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Twisted Group C^*-Algebras as Compact Quantum Metric Spaces

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Abstract. We construct a class of compact quantum metric spaces. We prove that twisted reduced group C^*-algebras for discrete groups with twisted rapid decay property are compact quantum metric spaces, which contain noncommutative tori, hyperbolic reduced group C^*-algebras and discrete Heisenberg group C^*-algebras, and that the compact quantum metric space structures depend only on the cohomology class of 2-cocycles in the Lipschitz isometric sense.

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1. Introduction

In [4], using a Dirac operator on a compact Riemannian spin manifold M, Connes constructed a seminorm on a dense subset of the C^*-algebra $C(M)$ of $\mathbb{C}$-valued continuous functions on M, and gave a characterization of the geodesic distance on M. More generally, he obtained a metric on the state space of a unital C^*-algebra from a spectral triple. A typical noncommutative example is the reduced group C^*-algebra of a discrete group with a length function. In 1998, Pavlović and Rieffel discussed the question of when "such" metrics

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induce the weak *-topology on the state space [14,16]. Rieffel studied the question in the much more general framework of order-unit spaces, and defined a compact quantum metric space as an order-unit space with a seminorn such that the induced metric topology on its state space is the weak *-topology (A seminorm with this property was called a Lip-norm) [17,19].

Rieffel proved in [18] that noncommutative tori $C_r^*(\mathbb{Z}^N, \sigma)$ can be equipped with compact quantum metric space structures, using length functions on $\mathbb{Z}^N$. He then asked the question of which other discrete groups could provide examples of compact quantum metric spaces. Ozawa and Rieffel proved that for groups with a Haagerup-type condition (in particular, hyperbolic groups), the reduced group C^*-algebras are compact quantum metric spaces [13]. Antonescu and Christensen considered a group Γ with rapid decay property, and constructed a large class of Lip-norms, which give compact quantum metric space structures on the reduced group C^*-algebra $C_r^*(\Gamma)$ [1]. Recently, Christ and Rieffel showed that for any length function of a strong form of polynomial growth on a discrete group, the topology from this metric coincides with the weak *-topology [3]. It is well known that $\mathbb{Z}^N$ and hyperbolic groups are groups with rapid decay property [5,7–9,13]. In [12], a σ-twisted rapid decay property was defined on twisted reduced group C^*-algebra $C_r^*(\Gamma, \sigma)$ for a finitely generated group Γ. Therefore, it is a natural question whether one may construct a compact quantum metric space structure over $C_r^*(\Gamma, \sigma)$ for a group Γ with the σ-twisted rapid decay property for a 2-cocycle σ on Γ.

In this paper we propose a solution to the question above. In Sect. 2 we introduce some basic concepts and results on twisted reduced group C^*-algebras, and extend σ-twisted rapid decay property to discrete groups. In Sect. 3, using a length function ℓ on a discrete group Γ, we construct seminorms L_D^k for $k \in \mathbb{N}$ on the twisted reduced group C^*-algebra $C_r^*(\Gamma, \sigma)$ of Γ. We endow the state space $\mathcal{S}(C_r^*(\Gamma, \sigma))$ with metrics induced by these Lipschitz seminorms, and then by a result of [13], we prove that when ℓ is proper, the twisted reduced group C^*-algebra $C_r^*(\Gamma, \sigma)$ is a compact quantum metric space under the Lip-norm L_D^k for all k large enough. Several typical examples are given, in Sect. 4, from this result, such as noncommutative tori, hyperbolic reduced group C^*-algebras and discrete Heisenberg group C^*-algebras. In Sect. 5 we discuss the relation between our compact quantum metric space structures on $C_r^*(\Gamma, \sigma)$ and the 2-cocycle σ cohomology class. We show that they depend only on the cohomology class of the 2-cocycle σ.

2. Twisted Reduced Group C^*-Algebras and Rapid Decay Property

Let Γ be a discrete group, and let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. A 2-*cocycle* on Γ is a function $\sigma : \Gamma \times \Gamma \mapsto \mathbb{T}$ such that

$$\sigma(s,t)\sigma(st,r) = \sigma(s,tr)\sigma(t,r),$$

for all $s, t, r \in \Gamma$. A 2-cocycle σ on Γ is called a *multiplier* if $\sigma(e, e) = 1$, where e is the identity of Γ. A 2-cocycle σ on Γ is called a *coboundary* if there is a function $\rho : \Gamma \mapsto \mathbb{T}$ such that

$$\sigma = d\rho,$$

where $d\rho(s, t) = \rho(s)\rho(t)\overline{\rho(st)}$ for $s, t \in \Gamma$. If two 2-cocycles σ_1 and σ_2 on Γ differ by a coboundary σ on Γ, that is, $\sigma_1 = \sigma \cdot \sigma_2$, we say that σ_1 and σ_2 are *cohomologous*.

Example 2.1. Let $\Gamma = \mathbb{Z}^N$. The map $\sigma_\Theta : \mathbb{Z}^N \times \mathbb{Z}^N \mapsto \mathbb{T}$, defined by

$$\sigma_\Theta(x, y) = \exp(ix^t\Theta y), \quad x, y \in \mathbb{Z}^N,$$

for some $N \times N$ real matrix Θ, is a 2-cocycle on Γ.

Let Γ be a discrete group, and let σ be a multiplier on Γ. Let $C_c(\Gamma, \sigma)$ denote the set of complex-valued functions with finite support on Γ. Define

$$\begin{aligned}
(f + g)(s) &= f(s) + g(s), \\
(\alpha f)(s) &= \alpha f(s), \\
(f *_\sigma g)(s) &= \sum_{t \in \Gamma} f(t) g(t^{-1}s)\sigma(t, t^{-1}s), \\
f^*(s) &= \overline{f(s^{-1})\sigma(s, s^{-1})}
\end{aligned}$$

for any $f, g \in C_c(\Gamma, \sigma)$, $\alpha \in \mathbb{C}$ and $s \in \Gamma$. Then $C_c(\Gamma, \sigma)$ becomes a *-algebra with the identity

$$\delta_e(s) = \begin{cases} 1, & s = e, \\ 0, & s \neq e \end{cases}$$

[22]. The left regular σ-projective representation λ_Γ of $C_c(\Gamma, \sigma)$ on $\ell^2(\Gamma)$ is given by

$$(\lambda_\Gamma(f)(\xi))(s) = (f *_\sigma \xi)(s) = \sum_{t \in \Gamma} f(t)\sigma(t, t^{-1}s)\xi(t^{-1}s), \quad s \in \Gamma,$$

for $f \in C_c(\Gamma, \sigma)$ and $\xi \in \ell^2(\Gamma)$. More precisely, let $\{\delta_s\}_{s \in \Gamma}$ be the canonical basis of $\ell^2(\Gamma)$. Then we have

$$\lambda_\Gamma(\delta_s)(\delta_t) = \sigma(s, t)\delta_{st}$$

for all $s, t \in \Gamma$, and

$$\lambda_\Gamma(f) = \sum_{s \in \Gamma} f(s)\lambda_\Gamma(\delta_s)$$

for $f = \sum_{s \in \Gamma} f(s)\delta_s \in C_c(\Gamma, \sigma)$. It is clear that λ_Γ is linear. For any $s, t, r \in \Gamma$, we have

$$\begin{aligned}
\lambda_\Gamma(\delta_s *_\sigma \delta_t)(\delta_r) &= \sigma(s, t)\sigma(st, r)\delta_{str} = \sigma(s, tr)\sigma(t, r)\delta_{str} \\
&= (\lambda_\Gamma(\delta_s)\lambda_\Gamma(\delta_t))(\delta_r),
\end{aligned}$$

i.e., $\lambda_\Gamma(\delta_s *_\sigma \delta_t) = \lambda_\Gamma(\delta_s)\lambda_\Gamma(\delta_t)$, and so

$$\begin{aligned}\lambda_\Gamma(f *_\sigma g) &= \sum_{s,t\in\Gamma} f(s)g(t)\lambda_\Gamma(\delta_s *_\sigma \delta_t)\\ &= \sum_{s,t\in\Gamma} f(s)g(t)\lambda_\Gamma(\delta_s)\lambda_\Gamma(\delta_t) = \lambda_\Gamma(f)\lambda_\Gamma(g),\end{aligned}$$

for any $f, g \in C_c(\Gamma, \sigma)$. Moreover, for any $s, t \in \Gamma$, we have

$$(\lambda_\Gamma(\delta_s))^*(\delta_t) = \overline{\sigma(s, s^{-1}t)}\delta_{s^{-1}t} = \lambda_\Gamma(\delta_s{}^*)(\delta_t).$$

Hence for any $f \in C_c(\Gamma, \sigma)$, we have

$$\lambda_\Gamma(f^*) = \sum_{s\in\Gamma} \overline{f(s)}\lambda_\Gamma(\delta_s{}^*) = \sum_{s\in\Gamma} \overline{f(s)}(\lambda_\Gamma(\delta_s))^* = (\lambda_\Gamma(f))^*.$$

Since $\lambda_\Gamma(\delta_s)$ is a unitary for all $s \in \Gamma$, we see that

$$\|\lambda_\Gamma(f)\| = \|\sum_{s\in\Gamma} f(s)\lambda_\Gamma(\delta_s)\| \le \sum_{s\in\Gamma} |f(s)| = \|f\|_1$$

for any $f = \sum_{s\in\Gamma} f(s)\delta_s \in C_c(\Gamma, \sigma)$. So for any $f \in C_c(\Gamma, \sigma)$, we have that $\lambda_\Gamma(f) \in B(\ell^2(\Gamma))$. We define

$$\|f\| = \|\lambda_\Gamma(f)\|, \quad f \in C_c(\Gamma, \sigma).$$

Then $\|\cdot\|$ is a C^*-norm on the *-algebra $C_c(\Gamma, \sigma)$. The *σ-twisted reduced group C^*-algebra* $C_r^*(\Gamma, \sigma)$ of Γ is the completion of $C_c(\Gamma, \sigma)$ for the norm $\|\cdot\|$. The map λ_Γ extends uniquely to a *-isomorphism, still denoted by λ_Γ, of $C_r^*(\Gamma, \sigma)$ into $B(\ell^2(\Gamma))$. In particular, if $\sigma \equiv 1$, $C_r^*(\Gamma, \sigma)$ is the reduced group C^*-algebra $C_r^*(\Gamma)$.

Remark 2.2. Let Γ be a discrete group, and let σ_1 be a 2-cocycle on Γ. As in the situation of multipliers, we can endow a *-algebraic structure on $C_c(\Gamma, \sigma_1)$ and define the left regular σ_1-projective representation of $C_c(\Gamma, \sigma_1)$ on $\ell^2(\Gamma)$. Then this representation gives a C^*-norm on $C_c(\Gamma, \sigma_1)$. Now we have the σ_1-twisted reduced group C^*-algebra $C_r^*(\Gamma, \sigma_1)$ from the completion of $C_c(\Gamma, \sigma_1)$ under this C^*-norm. This is a C^*-algebra with the identity $\overline{\sigma_1(e, e)}\delta_e$.

Example 2.3. Let $\Gamma = \mathbb{Z}^N$, and let σ_Θ be defined as in Example 2.1. We have the σ_Θ-twisted reduced group C^*-algebra $A_\Theta = C_r^*(\Gamma, \sigma_\Theta)$: the noncommutative N-torus [5,6,15].

A *length function* on a group Γ is a function

$$\ell : \Gamma \mapsto [0, +\infty)$$

such that

(i) $\ell(s) = 0$ if and only if $s = e$, where e is the identity of Γ.
(ii) $\ell(s) = \ell(s^{-1})$ for all $s \in \Gamma$.
(iii) $\ell(s_1 s_2) \le \ell(s_1) + \ell(s_2)$ for all $s_1, s_2 \in \Gamma$.

When a group G is endowed with a length function ℓ, a right-invariant metric on G can be defined by

$$d(s,t) = \ell(st^{-1}), \quad s,t \in G.$$

Let Γ be a discrete group endowed with a length function ℓ, and let σ be a 2-cocycle on Γ. For any $\alpha \geq 0$, the set

$$H_\ell^\alpha(\Gamma,\sigma) = \{\xi : \Gamma \mapsto \mathbb{C} \mid \sum_{s\in\Gamma} |\xi(s)(1+\ell(s))^\alpha|^2 < +\infty\}$$

is called the *Sobolev space* of order α with respect to ℓ [2,9].

The rapid decay property on twisted reduced group C^*-algebras for finitely generated groups is discussed in [12]. We extend it to the discrete group case.

Definition 2.4. Let Γ be a discrete group, and let σ be a 2-cocycle on Γ. We say that Γ has the σ-*twisted rapid decay property* with respect to a length function ℓ on Γ if there exist constants $C, \alpha > 0$ such that

$$\|\lambda_\Gamma(f)\| \leq C\|f\|_{2,\alpha,\ell},$$

for all $f \in C_c(\Gamma,\sigma)$, where $\|f\|_{2,\alpha,\ell} = (\sum_{s\in\Gamma} |f(s)|^2(1+\ell(s))^{2\alpha})^{\frac{1}{2}}$.

Remark 2.5. We just say that a discrete group Γ has the *rapid decay property*, with respect to a length function ℓ, if it has the σ-twisted rapid decay property with respect to ℓ for the constant 2-cocycle $\sigma \equiv 1$ [2,9].

Proposition 2.6. *Let σ be a 2-cocycle on a discrete group Γ, and let ℓ be a length function on Γ. Then the following statements are equivalent:*

(i) *Γ has the σ-twisted rapid decay property with respect to ℓ.*

(ii) *There exists a polynomial P such that for all $r > 0$*

$$\|\lambda_\Gamma(f)\| \leq |P(r)|\|f\|_2,$$

for any $f \in C_c(\Gamma,\sigma)$ with support in the ball of radius r and center e.

(iii) *There exists a polynomial P such that for all $r > 0$*

$$\|f *_\sigma g\|_2 \leq |P(r)|\|f\|_2\|g\|_2,$$

where f and g are elements in $C_c(\Gamma,\sigma)$ and f is supported on the ball of radius r and center e.

Proof. $(i) \Rightarrow (ii)$. Since Γ has the σ-twisted rapid decay property with respect to ℓ, there exist constants $C, \alpha > 0$ such that

$$\|\lambda_\Gamma(g)\| \leq C\|g\|_{2,\alpha,\ell},$$

for all $g \in C_c(\Gamma,\sigma)$. Suppose that $f \in C_c(\Gamma,\sigma)$ and f is supported on a ball of radius r centered at e. We then have

$$\begin{aligned}\|\lambda_\Gamma(f)\| &\leq C\|f\|_{2,\alpha,\ell}\\ &= C\cdot\sqrt{\sum_{\ell(s)\leq r}|f(s)|^2(1+\ell(s))^{2\alpha}}\\ &\leq C\cdot\sqrt{\sum_{\ell(s)\leq r}|f(s)|^2(1+r)^{2\alpha}}\\ &= C(1+r)^{\alpha}\|f\|_2.\end{aligned}$$

Thus (ii) is satisfied for the polynomial $P(r) = C(1+r)^{[\alpha]+1}$.

$(ii) \Rightarrow (i)$. For any $n \in \mathbb{N}\cup\{0\}$, we denote

$$S_n = \{s \in \Gamma \mid n \leq \ell(s) < n+1\}.$$

Assume that $P(x) = a_{k-1}x^{k-1} + \cdots + a_1x + a_0$. Then for $n \geq 1$ we have

$$\begin{aligned}|P(n+1)| &\leq \sup\{|a_i|\}((n+1)^{k-1} + (n+1)^{k-2} + \cdots + (n+1) + 1)\\ &= \sup\{|a_i|\}\frac{(n+1)^k - 1}{(n+1)-1}\\ &\leq \sup\{|a_i|\}\cdot(n+1)^k.\end{aligned}$$

When $n = 0$, we have

$$|P(1)| = |\sum_{i=0}^{k-1} a_i| \leq \sum_{i=0}^{k-1}|a_i| \leq k\cdot\sup\{|a_i|\}(0+1)^k.$$

Denote $C = k\cdot\sup\{|a_i|\}$. Then for any $n \in \mathbb{N}\cup\{0\}$, we have

$$|P(n+1)| \leq C(n+1)^k.$$

For any $f \in C_c(\Gamma,\sigma)$, we have

$$\begin{aligned}\|\lambda_\Gamma(f)\| &= \left\|\sum_{n=0}^{\infty}\lambda_\Gamma(f|_{S_n})\right\| \leq \sum_{n=0}^{\infty}\|\lambda_\Gamma(f|_{S_n})\|\\ &\leq \sum_{n=0}^{\infty}|P(n+1)|\|f|_{S_n}\|_2 \leq \sum_{n=0}^{\infty}C(n+1)^k\|f|_{S_n}\|_2\\ &= \sum_{n=0}^{\infty}C(n+1)^{-1}(n+1)^{k+1}\|f|_{S_n}\|_2\\ &\leq C\cdot\sqrt{\sum\nolimits_{n=0}^{\infty}(n+1)^{-2}}\sqrt{\sum\nolimits_{n=0}^{\infty}(n+1)^{2k+2}\|f|_{S_n}\|_2^2}\\ &\leq \frac{C\pi}{\sqrt{6}}\cdot\sqrt{\sum\nolimits_{n=0}^{\infty}\sum\nolimits_{s\in S_n}|f(s)|^2(1+\ell(s))^{2k+2}}\end{aligned}$$

$$= \frac{C\pi}{\sqrt{6}} \cdot \sqrt{\sum\nolimits_{s\in\Gamma} |f(s)|^2(1+\ell(s))^{2(k+1)}}$$
$$= \frac{C\pi}{\sqrt{6}} \|f\|_{2,k+1,\ell}.$$

Thus Γ has the σ-twisted rapid decay property with respect to the length function ℓ.

$(ii) \Leftrightarrow (iii)$. This follows from the definition of the operator norm. □

When σ is the constant multiplier 1, Proposition 2.6 recovers the characterizations of the property of rapid decay as discussed in [2].

Proposition 2.7. *Let ℓ be a length function on a discrete group Γ. If Γ has the rapid decay property with respect to ℓ, then for any 2-cocycle σ, it also has the σ-twisted rapid decay property with respect to ℓ.*

Proof. Suppose that $f, g \in C_c(\Gamma, \sigma)$ and f is supported on a ball of radius r and center e. Since Γ has the rapid decay property with respect to ℓ, there is a polynomial P such that

$$\|f * g\|_2 \leq |P(r)| \|f\|_2 \|g\|_2.$$

For any $s \in \Gamma$, we have

$$|(f *_\sigma g)(s)| = \left| \sum_{t\in\Gamma} f(t) g(t^{-1}s)\sigma(t, t^{-1}s) \right| \leq \sum_{t\in\Gamma} |f(t)||g(t^{-1}s)| = (|f| * |g|)(s).$$

Therefore,

$$\begin{aligned} \|f *_\sigma g\|_2^2 &= \sum_{s\in\Gamma} |(f *_\sigma g)(s)|^2 \leq \sum_{s\in\Gamma} ((|f| * |g|)(s))^2 \\ &= \||f| * |g|\|_2^2 \leq |P(r)|^2 \||f|\|_2^2 \||g|\|_2^2 \\ &= |P(r)|^2 \|f\|_2^2 \|g\|_2^2. \end{aligned}$$

Thus

$$\|f *_\sigma g\|_2 \leq |P(r)| \|f\|_2 \|g\|_2.$$

So Γ has the σ-twisted rapid decay property with respect to ℓ by Proposition 2.6. □

Example 2.8. There are many groups which have the twisted rapid decay property with respect to some length functions on them. We just enumerate some familiar examples.

(i) Finitely generated nonabelian free groups $\mathbb{F}_n$. In fact, if a finitely generated group Γ has the rapid decay property with respect to some length function on it, then it has the rapid decay property with respect to the word-length function for some finite generating set S of Γ [9]. Since $\mathbb{F}_n$ has the rapid decay property with respect to the word-length function on it [7,9], $\mathbb{F}_n$ has the σ-twisted rapid decay property with respect to the word-length function for any 2-cocycle σ on it by Proposition 2.7.

(ii) Finitely generated groups with polynomial growth. Because every finitely generated nilpotent group has polynomial growth by Theorem 3.2 in [20], finitely generated abelian groups and discrete Heisenberg groups are of polynomial growth. Finitely generated groups with polynomial growth have the rapid decay property with respect to the word-length functions on them [9], and so they have the σ-twisted rapid decay property with respect to the word-length functions for any 2-cocycle on them by Proposition 2.7.

(iii) Gromov word hyperbolic groups. Gromov word hyperbolic groups have the rapid decay property with respect to the word-length functions on them [5,8,13]. Therefore, by Proposition 2.7 they have the σ-twisted rapid decay property with respect to the word-length functions for any 2-cocycle on them.

3. Lip-Norms on Twisted Reduced Group C^*-Algebras

A *Lipschitz seminorm* on a C^*-algebra A with identity 1_A is a seminorm, L, on A that is permitted to take the value $+\infty$, and satisfies

(i) $L(a) = L(a^*)$ for all $a \in A$.

(ii) $L(a) = 0$ if and only if $a \in \mathbb{C}1_A$.

(iii) the set $\mathcal{A} = \{a \in A \mid L(a) < +\infty\}$ is a dense subset of A.

A *Lip-norm* on a unital C^*-algebra A is a Lipschitz seminorm L such that the topology, induced by the metric

$$\rho(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| \mid L(a) \leq 1\}, \quad \mu, \nu \in \mathcal{S}(A)$$

on the state space $\mathcal{S}(A)$ of A, coincides with the weak *-topology [19,21]. If there exists a Lip-norm L on A, we say that the pair (A, L) is a *compact quantum metric space*. The usual definition of Lipschitz seminorms and Lip-norms only require them to be defined densely on the self-adjoint part of the C^*-algebra [17,19]. Under the condition $L(a) = L(a^*)$ for all $a \in A$ it suffices to take the above supremum just over self-adjoint elements of A when defining ρ [19].

Let Γ be a discrete group endowed with a length function ℓ, and let σ be a 2-cocycle on Γ. We let M_ℓ denote the (usually unbounded) operator on $\ell^2(\Gamma)$ of pointwise multiplication by ℓ. We define the derivation by the Dirac operator $D = M_\ell$ as follows:

$$\Delta(a) = [D, \lambda_\Gamma(a)] = [M_\ell, \lambda_\Gamma(a)],$$

for all $a \in C_r^*(\Gamma, \sigma)$. The operator $\Delta(a)$ on $\ell^2(\Gamma)$ may not be bounded [4]. For $k \in \mathbb{N}$, we denote

$$\Delta^k(a) = \underbrace{[D, [D, \ldots, [D, \lambda_\Gamma(a)] \cdots]]}_{k},$$

for all $a \in C_r^*(\Gamma, \sigma)$.

For $s, t \in \Gamma$ and $\xi \in \ell^2(\Gamma)$, we have

$$\begin{aligned}\Delta(\delta_s)(\xi)(t) &= ([D, \lambda_\Gamma(\delta_s)](\xi))(t)\\ &= ((D\lambda_\Gamma(\delta_s) - \lambda_\Gamma(\delta_s)D)(\xi))(t)\\ &= D(\lambda_\Gamma(\delta_s)(\xi))(t) - \lambda_\Gamma(\delta_s)(D(\xi))(t)\\ &= \ell(t)(\lambda_\Gamma(\delta_s)\xi)(t) - (D\xi)(s^{-1}t)\sigma(s, s^{-1}t)\\ &= \ell(t)\xi(s^{-1}t)\sigma(s, s^{-1}t) - \ell(s^{-1}t)\xi(s^{-1}t)\sigma(s, s^{-1}t)\\ &= (\ell(t) - \ell(s^{-1}t))\xi(s^{-1}t)\sigma(s, s^{-1}t),\end{aligned}$$

and

$$\begin{aligned}\Delta^2(\delta_s)(\xi)(t) &= ([D, [D, \lambda_\Gamma(\delta_s)]](\xi))(t)\\ &= (D[D, \lambda_\Gamma(\delta_s)](\xi) - [D, \lambda_\Gamma(\delta_s)]D(\xi))(t)\\ &= \ell(t)([D, \lambda_\Gamma(\delta_s)](\xi))(t) - (\ell(t) - \ell(s^{-1}t))(D(\xi))(s^{-1}t)\sigma(s, s^{-1}t)\\ &= \ell(t)(\ell(t) - \ell(s^{-1}t))\xi(s^{-1}t)\sigma(s, s^{-1}t)\\ &\quad -(\ell(t) - \ell(s^{-1}t))\ell(s^{-1}t)\xi(s^{-1}t)\sigma(s, s^{-1}t)\\ &= (\ell(t) - \ell(s^{-1}t))(\ell(t) - \ell(s^{-1}t))\xi(s^{-1}t)\sigma(s, s^{-1}t)\\ &= (\ell(t) - \ell(s^{-1}t))^2\xi(s^{-1}t)\sigma(s, s^{-1}t).\end{aligned}$$

In general, we have

$$\begin{aligned}\Delta^k(\delta_s)(\xi)(t) &= \underbrace{[D, [D, \ldots, [D, \lambda_\Gamma(\delta_s)] \cdots]]}_{k}(\xi)(t)\\ &= (\ell(t) - \ell(s^{-1}t))^k\xi(s^{-1}t)\sigma(s, s^{-1}t),\end{aligned}$$

for all $s, t \in \Gamma$, $\xi \in \ell^2(\Gamma)$ and $k \in \mathbb{N}$. In particular, when $\xi = \delta_t$, we have

$$\Delta^k(\delta_s)(\xi) = \Delta^k(\delta_s)(\delta_t) = (\ell(st) - \ell(t))^k\sigma(s, t)\delta_{st}$$

for all $s, t \in \Gamma$ and $k \in \mathbb{N}$. Therefore, for any $k \in \mathbb{N}$ and $f \in C_c(\Gamma, \sigma)$ we have that $\Delta^k(f) \in B(\ell^2(\Gamma))$.

Definition 3.1. For any $k \in \mathbb{N}$, we define

$$L_D^k(a) = \begin{cases} \|\Delta^k(a)\|, & a \in C_c(\Gamma, \sigma),\\ +\infty, & a \in C_r^*(\Gamma, \sigma)\backslash C_c(\Gamma, \sigma).\end{cases}$$

For any $a \in C_r^*(\Gamma, \sigma)$, $\lambda_\Gamma(a)(\delta_e) \in \ell^2(\Gamma)$ has a Fourier series expansion

$$\lambda_\Gamma(a)(\delta_e) = \sum_{s\in\Gamma}\langle\lambda_\Gamma(a)(\delta_e), \delta_s\rangle\delta_s = \sum_{s\in\Gamma}(\lambda_\Gamma(a)(\delta_e))(s)\delta_s.$$

Now we can define a map as follows:

$$a \in C_r^*(\Gamma, \sigma) \mapsto \lambda_\Gamma(a)(\delta_e) \in \ell^2(\Gamma).$$

This is a continuous linear injection of $C_r^*(\Gamma,\sigma)$ into $\ell^2(\Gamma)$, i.e., if $\lambda_\Gamma(a)(\delta_e) = \lambda_\Gamma(b)(\delta_e)$, then $a = b$ for all $a, b \in C_r^*(\Gamma,\sigma)$. Indeed, we can define the right regular $\overline{\sigma}$-projective representation ρ_Γ of $C_c(\Gamma,\sigma)$ on $\ell^2(\Gamma)$ by

$$(\rho_\Gamma(\delta_s)(\xi))(t) = \overline{\sigma(t,s)}\xi(ts)$$

for $s, t \in \Gamma$ and $\xi \in \ell^2(\Gamma)$. Then we have

$$\rho_\Gamma(\delta_s)\rho_\Gamma(\delta_t) = \overline{\sigma(s,t)}\rho_\Gamma(\delta_{st}),$$

and

$$\rho_\Gamma(\delta_s)\lambda_\Gamma(\delta_t) = \lambda_\Gamma(\delta_t)\rho_\Gamma(\delta_s),$$

for $s, t \in \Gamma$. So

$$\rho_\Gamma(\delta_s)^*\lambda_\Gamma(\delta_t) = \lambda_\Gamma(\delta_t)\rho_\Gamma(\delta_s)^*,$$

for $s, t \in \Gamma$. For any $c \in C_r^*(\Gamma,\sigma)$, there is a sequence $\{c_n\}$ in $C_c(\Gamma,\sigma)$ such that $\lim_{n\to\infty} c_n = c$. Suppose that $c_n = \sum_{t\in\Gamma} c_n(t)\delta_t$ for $n \in \mathbb{N}$. Then for any $s \in \Gamma$ and any $n \in \mathbb{N}$, we have

$$\begin{aligned}\rho_\Gamma(\delta_s)^*\lambda_\Gamma(c_n) &= \sum_{t\in\Gamma} c_n(t)\rho_\Gamma(\delta_s)^*\lambda_\Gamma(\delta_t)\\ &= \sum_{t\in\Gamma} c_n(t)\lambda_\Gamma(\delta_t)\rho_\Gamma(\delta_s)^* = \lambda_\Gamma(c_n)\rho_\Gamma(\delta_s)^*.\end{aligned}$$

So $\rho_\Gamma(\delta_s)^*\lambda_\Gamma(c) = \lambda_\Gamma(c)\rho_\Gamma(\delta_s)^*$ for any $s \in \Gamma$. Thus for any $s \in \Gamma$ and $a, b \in C_r^*(\Gamma,\sigma)$ with $\lambda_\Gamma(a)(\delta_e) = \lambda_\Gamma(b)(\delta_e)$, we have

$$\begin{aligned}\lambda_\Gamma(a)(\delta_s) &= \lambda_\Gamma(a)(\rho_\Gamma(\delta_s)^*(\overline{\sigma(e,s)}\delta_e))\\ &= \overline{\sigma(e,s)}\rho_\Gamma(\delta_s)^*(\lambda_\Gamma(a)(\delta_e))\\ &= \overline{\sigma(e,s)}\rho_\Gamma(\delta_s)^*(\lambda_\Gamma(b)(\delta_e))\\ &= \lambda_\Gamma(b)(\rho_\Gamma(\delta_s)^*(\overline{\sigma(e,s)}\delta_e)) = \lambda_\Gamma(b)(\delta_s),\end{aligned}$$

that is, $a = b$. Therefore, we can identify $a \in C_r^*(\Gamma,\sigma)$ with $\lambda_\Gamma(a)(\delta_e) \in \ell^2(\Gamma)$.

Proposition 3.2. *Let Γ be a discrete group endowed with a length function ℓ, and let σ be a 2-cocycle on Γ. Then for any natural number k, L_D^k is a Lipschitz seminorm on $C_r^*(\Gamma,\sigma)$.*

Proof. It is clear that L_D^k is a seminorm on $C_r^*(\Gamma,\sigma)$ permitted to take the value $+\infty$. Since ℓ is a real-valued function, D is self-adjoint. Hence

$$\|[D,\lambda_\Gamma(a^*)]\| = \|[D,\lambda_\Gamma(a)]^*\| = \|[D,\lambda_\Gamma(a)]\|$$

and

$$L_D^k(a^*) = \|\Delta^k(a^*)\| = \|\Delta^k(a)\| = L_D^k(a),$$

for any $a \in C_c(\Gamma,\sigma)$. When $a \in C_r^*(\Gamma,\sigma)\backslash C_c(\Gamma,\sigma)$, we have that $a^* \in C_r^*(\Gamma,\sigma)\backslash C_c(\Gamma,\sigma)$, and so $L_D^k(a^*) = L_D^k(a) = +\infty$.

We have seen that for any $f \in C_c(\Gamma, \sigma)$ we have that $\Delta^k(f) \in B(\ell^2(\Gamma))$. Hence for any $f \in C_c(\Gamma, \sigma)$ we have that $L_D^k(f) < +\infty$, i.e.,

$$C_c(\Gamma, \sigma) = \{a \in C_r^*(\Gamma, \sigma) \mid L_D^k(a) < +\infty\}.$$

We know that $C_c(\Gamma, \sigma)$ is dense in $C_r^*(\Gamma, \sigma)$. So the set $\{a \in C_r^*(\Gamma, \sigma) \mid L_D^k(a) < +\infty\}$ is dense in $C_r^*(\Gamma, \sigma)$.

Suppose that $L_D^k(a) = 0$. Since the unit vector δ_e is in the kernel of D, we have

$$0 = L_D^k(a) = \|\Delta^k(a)\| \geq \|\Delta^k(a)(\delta_e)\|_2 = \|\sum_{s\in\Gamma} \ell(s)^k a(s)\sigma(s,e)\delta_s\|_2.$$

Hence for all $s \neq e$ we have that $a(s) = 0$, that is, $a = a(e)\delta_e \in \mathbb{C}(\overline{\sigma(e,e)}\delta_e)$. So L_D^k is a Lipschitz seminorm. □

From the proposition above we can endow the state space $\mathcal{S}(C_r^*(\Gamma, \sigma))$ with a metric

$$\rho_\ell^k : \mathcal{S}(C_r^*(\Gamma, \sigma)) \times \mathcal{S}(C_r^*(\Gamma, \sigma)) \mapsto [0, +\infty]$$

as follows:

$$\rho_\ell^k(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| \mid a \in C_r^*(\Gamma, \sigma), L_D^k(a) \leq 1\}$$

for all $\mu, \nu \in \mathcal{S}(C_r^*(\Gamma, \sigma))$.

Let Γ be a discrete group, and let ℓ be a length function on Γ. If $\ell^{-1}([0, \beta])$ is a finite set for each positive $\beta \in \mathbb{R}$, then we say that ℓ is *proper*.

Lemma 3.3. *Let Γ be a discrete group endowed with a proper length function ℓ, and let $\alpha > 0$. Let σ be a 2-cocycle on Γ. Then there exists a real number $\gamma > 0$ such that*

$$\|a\|_{2,\alpha,\ell} \leq \gamma L_D^k(a)$$

for any $a \in C_c(\Gamma, \sigma)$ with $a(e) = 0$ and any integer $k > \alpha$.

Proof. We have

$$\left\|\sum_{t\in\Gamma} \ell(t)^k a(t)\sigma(t,e)\delta_t\right\|_2 = \|\Delta^k(a)(\delta_e)\|_2 \leq \|\Delta^k(a)\| = L_D^k(a).$$

Hence

$$\sum_{t\in\Gamma} (\ell(t))^{2k}|a(t)|^2 = \left\|\sum_{t\in\Gamma} \ell(t)^k a(t)\sigma(t,e)\delta_t\right\|_2^2 \leq (L_D^k(a))^2.$$

Let $\beta \in \mathbb{R}$ with $\beta \geq 1$. Then for $t \in \Gamma$ with $\ell(t) \geq \beta \geq 1$ we have

$$(1 + \ell(t))^{2\alpha} \leq 2^{2\alpha}(\ell(t))^{2\alpha} \leq 2^{2\alpha}\beta^{2\alpha-2k}(\ell(t))^{2k}.$$

Thus

$$\begin{aligned}\sum_{t\in\Gamma}|a(t)|^2(1+\ell(t))^{2\alpha} &= \sum_{\ell(t)<\beta}|a(t)|^2(1+\ell(t))^{2\alpha}+\sum_{\ell(t)\geq\beta}|a(t)|^2(1+\ell(t))^{2\alpha}\\ &\leq \sum_{\ell(t)<\beta}|a(t)|^2(1+\ell(t))^{2\alpha}\\ &\quad+\sum_{\ell(t)\geq\beta}|a(t)|^2 2^{2\alpha}\beta^{2\alpha-2k}(\ell(t))^{2k}\\ &\leq \sum_{\ell(t)<\beta}|a(t)|^2(1+\ell(t))^{2k}+2^{2\alpha}\beta^{2\alpha-2k}(L_D^k(a))^2\\ &\leq \sum_{0<\ell(t)<\beta}|a(t)|^2(\ell(t))^{2k}(1+(\ell(t))^{-1})^{2k}\\ &\quad+2^{2\alpha}\beta^{2\alpha-2k}(L_D^k(a))^2\\ &\leq \sum_{0<\ell(t)<\beta}|a(t)|^2(\ell(t))^{2k}(\sup_{0<\ell(s)<\beta}(1+(\ell(s))^{-1})^{2k})\\ &\quad+2^{2\alpha}\beta^{2\alpha-2k}(L_D^k(a))^2\\ &\leq (\sup_{0<\ell(s)<\beta}(1+(\ell(s))^{-1})^{2k}+2^{2\alpha}\beta^{2\alpha-2k})(L_D^k(a))^2\\ &= \gamma^2(L_D^k(a))^2,\end{aligned}$$

where the constant

$$\begin{aligned}\gamma &= (\sup_{0<\ell(s)<\beta}(1+(\ell(s))^{-1})^{2k}+2^{2\alpha}\beta^{2\alpha-2k})^{\frac{1}{2}}\\ &= (\max_{0<\ell(s)<\beta}(1+(\ell(s))^{-1})^{2k}+2^{2\alpha}\beta^{2\alpha-2k})^{\frac{1}{2}}\in(0,+\infty)\end{aligned}$$

since ℓ is proper. So we have that $\|a\|_{2,\alpha,\ell}\leq\gamma L_D^k(a)$. □

Proposition 3.4. *Let Γ be a discrete group endowed with a proper length function ℓ, and let $\alpha>0$. Let σ be a 2-cocycle on Γ. If Γ has the σ-twisted rapid decay property with respect to ℓ and α, then for any integer $k>\alpha$ the diameter of the metric space $(\mathcal{S}(C_r^*(\Gamma,\sigma)),\rho_\ell^k)$ is finite.*

Proof. Because Γ has the σ-twisted rapid decay property with respect to ℓ and α, there is a constant $C>0$ such that

$$\|\lambda_\Gamma(a)\|\leq C\|a\|_{2,\alpha,\ell},$$

for any $a\in C_c(\Gamma,\sigma)$. By Lemma 3.3 there exists a constant $\gamma>0$ such that

$$\|a\|_{2,\alpha,\ell}\leq\gamma L_D^k(a),$$

for any $a\in C_c(\Gamma,\sigma)$ with $a(e)=0$. So for any $a\in C_c(\Gamma,\sigma)$, we have

$$\|\lambda_\Gamma(a-a(e)\overline{\sigma(e,e)}\delta_e)\|\leq C\gamma L_D^k(a-a(e)\overline{\sigma(e,e)}\delta_e)=C\gamma L_D^k(a).$$

Thus

$$\|\tilde{a}\|^{\sim} \leq C\gamma L_D^k(a),$$

for any $a \in C_c(\Gamma, \sigma)$, where $\| \cdot \|^{\sim}$ is the quotient norm on the quotient space $C_c(\Gamma, \sigma)/\mathbb{C}\delta_e$ with respect to the norm $\|\cdot\|$ on $C_c(\Gamma, \sigma)$. Hence from Proposition 2.2 in [17] we see that the diameter is finite. □

Let L be a Lipschitz seminorm on a unital C^*-algebra A, and let ϕ be a state of A, then L is a Lip-norm if and only if the set

$$\{a \in A \mid L(a) \leq 1 \text{ and } \phi(a) = 0\}$$

is a norm-totally-bounded subset of A [13]. For other equivalent characterizations of Lip-norm we may refer to [16]. Now we can prove the main result of this paper.

Theorem 3.5. *Let Γ be a discrete group endowed with a proper length function ℓ, and let σ be a 2-cocycle on Γ. If Γ has the σ-twisted rapid decay property with respect to ℓ, then there exists a $k_0 \in \mathbb{N}$ such that for any integer $k \geq k_0$, the seminorm L_D^k is a Lip-norm, that is, $(C_r^*(\Gamma, \sigma), L_D^k)$ is a compact quantum metric space.*

Proof. In order to prove that L_D^k is a Lip-norm, it's sufficient to show that the set

$$B_k = \{a \in C_r^*(\Gamma, \sigma) \mid L_D^k(a) \leq 1 \text{ and } \operatorname{tr}(a) = \langle \lambda_\Gamma(a)\delta_e, \delta_e \rangle = 0\}$$

is totally bounded for the norm on $C_r^*(\Gamma, \sigma)$.

Since Γ has the σ-twisted rapid decay property with respect to ℓ, there exist constants $C, \alpha > 0$ such that

$$\|\lambda_\Gamma(f)\| \leq C \left(\sum_{t \in \Gamma} (1 + \ell(t))^{2\alpha} |f(t)|^2 \right)^{\frac{1}{2}}$$

for all $f \in C_c(\Gamma, \sigma)$.

Let $k_0 = [\alpha] + 1$ and fix a $k \in \mathbb{N}$ such that $k \geq k_0 > \alpha$. For $a \in B_k$, we have

$$\left\| \sum_{t \in \Gamma} \ell(t)^k a(t) \sigma(t, e) \delta_t \right\|_2 = \|\Delta^k(a)(\delta_e)\|_2 \leq \|\Delta^k(a)\| = L_D^k(a) \leq 1.$$

Hence

$$\sum_{t \in \Gamma} \ell(t)^{2k} |a(t)|^2 = \left\| \sum_{t \in \Gamma} \ell(t)^k a(t) \sigma(t, e) \delta_t \right\|_2^2 \leq (L_D^k(a))^2 \leq 1$$

for all $a \in B_k$. Because ℓ is proper, the cardinal number of $\{t \in \Gamma \mid \ell(t) \leq n\}$ is finite for any $n \in \mathbb{N}$. The set

$$\left\{ \sum_{\ell(t) \leq n} a(t)\delta_t \mid a \in B_k \right\}$$

is totally bounded. Therefore, in order to prove that B_k is totally bounded it is sufficient to show that for any positive real ε there exists a real number $\beta \geq 1$ such that for any $a \in B_k$

$$\left\| \lambda_\Gamma \left(\sum_{\ell(t)\geq\beta} a(t)\delta_t \right) \right\| \leq \varepsilon.$$

Indeed, as the proof in Lemma 3.3, for any $\beta \geq 1$ and $t \in \Gamma$ with $\ell(t) \geq \beta$ we have

$$(1+\ell(t))^{2\alpha} \leq 2^{2\alpha}(\ell(t))^{2\alpha} \leq 2^{2\alpha}\beta^{2\alpha-2k}(\ell(t))^{2k}$$

and

$$\sum_{t\in\Gamma} |a(t)|^2(1+\ell(t))^{2\alpha} \leq \sum_{\ell(t)<\beta} |a(t)|^2(1+\beta)^{2\alpha} + 2^{2\alpha}\beta^{2\alpha-2k}.$$

Hence

$$\sum_{\ell(t)\geq\beta} (1+\ell(t))^{2\alpha}|a(t)|^2 \leq 2^{2\alpha}\beta^{2\alpha-2k}.$$

Since $2\alpha - 2k < 0$, there exists a $\beta \geq 1$ such that

$$2^{2\alpha}\beta^{2\alpha-2k} \leq \frac{\varepsilon^2}{C^2}.$$

For this β we then have

$$\begin{aligned}\left\| \lambda_\Gamma \left(\sum_{\ell(t)\geq\beta} a(t)\delta_t \right) \right\| &\leq C \left(\sum_{\ell(t)\geq\beta} (1+\ell(t))^{2\alpha}|a(t)|^2 \right)^{\frac{1}{2}} \\ &\leq C(2^{2\alpha}\beta^{2\alpha-2k})^{\frac{1}{2}} \leq C\cdot\frac{\varepsilon}{C} = \varepsilon,\end{aligned}$$

where the first inequality is due to the σ-twisted rapid decay property of Γ. So B_k is totally bounded in $C_r^*(\Gamma,\sigma)$, and the theorem follows by Proposition 1.3 in [13]. □

Corollary 3.6. *If Γ is a discrete group with rapid decay property with respect to a proper length function ℓ, then there exists a $k_0 \in \mathbb{N}$ such that $(C_r^*(\Gamma,\sigma), L_D^k)$ is a compact quantum metric space for any 2-cocycle σ and integer $k \geq k_0$.*

Proof. By Proposition 2.7 Γ has the σ-twisted rapid decay property for any 2-cocycle σ with respect to ℓ. Hence the conclusion follows by Theorem 3.5. □

In particular, we have the following result of Antonescu and Christensen on reduced group C^*-algebras for group with rapid decay property [1, Theorem 2.6].

Corollary 3.7. *Suppose that* Γ *is a discrete group endowed with a proper length function* ℓ*. If* Γ *has the rapid decay property with respect to* ℓ*, then there exists a* $k_0 \in \mathbb{N}$ *such that for any natural number* $k \geq k_0$ *the metric generated by* L_D^k *on* $S(C_r^*(\Gamma))$ *is bounded and* $(C_r^*(\Gamma), L_D^k)$ *is a compact quantum metric space.*

Proof. This is just the constant 2-cocycle $\sigma \equiv 1$ case for $C_r^*(\Gamma, \sigma)$ by Proposition 3.4 and Corollary 3.6. □

4. Examples

In this section we will give some examples of groups with twisted rapid decay property. Hence we will obtain compact quantum metric space structures on the resulting twisted reduced group C^*-algebras. In Proposition 2.7 we proved that any group with rapid decay property with respect to a length function has the σ-twisted rapid decay property with respect to the length function for any 2-cocycle σ on the group. Thus groups with rapid decay property (as in Example 2.8) are the main concerned resources to groups with twisted rapid decay property.

4.1. Hyperbolic Groups

A metric space (X, ρ) is *hyperbolic* if there is a constant $\delta \geq 0$ such that for any four points $x, y, z, w \in X$ we have

$$\rho(x, y) + \rho(z, w) \leq \max\{\rho(x, z) + \rho(y, w), \rho(x, w) + \rho(y, z)\} + \delta.$$

Let Γ be a finitely generated discrete group, and let S be a finite generating subset for Γ, with $S = S^{-1}$. Let ℓ be the word-length function on Γ determined by S, and let ρ be the corresponding left-invariant metric on Γ defined by $\rho(x, y) = \ell(x^{-1}y)$. Then Γ is said to be *hyperbolic* if the metric space (Γ, ρ) is hyperbolic. For a hyperbolic group Γ, Ozawa and Rieffel proved that $(C_r^*(\Gamma), L_D^1)$ is a compact quantum metric space [13, Corollary 4.4]. Now we turn to the twisted case.

Proposition 4.1. *Let* Γ *be a hyperbolic group endowed with the word-length function* ℓ*, and let* σ *be a* 2*-cocycle on* Γ*. Then for the* σ*-twisted reduced group* C^**-algebra* $C_r^*(\Gamma, \sigma)$*, there is a* $k_0 \in \mathbb{N}$ *such that for any* $k \geq k_0$ $(C_r^*(\Gamma, \sigma), L_D^k)$ *is a compact quantum metric space.*

Proof. Since hyperbolic group has the rapid decay property with respect to the word-length function [5,8,13], the proposition follows by Corollary 3.6. □

4.2. Finitely Generated Free Non-abelian Groups

Let $\mathbb{F}_n$ be the nonabelian free group of rank $n \geq 2$. By Lemma 1.5 in [7] $\mathbb{F}_n (n \geq 2)$ has the rapid decay property with respect to the word-length function. Moreover, we even can take $C = 2$ and $\alpha = 2$ there.

Proposition 4.2. *For any $n, k \in \mathbb{N}$ with $n \geq 2$ and $k \geq 3$ and any 2-cocycle σ on $\mathbb{F}_n$, $(C_r^*(\mathbb{F}_n, \sigma), L_D^k)$ is a compact quantum metric space with respect to the word-length function ℓ.*

Proof. As in the proof of Theorem 3.5, we can take $k_0 = [\alpha] + 1 = 3$, so the conclusion follows by Corollary 3.6. □

4.3. Finitely Generated Groups with Polynomial Growth

Let Γ be a finitely generated group with the word-length function ℓ for a finite generating set S. If there exist $c > 0$ and $r > 0$ such that

$$\text{the cardinal number of } \{t \in \Gamma \mid \ell(t) \leq k\} \leq c(1+k)^r$$

for every $k \geq 0$, we say that Γ is of *polynomial growth*.

Proposition 4.3. *If Γ is a finitely generated group with polynomial growth for a finite generating set S, then there exists a $k_0 \in \mathbb{N}$ such that $(C_r^*(\Gamma, \sigma), L_D^k)$ is a compact quantum metric space with respect to the word-length function for any $k \geq k_0$ and any 2-cocycle σ.*

Proof. By Theorem 3.1.7 (2) in [9], every finitely generated group with polynomial growth for a finite generating set S has the rapid decay property with respect to the word-length function. So the proposition follows from Corollary 3.6. □

4.4. Finitely Generated Free Abelian Groups

Rieffel proved, in [18], that for any 2-cocycle σ, $(C_r^*(\mathbb{Z}^N, \sigma), L_D^1)$ is a compact quantum metric space with respect to a length function ℓ on $\mathbb{Z}^N$ which is either the word-length function for some finite generating subset of $\mathbb{Z}^N$ or the restriction to $\mathbb{Z}^N$ of some norm on $\mathbb{R}^N$. The following proposition gives other compact quantum metric space structures on the twisted reduced group C^*-algebra $C_r^*(\mathbb{Z}^N, \sigma)$.

Proposition 4.4. *There exists a $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ and any 2-cocycle σ on $\mathbb{Z}^n$, $(C_r^*(\mathbb{Z}^N, \sigma), L_D^k)$ is a compact quantum metric space with respect to the word-length function ℓ.*

Proof. Since $\mathbb{Z}^N$ is finitely generated and nilpotent, by Theorem 3.2 in [20] it is of polynomial growth. Thus by Theorem 3.1.7 (2) in [9] it has the rapid decay property with respect to the word-length function ℓ, and the conclusion follows by Corollary 3.6. □

4.5. Heisenberg Groups

For $n \in \mathbb{N}$, the $2n+1$-dimensional discrete (integer) Heisenberg group is the set

$$H_n = \left\{ \begin{pmatrix} 1 & x_1 & \cdots & x_n & z \\ & 1 & & 0 & y_1 \\ & & \ddots & & \vdots \\ & 0 & & 1 & y_n \\ & & & & 1 \end{pmatrix} | x_1, \ldots, x_n, y_1, \ldots, y_n, z \in \mathbb{Z} \right\}$$

with the matrix multiplication [11].

Proposition 4.5. *Let σ be a 2-cocycle on H_n. For the σ-twisted reduced group C^*-algebra $C_r^*(H_n, \sigma)$, there is a $k_0 \in \mathbb{N}$ such that $(C_r^*(H_n, \sigma), L_D^k)$ is a compact quantum metric space with respect to the word-length function for any integer $k \geq k_0$.*

Proof. Heisenberg group is a 2-step nilpotent and finitely generated group. It is of polynomial growth by Theorem 3.2 in [20], and so it has the rapid decay property with respect to the word-length function by Theorem 3.1.7 (2) in [9]. The proposition follows easily. $\square$

5. Lipschitz Isomorphisms on 2-Cocycles

In this section we will investigate the compact quantum metric space structure $(C_r^*(\Gamma, \sigma), L_D^k)$ with respect to the 2-cocycle σ cohomology class. Recall that two 2-cocycles σ_1 and σ_2 on Γ are said to be *cohomologous* if there exists a map $\rho : \Gamma \mapsto \mathbb{T}$ such that $\sigma_1 = d\rho \cdot \sigma_2$, where $d\rho(s,t) = \rho(s)\rho(t)\overline{\rho(st)}$ for $s, t \in \Gamma$ [22]. The following proposition is not a new result [12,22]. We include the proof here since we will use the concrete isomorphism in the sequel.

Proposition 5.1. *If σ_1 and σ_2 are two cohomologous 2-cocycles on Γ, then $C_r^*(\Gamma, \sigma_1)$ and $C_r^*(\Gamma, \sigma_2)$ are *-isomorphic.*

Proof. Suppose that $\rho : \Gamma \mapsto \mathbb{T}$ is the map such that $\sigma_1 = d\rho \cdot \sigma_2$. Define the map $\Phi : C_c(\Gamma, \sigma_1) \mapsto C_c(\Gamma, \sigma_2)$ by

$$\Phi(f) = \sum_{s \in \Gamma} \rho(s) f(s) \delta_s, \quad f \in C_c(\Gamma, \sigma_1).$$

We then have

$$\begin{aligned} \Phi(\delta_s *_{\sigma_1} \delta_t) &= \Phi(\sigma_1(s,t)\delta_{st}) \\ &= \sigma_1(s,t)\Phi(\delta_{st}) \\ &= \sigma_1(s,t)\rho(st)\delta_{st} \\ &= \sigma_1(s,t)\rho(st)\overline{\sigma_2(s,t)}\delta_s *_{\sigma_2} \delta_t \end{aligned}$$

$$
\begin{aligned}
&= \rho(s)\rho(t)\delta_s *_{\sigma_2} \delta_t \\
&= \Phi(\delta_s) *_{\sigma_2} \Phi(\delta_t),
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi({\delta_s}^*) &= \Phi(\overline{\sigma_1(s,s^{-1})}\delta_{s^{-1}}) = \overline{\sigma_1(s,s^{-1})}\rho(s^{-1})\delta_{s^{-1}} \\
&= \overline{\sigma_1(s,s^{-1})}\rho(s^{-1})\sigma_2(s,s^{-1}){\delta_s}^* \\
&= \overline{\rho(s)}{\delta_s}^* = \Phi(\delta_s)^*
\end{aligned}
$$

for all $s, t \in \Gamma$.

By linearity Φ is an isometrical *-isomorphism from $C_c(\Gamma, \sigma_1)$ onto $C_c(\Gamma, \sigma_2)$. Thus by density this map can be extended to a *-isomorphism between $C_r^*(\Gamma, \sigma_1)$ and $C_r^*(\Gamma, \sigma_2)$. □

From this proposition we find that two twisted reduced group C^*-algebras are *-isomorphic for any two cohomologous 2-cocycles. So a natural question is whether σ-twisted rapid decay property depends only on the cohomology class of σ. The following proposition gives a positive answer to this question.

Proposition 5.2. *If σ_1 and σ_2 are two cohomologous 2-cocycles on Γ, then the group Γ has the σ_1-twisted rapid decay property if and only if it has the σ_2-twisted rapid decay property.*

Proof. By Proposition 5.1 $C_r^*(\Gamma, \sigma_1)$ and $C_r^*(\Gamma, \sigma_2)$ are *-isomorphic. Since they have the same operator norm, the group has the σ_1-twisted rapid decay property if and only if it has the σ_2-twisted rapid decay property. □

Let Γ be a discrete group, and let σ_1 and σ_2 be two cohomologous 2-cocycles on Γ. Let $\lambda_\Gamma^{\sigma_1}$ be the left regular σ_1-projective representation of $C_c(\Gamma, \sigma_1)$ on $\ell^2(\Gamma)$, and let $\lambda_\Gamma^{\sigma_2}$ be the left regular σ_2-projective representation of $C_c(\Gamma, \sigma_2)$ on $\ell^2(\Gamma)$. If Γ has the σ_1-twisted rapid decay property, it also has the σ_2-twisted rapid decay property by Proposition 5.2. So there exist constants $C, \alpha > 0$ such that

$$
\|\lambda_\Gamma^{\sigma_1}(f)\| \leq C\|f\|_{2,\alpha,\ell}, \quad \|\lambda_\Gamma^{\sigma_2}(g)\| \leq C\|g\|_{2,\alpha,\ell}
$$

for all $f \in C_c(\Gamma, \sigma_1)$ and $g \in C_c(\Gamma, \sigma_2)$. By Theorem 3.5 there exists an integer k_0 such that for any $k \geq k_0$, $(C_r^*(\Gamma, \sigma_1), L_D^k)$ and $(C_r^*(\Gamma, \sigma_2), L_D^k)$ are compact quantum metric spaces. In the remaining part of this section we will discuss the relations between compact quantum metric spaces $(C_r^*(\Gamma, \sigma_1), L_D^k)$ and $(C_r^*(\Gamma, \sigma_2), L_D^k)$, and prove that the compact quantum metric space structures do not depend on the choice of a 2-cocycle in the cohomology class of σ.

Let A and B be two unital C^*-algebras with Lipschitz seminorms L_A and L_B, respectively. A map $\Phi : A \mapsto B$ is said to be *Lipschitz* if there exists a constant $\gamma \geq 0$ such that

$$
L_B(\Phi(a)) \leq \gamma L_A(a)
$$

for all $a \in A$. When Φ is invertible and both Φ and Φ^{-1} are Lipschitz we say that Φ is *bi-Lipschitz*. If

$$L_B(\Phi(a)) = L_A(a)$$

for all $a \in A$, then we say that Φ is *Lipschitz isometric* [10,21]. For two compact quantum metric spaces (A, L_A) and (B, L_B), if there is a *-isomorphism Φ from A onto B such that Φ and Φ^{-1} are Lipschitz isometric, we say that (A, L_A) and (B, L_B) are *Lipschitz isometric*.

Theorem 5.3. *Let Γ be a discrete group endowed with a proper length function ℓ. Assume σ_1 and σ_2 be two 2-cocycles on Γ. If Γ has the σ_1-twisted rapid decay property with respect to ℓ and σ_1 and σ_2 are cohomologous, then there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, the compact quantum metric spaces $(C_r^*(\Gamma, \sigma_1), L_D^k)$ and $(C_r^*(\Gamma, \sigma_2), L_D^k)$ are Lipschitz isometric. Therefore, the compact quantum metric space structures $(C_r^*(\Gamma, \sigma), L_D^k)$ depend only on the cohomology class of σ.*

Proof. Let Φ and ρ be defined as in Proposition 5.1, and then Φ is a *-isomorphism from $C_r^*(\Gamma, \sigma_1)$ onto $C_r^*(\Gamma, \sigma_2)$.

Define unitary operator U on $\ell^2(\Gamma)$ as follows:

$$U : f = \sum_{s\in\Gamma} f(s)\delta_s \in \ell^2(\Gamma) \mapsto \sum_{s\in\Gamma} \rho(s)f(s)\delta_s \in \ell^2(\Gamma).$$

Since D and U are pointwise multiplication operator by the length function ℓ and ρ, respectively, they are commutative, that is, $UD = DU$.

Let $\lambda_\Gamma^{\sigma_1}$ be the left regular σ_1-projective representation of $C_c(\Gamma, \sigma_1)$ on $\ell^2(\Gamma)$, and let $\lambda_\Gamma^{\sigma_2}$ be the left regular σ_2-projective representation of $C_c(\Gamma, \sigma_2)$ on $\ell^2(\Gamma)$. For $\xi \in \ell^2(\Gamma)$ and $s, t \in \Gamma$, we have

$$(\lambda_\Gamma^{\sigma_1}(\delta_s)(\xi))(t) = \sigma_1(s, s^{-1}t)\xi(s^{-1}t)$$

and

$$\begin{aligned}(U^*\lambda_\Gamma^{\sigma_2}(\Phi(\delta_s))U(\xi))(t) &= \overline{\rho(t)}(\lambda_\Gamma^{\sigma_2}(\Phi(\delta_s))U(\xi))(t)\\ &= \overline{\rho(t)}\rho(s)\sigma_2(s, s^{-1}t)(U(\xi))(s^{-1}t)\\ &= \overline{\rho(t)}\rho(s)\sigma_2(s, s^{-1}t)\rho(s^{-1}t)\xi(s^{-1}t)\\ &= \sigma_1(s, s^{-1}t)\xi(s^{-1}t).\end{aligned}$$

So

$$U^*\lambda_\Gamma^{\sigma_2}(\Phi(\delta_s))U = \lambda_\Gamma^{\sigma_1}(\delta_s)$$

for all $s \in \Gamma$. Hence Φ is implemented by U. For $a \in C_r^*(\Gamma, \sigma_1)$ with $L_D^k(a) < +\infty$, we have

$$
\begin{aligned}
L_D^k(\Phi(a)) &= \|\Delta^k(\Phi(a))\| = \| \underbrace{[D,[D,\ldots,[D,\lambda_\Gamma^{\sigma_2}(\Phi(a))]\cdots]]}_{k} \| \\
&= \| \underbrace{[D,[D,\ldots,[D,U\lambda_\Gamma^{\sigma_1}(a)U^*]\cdots]]}_{k} \| \\
&= \| \underbrace{[D,[D,\ldots,U[D,\lambda_\Gamma^{\sigma_1}(a)]U^*\cdots]]}_{k} \| \\
&= \|U \underbrace{[D,[D,\ldots,[D,\lambda_\Gamma^{\sigma_1}(a)]\cdots]]}_{k} U^*\| \\
&= \| \underbrace{[D,[D,\ldots,[D,\lambda_\Gamma^{\sigma_1}(a)]\cdots]]}_{k} \| = L_D^k(a).
\end{aligned}
$$

Thus Φ is Lipschitz isometric. Obviously Φ^{-1} is also Lipschitz isometric. Therefore, Φ is a Lipschitz isometric map from $(C_r^*(\Gamma,\sigma_1), L_D^k)$ onto $(C_r^*(\Gamma,\sigma_2), L_D^k)$. □

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