Results Math 71 (2017), 1073–1082

1422-6383/17/031073-10
*published online* May 12, 2016
DOI 10.1007/s00025-016-0555-6

**Results in Mathematics**

# "2CM+1IM" Theorem for Periodic Meromorphic Functions

Shengjiang Chen

**Abstract.** Generally, the concrete relations between two nonconstant meromorphic functions that share two values CM and one value IM are hard to determine. However, for the class $\mathcal{F}$ of all nonconstant meromorphic functions with the same period $c \neq 0$, we prove a result in this paper that: let $f(z), g(z) \in \mathcal{F}$ such that the hyper-order $\rho_2(f) < 1$, if $f(z), g(z)$ share $0, \infty$ CM and 1 IM, then either $f(z) \equiv g(z)$ or $f(z) = e^{az+b}g(z)$ and $\mu(f) = \mu(g) = 1$, where $a = \frac{2k\pi i}{c}$ and $k$ is some integer. As an application of this result, we obtain an uniqueness theorem for elliptic meromorphic functions. Moreover, examples are given to illustrate that all the conditions are necessary and sharp.

**Mathematics Subject Classification.** Primary 30D35, 33E05.

**Keywords.** Meromorphic function, periodicity, sharing value.

## 1. Introduction and Main Results

In this paper, a meromorphic function will always mean meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with the fundamental concepts of Nevanlinna's value distribution theory (see [5,8]) and in particular with the most usual of its symbols: $m(r, f)$, $N(r, f)$, $T(r, f)$, $S(r, f)$, where $S(r, f)$ satisfies $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ outside of a possible exceptional set $E$ of finite linear measure. Given $a \in \mathbb{C}$, we say that two meromorphic functions $f(z)$ and $g(z)$ share $a$ IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same

This work was supported by NNSFs of China (No. 11301076, No. 11371225), NSFs of Fujian Province (No. 2011J01006, No. 2014J01004) and the Scientific Research Project of Fujian Provincial Education Department (JA15562).

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zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share $a$ CM (counting multiplicities). Moreover, we say that $f, g$ share $\infty$ IM (resp. CM) if only if $\frac{1}{f}, \frac{1}{g}$ share 0 IM (resp. CM). Meanwhile, the order $\rho(f)$, hyper-order $\rho_2(f)$, lower order $\mu(f)$ and lower hyper-order $\mu_2(f)$ of a meromorphic function $f$ are defined in turn as follows:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r},$$
$$\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \mu_2(f) = \liminf_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r}.$$

It is well known that the classical results of the value distribution theory of meromorphic functions are the five-point, resp. four-point, theorems due to Nevanlinna: if two meromorphic functions $f, g$ share five distinct values in the extended complex plane IM, then $f = g$. Similarly, if two meromorphic functions $f, g$ share four distinct values in the extended complex plane CM, then $g$ is a Möbius transformation of $f$. The assumption "4 CM" in the four-point theorem has been improved to "2 CM + 2 IM" by Gundersen (see [4]). However, "4 CM" cannot be improved to "4 IM" (see [3]) and "1 CM + 3 IM" remains an open problem.

For the case when two nonconstant meromorphic functions $f$ and $g$ share three values, the concrete relations between $f$ and $g$, even if $f$ and $g$ share three values CM, are hard to determine in general. We refer the reader to [8] in this respect.

However, it is interesting to ask: what can be said if two nonconstant periodic meromorphic functions with the same period $c \neq 0$ share three values?

In 1992, Zheng (see [9]) proved the following result.

**Theorem 1.1.** *Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions with the same period $c(\neq 0)$. Suppose the lower hyper-order $\mu_2(f)$ of $f(z)$ is less than 1. If $f(z)$ and $g(z)$ share $0, 1, \infty$ CM, then $f(z) \equiv g(z)$ or $f(z)$ and $g(z)$ assume the following form $f(z) = \frac{e^{a_1 z + b_1} - 1}{e^{a_2 z + b_2} - 1}$, $g(z) = \frac{e^{-a_1 z - b_1} - 1}{e^{-a_2 z - b_2} - 1}$, where $a_1 = \frac{2m\pi i}{c}$, $a_2 = \frac{2k\pi i}{c}$, $b_1, b_2$ are constants, and $m, k$ are integers.*

In 2011, the author recently [1] obtained a result related to Theorem 1.1 in the case of "2CM+1IM".

**Theorem 1.2.** *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions with the same period $c(\neq 0)$ and let $a_1, a_2, a_3$ be three distinct numbers in the extended complex plane. Suppose that $1 < \mu(f) \leq \rho(f) < +\infty$ and that $\limsup_{r\to+\infty} \frac{\overline{N}(r, \frac{1}{f-a_1})}{T(r,f)} < 1$. If $f(z)$ and $g(z)$ share $a_1, a_2$ CM and $a_3$ IM, then $f(z) \equiv g(z)$.*

In this paper, we propose another result related to the previous theorems.

**Theorem 1.3.** *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions with the same period $c(\neq 0)$. Suppose that $\rho_2(f) < 1$. If $f(z)$ and $g(z)$ share*

$0, \infty$ *CM and* 1 *IM, then either* (i) $f(z) \equiv g(z)$*; or* (ii) $f(z) = e^{az+b}g(z)$ *and* $\mu(f) = \mu(g) = 1$*, where* $a = \frac{2k\pi i}{c}$*,* $b$ *are constants, and* $k$ *is some integer.*

*Remark* 1.4. The conclusion (ii) of Theorem 1.3 could occur. For example, let $f(z) = \frac{e^z}{e^z+1-e^{2z}}$ and $g(z) = \frac{e^{-z}}{e^z+1-e^{2z}} = e^{-2z}f(z)$. Clearly, $f, g$ are periodic functions with the same period $2\pi i$ and $\mu(f) = \mu(g) = 1$. By a simple calculation, it is easy to see that $f, g$ share $0, \infty$ CM and 1 IM.

*Remark* 1.5. The assumption "$\rho_2(f) < 1$" is necessary and sharp as shown by a result due to Ozawa (see [7]): For an arbitrary number $\sigma \in [1, \infty)$, there exists a periodic entire function $D(z)$ with period $d \neq 0$ such that $\rho(D) = \sigma$. Let

$$f(z) = \frac{e^{D(z)}}{e^{D(z)} + 1 - e^{2D(z)}}, \qquad g(z) = \frac{e^{-D(z)}}{e^{D(z)} + 1 - e^{2D(z)}}.$$

Clearly, $f, g$ are both periodic functions with the same period $d$ and $\rho_2(f) = \sigma \in [1, \infty)$. By a simple calculation, it is easy to see that $f, g$ share $0, \infty$ CM and 1 IM. However, both conclusions (i) and (ii) of Theorem 1.3 do not hold.

*Remark* 1.6. The assumption "2CM+1IM" cannot be replaced by "1CM+2IM". We give a counterexample as follows: Let $D(z)$ be a periodic entire function with period $d \neq 0$ such that $\rho(D) \in (1, \infty)$. Set $f = D$ and $g = \frac{2D}{D^2+1}$. Then, we can see that $f, g$ share 0 CM, $1, -1$ IM. However, both conclusions (i) and (ii) of Theorem 1.3 do not hold.

From Theorem 1.3, we can get

**Corollary 1.7.** *Let* $f(z)$ *and* $g(z)$ *be two nonconstant meromorphic functions with the same period* $c(\neq 0)$*. Suppose that* $\mu(f) \neq 1$ *and* $\rho_2(f) < 1$*. If* $f(z)$ *and* $g(z)$ *share* $0, \infty$ *CM and* 1 *IM, then* $f(z) \equiv g(z)$*.*

As an application of Theorem 1.3, we obtain the following result for nonconstant elliptic meromorphic functions.

**Corollary 1.8.** *Let* $f(z)$ *and* $g(z)$ *be two nonconstant elliptic meromorphic functions having one same period* $c(\neq 0)$*. If* $f(z)$ *and* $g(z)$ *share* $0, \infty$ *CM and* 1 *IM, then either (i)* $f(z) \equiv g(z)$*; or (ii)* $f(z) = e^{az+b}g(z)$ *and* $\mu(f) = \mu(g) = 1$*, where* $a = \frac{2k\pi i}{c}$*,* $b$ *are constants, and* $k$ *is some integer.*

*Remark* 1.9. Because that the order of a nonconstant elliptic meromorphic function is of finite, hence Corollary 1.8 holds according to Theorem 1.3.

## 2. Some Lemmas

**Lemma 2.1.** *If* $f(z)$ *is a nonconstant periodic meromorphic function, then* $\rho(f) \geq 1$*,* $\mu(f) \geq 1$*.*

**Lemma 2.2.** *Suppose* $h(z)$ *is a nonconstant entire function and* $f(z) = e^{h(z)}$*, then* $\rho_2(f) = \rho(h)$*.*

**Lemma 2.3.** *Let $f(z)$ be a meromorphic function. If*

$$g = \frac{af+b}{cf+d},$$

*where $a, b, c, d$ are small functions with respect to $f$ and $ad - bc \not\equiv 0$, then*

$$T(r,g) = T(r,f) + S(r,f).$$

**Lemma 2.4.** *Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions. If $f(z)$ and $g(z)$ share distinct three values $a_1, a_2, a_3$ IM, then*

$$T(r,f) \leq 3T(r,g) + S(r,f), \quad T(r,g) \leq 3T(r,f) + S(r,g).$$

**Lemma 2.5.** *Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions. If*

$$T(r,f) = O(T(r,g)), \quad (r \to \infty,\ r \notin E,\ mesE < \infty),$$

*then $\mu(f) \leq \mu(g)$, $\rho(f) \leq \rho(g)$, $\mu_2(f) \leq \mu_2(g)$ and $\rho_2(f) \leq \rho_2(g)$.*

**Lemma 2.6.** *Suppose $f(z)$ and $g(z)$ are nonconstant meromorphic functions with $\rho_2(f)$ and $\rho_2(g)$ as their hyper-orders, respectively. Then*

$$\rho_2(fg) \leq \max\{\rho_2(f), \rho_2(g)\}.$$

*Proof.* Without loss of generality, we assume that

$$\rho_2(f) \leq \rho_2(g) < \infty.$$

From the definition of the hyper-order, for any $\varepsilon > 0$, there exists a positive number $R$ such that

$$T(r,f) < \exp\left(r^{\rho_2(f)+\varepsilon}\right), \quad T(r,g) < \exp\left(r^{\rho_2(g)+\varepsilon}\right), \quad r > R.$$

Noting $T(r,fg) \leq T(r,f) + T(r,g)$, we have

$$T(r,fg) \leq 2\exp\left(r^{\rho_2(g)+\varepsilon}\right), \quad r > R,$$

which implies that

$$\limsup_{r\to\infty} \frac{\log^+ \log^+ T(r,fg)}{\log r} \leq \rho_2(g) + \varepsilon$$

holds for any $\varepsilon > 0$. Therefore, the assertion of Lemma 2.6 follows. □

**Lemma 2.7** (see [2,6]). *Let $f(z)$ be a meromorphic function of finite order $\rho$ and let $\eta$ be a nonzero complex number. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O(r^{\rho-1+\varepsilon}).$$

## 3. Proof of Theorem 1.3

*Proof.* Suppose that $f(z) \not\equiv g(z)$.

According to the assumptions of Theorem 1.3, Lemma 2.1, Lemma 2.4 and Lemma 2.5, we have $S(r,f) = S(r,g)$ and

$$1 \leq \mu(g) = \mu(f), \quad \rho_2(g) = \rho_2(f) < 1. \tag{3.1}$$

For convenience, we set $S(r) := S(r,f) = S(r,g)$ below.

Since $f$ and $g$ share $0, \infty$ CM, together with (3.1), we have

$$\frac{f(z)}{g(z)} = e^{Q(z)}, \tag{3.2}$$

where $Q(z)$ is an entire function with $\rho(Q) < 1$. It follows from (3.2) that $\frac{f(z+c)}{g(z+c)} = e^{Q(z+c)}$. Noting $f(z)$ and $g(z)$ have the same period $c(\neq 0)$, we have $e^{Q(z+c)} \equiv e^{Q(z)}$. Namely, $Q'(z+c) - Q'(z) \equiv 0$. Noting $\rho(Q') = \rho(Q) < 1$, we could deduce from Lemma 2.1 that $Q'(z) \equiv a$, where $a$ is a constant. Thus, we can assume that $Q(z) = az + b$, where $b$ is a constant. Substituting this into (3.2), we have

$$\frac{f(z)}{g(z)} = e^{az+b}. \tag{3.3}$$

If $\mu(f) = 1$, then we can arrive at the conclusion (ii) of Theorem 1.3 due to (3.1). Next, we shall show that the case $\mu(f) > 1$ cannot occur. Suppose on the contrary that $\mu(f) > 1$. Together with $\rho(e^{az+b}) = 1$, we have

$$T(r, e^{az+b}) = S(r). \tag{3.4}$$

Furthermore, we deduce from (3.3) and (3.4) that

$$\overline{N}\left(r, \frac{1}{f-1}\right) = \overline{N}\left(r, \frac{1}{g-1}\right) \leq N\left(r, \frac{1}{e^{az+b}-1}\right) = O(r) = S(r). \tag{3.5}$$

Next, we discuss the following two cases.

**Case 1.** Suppose that $a = 0$.

Then, we can see from (3.3) that $b \neq 0$ and 1 is the Picard value of $f$ and $g$ since $f(z) \not\equiv g(z)$. Using the same arguments as the proof of (3.3), we have

$$\frac{f(z)-1}{g(z)-1} = e^{a_1 z + b_1}, \tag{3.6}$$

which yields

$$\overline{N}\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{e^{a_1 z + b_1} - 1}\right) = S(r). \tag{3.7}$$

Rewriting (3.3) as $f(z) - e^{az+b} = e^{az+b}(g(z)-1)$, we deduce from (3.5) and (3.7) that

$$T(r,f) \leq \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f - e^{az+b}}\right) + S(r) = S(r).$$

It's impossible.

**Case 2.** Suppose that $a \neq 0$. We shall distinguish the following seven steps. Firstly, it is clear that $e^{a(z+c)+b} \equiv e^{az+b}$ due to (3.3). Thus, there exists a nonzero integer $k_0$ such that

$$c = \frac{2\pi k_0 i}{a}. \tag{3.8}$$

Secondly, we introduce the following notation: denote by $\overline{E}_h$ the set of the all distinct zeros of the meromorphic function $h$. Therefore, it is easy to see that

$$\overline{E}_{f-1} \subset \overline{E}_{e^{az+b}-1}. \tag{3.9}$$

Obviously,

$$\overline{E}_{e^{az+b}-1} = \left\{ z_n : z_n = \frac{2n\pi i}{a} - \frac{b}{a}, \quad n \in \mathbb{Z} \right\}. \tag{3.10}$$

Here and throughout this paper, $\mathbb{Z}$ is the set of all integers.

Thirdly, let the set of all congruence class of the integers for the modulus $|k_0|$ be $\{[0], [1], \dots, [|k_0| - 1]\}$. Set

$$z_m = \frac{2m\pi i}{a} - \frac{b}{a}, \quad m = 0, 1, \dots, |k_0| - 1. \tag{3.11}$$

For each $m = 0, 1, \dots, |k_0| - 1$, we define

$$[z_m] = \left\{ z_n = \frac{2n\pi i}{a} - \frac{b}{a} \in \overline{E}_{e^{az+b}-1} : \ n \in [m] \right\}. \tag{3.12}$$

Clearly, for any $z_n \in [z_m]$ ($m = 0, 1, \dots, |k_0| - 1$), there exists an integer $k_m$ such that

$$z_n = z_m + k_m c. \tag{3.13}$$

In fact, $z_n - z_m = \frac{2\pi(n-m)i}{a} = \frac{2\pi k|k_0|i}{a} = \frac{2\pi k_m k_0 i}{a} = k_m c$ for some $k, k_m \in \mathbb{Z}$. Conversely, for all integers $k$, we have $z_m + kc \in [z_m]$. Thus, we see that

$$[z_m] = \{z : z_m + kc, \ k \in \mathbb{Z}\}, \ m = 0, 1, \dots, |k_0| - 1.$$

What's more, if $\xi_0 \in [z_m]$ ($m = 0, 1, \dots, |k_0| - 1$), then

$$\xi_0 + kc \in [z_m] \tag{3.14}$$

holds for all integers $k$.

Next, we shall prove some properties of $[z_m]$ ($m = 0, 1, \dots, |k_0| - 1$).

**Proposition 3.1.** *If $z_m$, $m \in \{0, 1, \dots, |k_0| - 1\}$, is a zero of $f(z) - 1$ (resp. $g(z) - 1$) with multiplicity $p$ (resp. $q$), then each element $z_n \in [z_m]$ is also a zero of $f(z) - 1$ (resp. $g(z) - 1$) with multiplicity $p$ (resp. $q$).*

*Proof.* We only prove the case for $f(z) - 1$. It follows from (3.13) that $z_n = z_m + k_m c$ holds for some integer $k_m$. Noting $f(z+c) \equiv f(z)$, we have $f(z_n) - 1 = f(z_m + k_m c) - 1 = f(z_m) - 1$ and $f^{(s)}(z_n) = f^{(s)}(z_m + k_m c) = f^{(s)}(z_m)$, $\forall s \in \mathbb{Z}^+$. It follows that the conclusion of Proposition 3.1 holds. □

**Proposition 3.2.** *If $\xi_0 \in [z_m]$ is a zero of $f(z)-1$ (resp. $g(z)-1$) with multiplicity $p$ (resp. $q$), then each element $\xi_n \in [z_m]$ is also a zero of $f(z)-1$ (resp. $g(z)-1$) with multiplicity $p$ (resp. $q$).*

*Proof.* We only prove the case for $f(z)-1$. From (3.13), we have $\xi_0 - z_m = k_1 c$ and $\xi_n - z_m = k_2 c$, where $k_1$ and $k_2$ are some integers. Thus, we obtain $\xi_0 - \xi_n = (k_1 - k_2)c$ which implies that $f(\xi_n) - 1 = f(\xi_0) - 1$. Moreover, we also have $f^{(s)}(\xi_0) = f^{(s)}(\xi_n + (k_1 - k_2)c) = f^{(s)}(\xi_n)$ for all positive integer $s$. Hence, the conclusion of Proposition 3.2 holds. □

**Proposition 3.3.** *Suppose that $m_1, m_2 \in \{0, 1, \ldots, |k_0|-1\}$ such that $m_1 \neq m_2$. Then $[z_{m_1}] \cap [z_{m_2}] = \emptyset$.*

*Proof.* If there exists an element $\xi \in [z_{m_1}] \cap [z_{m_2}]$, then we can find two integers $k_1$ and $k_2$ such that $\xi = z_{m_1} + k_1 c$ and $\xi = z_{m_2} + k_2 c$ according to (3.13). Thus, we have $z_{m_1} - z_{m_2} = (k_1 - k_2)c$ which implies that $m_1 - m_2 = (k_1 - k_2)k_0$. However, this contradicts with $m_1 \notin [m_2]$. This completes the proof of Proposition 3.3. □

**Proposition 3.4.** $\overline{E}_{e^{az+b}-1} = \bigcup_{j=0}^{|k_0|-1}[z_j]$.

*Proof.* We only need to prove that $\overline{E}_{e^{az+b}-1} \subset \bigcup_{j=0}^{|k_0|-1}[z_j]$. For an arbitrary element $z_n \in \overline{E}_{e^{az+b}-1}$, we know that $z_n = \frac{2n\pi i}{a} - \frac{b}{a}$. Set $n = kk_0 + m$ holds for some $m \in \{0, 1, \ldots, |k_0|-1\}$ and $k \in \mathbb{Z}$. Thus, it follows from the definition of $[z_m]$ that $z_n \in [z_m]$, and hence, the conclusion of Proposition 3.4 is true. □

**Proposition 3.5.** *Suppose that $\Omega = \{z_m : f(z_m) = 1,\ m = 0, 1, \ldots, |k_0|-1\}$. Then $\overline{E}_{f-1} = \bigcup_{z_j \in \Omega}[z_j]$.*

*Proof.* By Proposition 3.1, we know that $\overline{E}_{f-1} \supset \bigcup_{z_j \in \Omega}[z_j]$. Next, we turn to prove $\overline{E}_{f-1} \subset \bigcup_{z_j \in \Omega}[z_j]$. If there exists an element $\xi \in \overline{E}_{f-1}$ but $\xi \notin \bigcup_{z_j \in \Omega}[z_j]$, then it follows from (3.9) and Proposition 3.4 that there exists a point $z_{m_0} \notin \Omega$ such that $\xi \in [z_{m_0}]$, where $m_0 \in \{0, 1, \ldots, |k_0|-1\}$. By Proposition 3.2, we deduce that $z_{m_0}$ is a zero of $f(z)-1$. But, this contradicts with $z_{m_0} \notin \Omega$. Hence, the proof of Proposition 3.5 is completed. □

Now, we proceed to prove Theorem 1.3. Without loss of generality, we assume that $\Omega = \{z_m : f(z_m) = 1,\ m = 0, 1, \ldots, |k_0|-1\} = \{z_0, z_1, \ldots, z_s\}$ $(0 \leq s \leq |k_0|-1)$ with multiplicities $p_0, p_1, \ldots, p_s$, respectively. Set $P = \max\{p_0, p_1, \ldots, p_s\}$.

Fourthly, we shall prove the following claim.

**Claim:** $N\left(r, \frac{1}{f-1}\right) = N\left(r, \frac{1}{g-1}\right) = O(r) = S(r)$.

In fact, by Proposition 3.5, we have $\overline{E}_{g-1} = \overline{E}_{f-1} = \bigcup_{z_j \in \Omega}[z_j] = \bigcup_{j=0}^{s}[z_j]$. Together (3.5) with Proposition 3.1, it follows that

$$N\left(r, \frac{1}{f-1}\right) \leq P \cdot \overline{N}\left(r, \frac{1}{f-1}\right) = O(r) = S(r)$$

and

$$N\left(r, \frac{1}{g-1}\right) \leq P \cdot \overline{N}\left(r, \frac{1}{g-1}\right) = O(r) = S(r).$$

Fifthly, we set

$$\frac{f(z)-1}{g(z)-1} = \phi(z). \tag{3.15}$$

Clearly, it follows from (3.3) that $\phi(z) = \frac{g(z)e^{az+b}-1}{g(z)-1}$ and hence

$$T(r, \phi) = T(r, g) + S(r), \tag{3.16}$$

according to (3.4) and Lemma 2.3. Combining (3.1), (3.16) and Lemma 2.5, we have

$$1 < \mu(\phi), \quad \rho_2(\phi) < 1. \tag{3.17}$$

Furthermore, we deduce from $f(z+c) \equiv f(z)$ and $g(z+c) \equiv g(z)$ that

$$\phi(z+c) \equiv \phi(z). \tag{3.18}$$

Sixthly, by changing $f$ and $g$, if needed, one could simply write

$$\phi = \frac{P_1}{P_2} e^{\alpha}, \quad d \geq 0 \tag{3.19}$$

where $\alpha$ is an entire function, $P_1(z)$ and $P_2(z)$ are the canonical products formed with the zeros, resp. poles, of $\phi$.

Next, we shall prove the following properties of $P_j(z)$, $j = 1, 2$.

**Property 1.** $\rho(P_j) \leq 1$, $j = 1, 2$.

**Property 2.** $P_j(z)$ and $P_j(z+c)$ share 0 CM, $j = 1, 2$.

**Property 3.** $P_j(z) \equiv A_j P_j(z+c)$, $A_j \neq 0$, $j = 1, 2$.

To prove Property 1, by the well-know result due to Borel (see [8], Theorem 2.3), we can see that the order $\rho(P_j)$ of $P_j(z)$ $(j = 1, 2)$ is equal to the exponent of convergence $\lambda(P_j)$ of the zeros of $P_j(z)$ $(j = 1, 2)$. Thus, we can deduce from (3.15) and Claim that

$$\rho(P_j) = \lambda(P_j) = \limsup_{r \to \infty} \frac{\log N(r, \frac{1}{P_j})}{\log r} \leq \limsup_{r \to \infty} \frac{\log N(r, \frac{1}{f-1})}{\log r} \leq 1, \quad j = 1, 2.$$

Therefore, the Property 1 is true.

To prove Property 2, let $\xi_0$ be a zero of $P_1(z)$ with multiplicity $t_0$, then $\xi_0$ is the zero of $f(z) - 1$ and $g(z) - 1$ with multiplicity $p_{j_0}$ and $q_{j_0}$ $(j_0 \in \{0, 1, \ldots, s\})$, respectively, such that $p_{j_0} - q_{j_0} = t_0$. By (3.14) and Proposition 3.2, $\xi_0 + c$ is also the zero of $f(z) - 1$ and $g(z) - 1$ with multiplicity $p_{j_0}$ and $q_{j_0}$, respectively. Thus, $\xi_0$ is the zero of $P_1(z+c)$ with multiplicity $t_0$. Conversely, let $\xi_1$ be a zero of $P_1(z+c)$ with multiplicity $t_1$, then $\xi_1 + c$ is the zero of $f(z) - 1$ and $g(z) - 1$ with multiplicity $p_{j_1}$ and $q_{j_1}$ $(j_1 \in \{0, 1, \ldots, s\})$, respectively, such that $p_{j_1} - q_{j_1} = t_1$. Again by (3.14) and Proposition 3.2, we have $(\xi_1 + c) - c$ is also the zero of $f(z) - 1$ and $g(z) - 1$ with multiplicity $p_{j_1}$ and $q_{j_1}$, respectively. Thus, $\xi_1$ is the zero of $P_1(z)$ with multiplicity $t_1$. Hence,

$P_1(z)$ and $P_1(z+c)$ share 0 CM. Similarly, the conclusion for the case $P_2(z)$ is also true.

To prove Property 3, we can deduce from Properties 1-2 that

$$\frac{P_j(z)}{P_j(z+c)} = e^{a_j z + b_j}, \quad j = 1, 2, \tag{3.20}$$

for some constants $a_j, b_j$ $(j = 1, 2)$.

If $a_j \neq 0$ $(j = 1, 2)$, then we know from Lemma 2.7 and (3.20) that

$$\frac{r}{\pi} = m\left(r, e^{a_j z + b_j}\right) = O\left(r^{\rho(P_j)-1+\varepsilon}\right) = O(r^{\varepsilon}), \quad j = 1, 2, \quad 0 < \varepsilon < 1,$$

which is impossible. Hence, the Property 3 is true.

Finally, from (3.17), (3.19) and Property 1, it follows that $\alpha$ is an entire function with $\mu(e^{\alpha}) > 1$ and $\rho(\alpha) < 1$. Moreover, from (3.18), (3.19) and Property 3, we can deduce that $e^{\alpha(z+c)} \equiv \frac{A_1}{A_2} e^{\alpha(z)}$, which implies that $\alpha'(z+c) - \alpha'(z) \equiv 0$. Hence, it follows from Lemma 2.1 that $\alpha(z) = a_3 z + b_3$ for some constants $a_3, b_3$. However, it leads to a contradiction with $\mu(e^{\alpha}) > 1$.

Therefore, we derive the desired conclusions of Theorem 1.3. □

## Acknowledgments

The author is grateful to the referee for his/her valuable suggestions and comments towards the improvement of this paper.

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Shengjiang Chen
Department of Mathematics
Fujian Normal University
Fuzhou 350007, Fujian
People's Republic of China
e-mail: chentrent@126.com

and

Department of Mathematics,
Ningde Normal University,
Ningde 352100, Fujian
People's Republic of China

Received: December 31, 2015.
Accepted: May 1, 2016.