta$ and, moreover,

$$\lim_{\mathrm{Im}\lambda \to \infty} F(\lambda) = 1. \tag{14}$$

By (13) we get

$$\Delta(n) = (-1)^{n+1}\pi\varkappa_n b_n, \quad b_n = \prod_{k \neq n} \left(1 - \frac{\varkappa_k}{n-k}\right), \quad n \in \mathbb{Z}.$$

Since the sequence $\{b_n\}$ is bounded, we have $\{\Delta(n)\} \in l_2$. Thus, one can uniquely construct the functions $w_1(x)$, $w_2(x) \in L_2(0, \pi)$, such that

$$\int_0^\pi w_1(t) \sin nt\, dt = \frac{\Delta(n) - \Delta(-n)}{2}, \quad n \geq 1,$$
$$\int_0^\pi w_2(t) \cos nt\, dt = \frac{\Delta(n) + \Delta(-n)}{2}, \quad n \geq 0. \tag{15}$$

Put

$$\theta(\lambda) = \int_0^\pi w_1(t) \sin \lambda t\, dt + \int_0^\pi w_2(t) \cos \lambda t\, dt,$$

and hence $\theta(n) = \Delta(n)$, $n \in \mathbb{Z}$. Thus, the function

$$R(\lambda) := \frac{\Delta(\lambda) - \theta(\lambda)}{\sin \lambda\pi} = F(\lambda) - \frac{\theta(\lambda)}{\sin \lambda\pi}$$

is entire in λ and bounded in G_δ. By the maximum modulus principle and Liouville's theorem, $R(\lambda) \equiv const$. Moreover, according to (14), $R(\lambda) \equiv 1$, and hence $\Delta(\lambda)$ has the form (6). □

Proof of Theorem 4. Let a complex sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ having the asymptotics (11) be given. Construct the function $\Delta(\lambda)$ by formula (12) and find the functions $w_1(x)$, $w_2(x) \in L_2(0, \pi)$ from the representation (6). Consider the system of main equations (7) with these functions $w_1(x)$, $w_2(x)$. By Theorem 1, it has a unique solution $p(x)$, $q(x)$, such that $(\pi - x)p(x)$, $(\pi - x)q(x) \in L_2(0, \pi)$. On the other hand, the characteristic function of the corresponding boundary value problem $D = D(p, q)$ by necessity has the form (6) with $w_1(x)$, $w_2(x)$ determined by (7). Thus, the characteristic function of D coincides with the function $\Delta(\lambda)$, constructed via (12), and hence the spectrum of D coincide with $\{\lambda_k\}_{k\in\mathbb{Z}}$.

The uniqueness of the problem D follows from the uniqueness of the functions $w_1(x)$, $w_2(x)$ determined by (15) along with uniqueness of the solution of system (7). □

The proof of Theorem 4 is constructive and generates the following algorithm for solving Inverse Problem 1.

Algorithm 1. Let the spectrum $\{\lambda_k\}_{k\in\mathbb{Z}}$ be given.

(i) Construct the characteristic function $\Delta(\lambda)$ by (12).
(ii) Find $w_1(t)$, $w_2(t)$ from (6) inverting the Fourier transform.
(iii) Solve the system of nonlinear main equations (7), and find $p(x)$, $q(x)$.

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