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Results in Mathematics

Approximation of Functions by Baskakov-Kantorovich Operator

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Abstract. We characterize the approximation of functions in L_p norm by Baskakov-Kantorovich operator. We define an appropriate K-functional and prove a direct and strong converse inequality of type B in terms of the K-functional.

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1. Introduction

The classical Bernstein operator is defined for every function $f \in C[0,1]$ by the formula [10]

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right) \quad \text{where} \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \tag{1}$$

and the classical Baskakov operator is defined for every function $f \in C[0,\infty)$ by [1]

$$V_n(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) V_{n,k}(x) \tag{2}$$

where

$$V_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}. \tag{3}$$

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In order to approximate functions in L_p-norm Kantorovich introduced a modification of B_n:

$$B_n^*(f,x)=\sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)du.$$

Analogously, in [5] Ditzian and Totik defined two Kantorovich modifications of V_n.

For $0\le x<\infty$ they introduced

$$V_n^*(f,x)=\sum_{k=0}^{\infty} V_{n,k}(x)\, n\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)du$$

and

$$\tilde{V}_n(f,x)=\sum_{k=0}^{\infty} V_{n,k}(x)\,(n-1)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(u)du. \tag{4}$$

The reason for introducing the second one is that the first one is not a contraction and because of that it is not very suitable for approximating of functions in L_p norm for $p<\infty$.

For all these operators they proved [5, Theorem 9.3.2] (for $1\le p<\infty$) a direct inequality

$$\|L_nf-f\|_p\le M\left[w_\varphi^2\left(f,n^{-1/2}\right)_p+n^{-1}\|f\|_p\right]$$

and a week converse inequality

$$\|L_nf-f\|_p=O(n^{-\alpha/2})\Leftrightarrow w_\varphi^2\,(f,h)_p=O(h^\alpha),\quad \alpha<2,$$

where L_n is B_n^*, V_n^* or $\tilde{V}_n$ and

$$\omega_\varphi^2(f,h)_p=\sup_{|t|\le h}\|f(x-\varphi(x)t)-2f(x)+f(x+\varphi(x)t)\|_p$$

is the second order modulus of smoothness of Ditzian-Totik, $\varphi(x)=\sqrt{x(1-x)}$ for B_n^* and $\varphi(x)=\sqrt{x(1+x)}$ for V_n^* and $\tilde{V}_n$.

In [3] Berens and Xu using the K-functional

$$K^*(f,t)=\inf\left\{\|f-g\|_p+t\,\|P(D)g\|_p : f-g, P(D)g\in L_p[0,1]\right\}$$

where

$$P(D)=\frac{d}{dx}\left(x(1-x)\frac{d}{dx}\right),$$

proved the direct and a week converse inequality for B_n^* in terms of $K^*(f,t)$. Later, Chen and Ditzian [11] proved the strong converse inequality of type B, and Gonska and Zhou [9] of type A in terminology of [4].

In this article we investigate the approximation of functions in L_p norm by $\tilde{V}_n(f,x)$. We define an appropriate K-functional $\tilde{K}(f,t)_p$ (used for the first time by Berdisheva [2] in order to prove the direct theorem for Baskakov-Durrmeyer operator and later on by Heilman and Wagner [8] to prove the converse theorem) and prove a direct inequality with no other terms on the right-hand side than $\tilde{K}(f,t)_p$ and the strong converse inequality of type B.

Before stating our main result, let us introduce the needed notations.

By $\psi(x) = x(1+x)$ we denote the weight which is naturally connected with the second derivative of Baskakov operator. The first derivative operator is denoted by $D = \frac{d}{dx}$. Thus, $Dg(x) = g'(x)$ and $D^k g(x) = g^{(k)}(x)$ for every natural k. We define a differential operator $\tilde{D}$ by the formula

$$\tilde{D} = \frac{d}{dx}\left(\psi(x)\frac{d}{dx}\right) = D\psi D.$$

The space $AC_{loc}(0,\infty)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b] \subset (0,\infty)$.

$$\begin{aligned} W_p^k[0,\infty) &= \{f : D^r f \in AC_{loc}(0,\infty), r = 0,1,\dots k-1, D^k f \in L_p[0,\infty)\}, \\ W_p^k(\psi)[0,\infty) &= \{f : D^{r-1} f \in AC_{loc}(0,\infty), \psi^{k/2} D^k f \in L_p[0,\infty)\}, \\ \tilde{W}_p[0,\infty) &= \{f : f, Df \in AC_{loc}(0,\infty), \tilde{D}f \in L_p[0,\infty), \lim_{x\to 0_+} xDf(x) = 0\}, \end{aligned}$$

$$L_p[0,\infty) + \tilde{W}_p[0,\infty) = \left\{f : f = f_1 + f_2, f_1 \in L_p[0,\infty), f_2 \in \tilde{W}_p[0,\infty)\right\}.$$

Also, we define the K-functional $\tilde{K}(f,t)_p$ by the formula

$$\tilde{K}(f,t)_p = \inf\left\{\|f-g\|_p + t\left\|\tilde{D}g\right\|_p : f-g \in L_p[0,\infty), g \in \tilde{W}_p[0,\infty)\right\}. \quad (5)$$

And the relation $\theta_1(f,t)$ is equivalent to $\theta_2(f,t)$, i.e. $\theta_1(f,t) \sim \theta_2(f,t)$ means that there exists a positive constant independent of f and t such that

$$C^{-1}\theta_1(f,t) \le \theta_2(f,t) \le C\theta_1(f,t).$$

Our main result is the following theorem.

Theorem 1. *For $\tilde{V}_n$ defined by (4), the K-functional given by (5), $1 < p \le \infty$ and for every $f \in L_p[0,\infty) + \tilde{W}_p[0,\infty)$, there exist absolute constants $R, C > 0$ such that for every natural $l \ge Rn$*

$$C^{-1}\|\tilde{V}_n f - f\|_p \le \tilde{K}\left(f, \frac{1}{n}\right)_p \le C\frac{l}{n}\left(\|\tilde{V}_n f - f\|_p + \|\tilde{V}_l f - f\|_p\right). \quad (6)$$

The left inequality is true for $p = 1$ as well.

Both inequalities in Theorem 1 are stronger than the results mentioned above. Indeed, from the simple inequality

$$\left\|\tilde{D}f\right\|_p \le \|D\psi Df\|_p + \left\|\psi D^2 f\right\|_p$$

and from [5, Theorem 9.5.3-a), c)] it follows that

$$\left\|\tilde{D}f\right\|_p \le C\left(\left\|\psi D^2 f\right\|_p + \|f\|_p\right)$$

and consequently

$$\tilde{K}\left(f, \frac{1}{n}\right)_p \le C\left[w^2_{\sqrt{\psi}}\left(f, n^{-1/2}\right)_p + n^{-1}\|f\|_p\right].$$

At the same time, $\tilde{K}\left(f, \frac{1}{n}\right)_p$ is not equivalent to $w^2_{\sqrt{\psi}}\left(f, n^{-1/2}\right)_p + n^{-1}\|f\|_p$. For instance, for $p = 1$ and $f(x) = (1 + x)^{-1}$ we have $\tilde{K}\left(f, \frac{1}{n}\right)_1 \le n^{-1}$ and $\|f\|_1 = \infty$.

Remark. Another way to state Theorem 1 is: there exists an integer k such that

$$\tilde{K}\left(f, \frac{1}{n}\right)_p \sim \|\tilde{V}_n f - f\|_p + \|\tilde{V}_{kn} f - f\|_p, \quad p > 1.$$

For the rest of this paper the constants C and C_i will always be absolute constants, which means they do not depend on f, l and n. The constant C may be different on each occurrence.

2. Auxiliary Results

In this section we gather some properties of V_n, $\tilde{V}_n$ and $V_{n,k}$, which can be found in [1,5,7] and prove all the needed lemmas.

$$V_n \quad \text{and} \quad \tilde{V}_n \quad \text{are linear, positive operators with} \quad \|V_n f\|_\infty \le \|f\|_\infty \tag{7}$$

and

$$\|\tilde{V}_n f\|_p \le \|f\|_p. \tag{8}$$

$$V_n(1, x) = 1, \quad V_n(t - x, x) = 0, \quad V_n\left((t - x)^2, x\right) = \frac{\psi(x)}{n}. \tag{9}$$

$$\tilde{V}_n(1, x) = 1, \quad \tilde{V}_n(t - x, x) = \frac{2x + 1}{2(n - 1)} = \frac{1}{2(n - 1)} D\psi(x). \tag{10}$$

$$(DV_{n,k})(x) = DV_{n,k}(x) = n\Big(V_{n+1,k-1}(x) - V_{n+1,k}(x)\Big). \tag{11}$$

$$DV_{n,k}(x) = \frac{n}{\psi(x)}\left(\frac{k}{n} - x\right)V_{n,k}(x). \tag{12}$$

The next three inequalities are valid for all integers m and (16) for every natural m [5]. The constant C depends only on m.

$$\sum_{k=1}^{\infty}\left(\frac{n}{k}\right)^m V_{n,k}(x) \le Cx^{-m}, \tag{13}$$

$$\sum_{k=0}^{\infty}\left(1 + \frac{k}{n}\right)^m V_{n,k}(x) \le C(1 + x)^m, \tag{14}$$

$$V_n\left((t - x)^{2m}, x\right) \le C\left(\frac{\psi(x)}{n}\right)^m \quad \text{for} \quad x \ge \frac{1}{n}, \tag{15}$$

$$D^m\tilde{V}_n(f, x) = \frac{(n + m - 1)!}{(n - 1)!}\sum_{k=0}^{\infty}\varDelta^m a_k(n - 1)V_{n+m,k}(x) \tag{16}$$

where

$$a_k(n) = n\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)du, \quad \Delta a_k = a_{k+1} - a_k, \quad \Delta^m a_k = \Delta\left(\Delta^{m-1} a_k\right).$$

By simple computations one can verify the next identities.

$$\int_0^\infty V_{n,k}(x)dx = \frac{1}{n-1}, \tag{17}$$

$$\int_0^\infty xV_{n,k}(x)dx = \frac{k+1}{(n-1)(n-2)}, \tag{18}$$

$$\int_0^\infty x^2V_{n,k}(x)dx = \frac{(k+1)(k+2)}{(n-1)(n-2)(n-3)}, \tag{19}$$

$$\int_0^\infty \left(x-\frac{k}{n}\right)^2 V_{n,k}(x)dx = \frac{1}{n^2}\psi\left(\frac{k}{n}\right) + O\left(\frac{1}{n^3}\right). \tag{20}$$

For every natural $r \geq 1$ [5, p. 118, 9.3.5]

$$\|\psi^r D^{2r}\tilde{V}_n f\|_p \leq Cn^r\|f\|_p \quad \text{for} \quad f \in L_p[0,\infty). \tag{21}$$

The next lemma is crucial for the proof of the direct inequality of Theorem 1.

Lemma 1. *For $V_{n,k}(x)$ defined by (3) the next identity is true:*

$$\sum_{k=1}^{n-1}\frac{1}{k(1+x)^k} = \sum_{k=0}^{\infty}\left(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{n+k-1}\right)V_{n,k}(x). \tag{22}$$

Proof. Let

$$S(y) = \sum_{k=0}^{\infty}\left(\frac{y^{k+1}}{k+1}+\frac{y^{k+2}}{k+2}+\cdots+\frac{y^{n+k-1}}{n+k-1}\right)V_{n,k}(x).$$

Then we get for the derivative of $S(y)$ (after straightforward computations):

$$DS(y) = \frac{1-y^{n-1}}{1-y}\frac{1}{(1+x-xy)^n}.$$

Computing the integral we obtain

$$S(y) = -\sum_{k=1}^{n-1}\frac{1}{k(1+x-xy)^k} + \sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{y}{1+x-xy}\right)^k + \ln\frac{1+x}{x} + constant.$$

From $S(0) = 0$ it follows that

$$S(y) = -\sum_{k=1}^{n-1}\frac{1}{k(1+x-xy)^k} + \sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{y}{1+x-xy}\right)^k + \sum_{k=1}^{n-1}\frac{1}{k(1+x)^k}$$

and

$$S(1) = \sum_{k=0}^{\infty}\left(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{n+k-1}\right)V_{n,k}(x) = \sum_{k=1}^{n-1}\frac{1}{k(1+x)^k}.$$

The lemma is proved. □

Lemma 2. *For $p > 1$ and every $g \in \tilde{W}_p[0,\infty)$ we have*

$$\|Dg\|_p \le C \left\|\tilde{D}g\right\|_p, \tag{23}$$

$$\|D\psi Dg\|_p \le C \left\|\tilde{D}g\right\|_p, \tag{24}$$

$$\left\|\psi D^2 g\right\|_p \le C \left\|\tilde{D}g\right\|_p. \tag{25}$$

Proof. We have

$$(1+x)Dg(x) = \frac{1}{x}\int_0^x \tilde{D}g(t)dt$$

and consequently

$$\left|(1+x)Dg(x)\right| \le \frac{1}{x}\int_0^x \left|\tilde{D}g(t)\right| dt \le M(\tilde{D}g, x)$$

where

$$M(h,x) = \sup_{x\in\Delta} \frac{1}{|\Delta|}\int_\Delta |h(t)|dt$$

is the Hardy's maximal function. Now, from the Hardy's inequality (for $p > 1$) we get

$$\|(1+x)Dg(x)\|_p \le C \left\|M(\tilde{D}g, x)\right\|_p \le C \left\|\tilde{D}g\right\|_p.$$

And (23) and (24) follow from the obvious

$$\|Dg\|_p \le \|(1+x)Dg(x)\|_p$$

and

$$\|D\psi Dg\|_p \le 2\,\|(1+x)Dg(x)\|_p\,.$$

We obtain (25) from (24) and the simple inequality

$$\left\|\psi D^2 g\right\|_p \le \left\|\psi D^2 g + D\psi Dg\right\|_p + \|D\psi Dg\|_p\,.$$

□

We will prove the converse inequality of Theorem 1 by method, suggested by Ditzian and Ivanov in [4] and based on using the second iteration of Baskakov-Kantorovich operator in the K-functional. But before using it we need to prove all needed inequalities some of which are important of their own.

Lemma 3. *For $1 \le p \le \infty$ we have*

$$\left\|D\tilde{V}_n f\right\|_p \le Cn\,\|f\|_p \quad \textit{for} \quad f \in L_p[0,\infty), \tag{26}$$

$$\left\|D^2\tilde{V}_n f\right\|_p \le Cn\,\|Df\|_p \quad \textit{for} \quad f \in W_p^1[0,\infty), \tag{27}$$

$$\left\|D^3\tilde{V}_n f\right\|_p \le Cn^2\,\|Df\|_p \quad \textit{for} \quad f \in W_p^1[0,\infty). \tag{28}$$

Proof. Equation (26) follows from (16) and (8).

The proofs of (27) and (28) are analogous. We prove them for $p=1$ and $p=\infty$ and apply the Riesz-Thorin theorem.

Let us prove (28).

1. $p=1$. From (16) and (17) we have

$$\begin{aligned}\|D^3\tilde{V}_n(f,x)\|_1 &= \left\| n(n+1)(n+2)\sum_{k=0}^{\infty}\Delta^3 a_k(n-1)V_{n+3,k}(x)\right\|_1\\ &\le Cn^3\sum_{k=0}^{\infty}\left|\Delta^3 a_k(n-1)\right|\int_0^{\infty}V_{n+3,k}(x)dx \le Cn^2\sum_{k=0}^{\infty}\left|\Delta^3 a_k(n-1)\right|\\ &\le 4Cn^2\sum_{k=0}^{\infty}\left|\Delta a_k(n-1)\right| = Cn^2\sum_{k=0}^{\infty}\left|\Delta a_k(n-1)\right|.\end{aligned}$$

Now,

$$\begin{aligned}\left|\Delta a_k(n-1)\right| &= (n-1)\left|\int_{\frac{k+1}{n-1}}^{\frac{k+2}{n-1}} f(t)dt - \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)dt\right|\\ &= (n-1)\left|\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\left(\int_0^{\frac{1}{n-1}} Df(u+v)dv\right)du\right| \le \int_{\frac{k}{n-1}}^{\frac{k+2}{n-1}}|Df(u)|du.\end{aligned}$$

Consequently,

$$\sum_{k=0}^{\infty}\left|\Delta a_k(n-1)\right| \le 2\left\|Df\right\|_1$$

and (28) is proved for $p=1$.

2. $p=\infty$.

$$\|D^3\tilde{V}_n(f,x)\|_\infty \le Cn^3\max_k|\Delta a_k(n-1)|.$$

The proof is similar to the case $p=1$. □

The next lemma is a Voronovkaya type of inequality.

Lemma 4. *For $p>1$ and every function $f\in C^3[0,\infty)$ such that the right-hand side of the inequality is finite, we have*

$$\begin{aligned}&\left\|\tilde{V}_n f - f - \frac{1}{2(n-1)}\tilde{D}f\right\|_p\\ &\quad\le C\left\{n^{-3/2}\|\psi^{3/2}D^3f\|_p + n^{-2}\|D^2f\|_p + n^{-2}\|\psi D^2 f\|_p + n^{-3}\|D^3f\|_p\right\}.\end{aligned}\tag{29}$$

Remark. Later on we will apply this lemma for $\tilde{V}_n^2 f$ where $f\in L_p[0,\infty)+\tilde{W}_p[0,\infty)$ for which the above conditions hold.

Proof. By Taylor's formula

$$f(u) = f(x) + (u-x)Df(x) + \frac{1}{2}(u-x)^2D^2f(x) + \frac{1}{2}\int_x^u (u-v)^2D^3f(v)dv.$$

After integrating from $\frac{k}{n-1}$ to $\frac{k+1}{n-1}$ with respect to u, multiplying by $(n-1)V_{n,k}(x)$ and summing with respect to k, we obtain:

$$\begin{aligned}\tilde{V}_n(f,x) &= f(x) + \frac{\psi'(x)}{2(n-1)}Df(x) + \frac{(n+1)\psi(x)+1/3}{2(n-1)^2}D^2f(x) + I_n\\ &= f(x) + \frac{1}{2(n-1)}\tilde{D}^2f(x) + \frac{\psi(x)+1/6}{(n-1)^2}D^2f(x) + I_n,\end{aligned}$$

where

$$I_n = \frac{n-1}{2}\sum_{k=0}^{\infty}V_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\left(\int_x^u (u-v)^2D^3f(v)dv\right)du.$$

Now we estimate I_n.

Case 1. $x < \frac{1}{n}$.

We will estimate terms in the sum of I_n separately for $k = 0, 1, 2$ and for $k \geq 3$.

$$\begin{aligned}&\frac{n-1}{2}V_{n,0}(x)\left|\int_0^{\frac{1}{n-1}}\left(\int_x^u (u-v)^2D^3f(v)dv\right)du\right|\\ &\quad\leq \frac{n-1}{2(1+x)^n}\int_0^{\frac{1}{n-1}}(u-x)^2du\left|\int_x^u D^3f(v)dv\right|\\ &\quad\leq \frac{n-1}{2}\int_0^{\frac{1}{n-1}}|u-x|^3du\left|\frac{1}{u-x}\int_x^u D^3f(v)dv\right| \leq \frac{C}{n^3}M(D^3f,x),\\ &\frac{n-1}{2}V_{n,1}(x)\left|\int_{\frac{1}{n-1}}^{\frac{2}{n-1}}\left(\int_x^u (u-v)^2D^3f(v)dv\right)du\right|\\ &\quad\leq \frac{(n-1)nx}{2(1+x)^n}\int_{\frac{1}{n-1}}^{\frac{2}{n-1}}|u-x|^3du\left|\frac{1}{u-x}\int_x^u D^3f(v)dv\right| \leq \frac{C}{n^3}M(D^3f,x).\end{aligned}$$

Analogously for $k = 2$.

$$\begin{aligned}&\frac{n-1}{2}\sum_{k=3}^{\infty}V_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\left(\int_x^u (u-v)^2D^3f(v)dv\right)du\\ &\quad\leq \frac{n-1}{2}\sum_{k=3}^{\infty}V_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}(u-x)^3\left|\frac{1}{u-x}\int_x^u D^3f(v)dv\right|du\\ &\quad\leq C\sum_{k=3}^{\infty}V_{n,k}(x)\left(\frac{k}{n}\right)^3M(D^3f,x)\\ &\quad= \frac{9(n+1)(n+2)}{n^2}x^3\sum_{k=3}^{\infty}V_{n,k}(x)M(D^3f,x) \leq \frac{C}{n^3}M(D^3f,x).\end{aligned}$$

Case 2. $x \geq \frac{1}{n}$.

(a) $u \geq x$.

$$\begin{aligned} I_n &= \frac{n-1}{2}\sum_{k=0}^{\infty} V_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\left(\int_x^u \frac{(u-v)^2}{\psi^{3/2}(v)}\left|\psi^{3/2}(v)D^3f(v)\right|dv\right)du \\ &\leq \frac{n-1}{2}\sum_{k=0}^{\infty} V_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\left(\int_x^u \frac{(u-x)^2}{\psi^{3/2}(x)}\left|\psi^{3/2}(v)D^3f(v)\right|dv\right)du \\ &= \frac{n-1}{2}\sum_{k=0}^{\infty}\psi^{-3/2}(x)V_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}(u-x)^3\left|\frac{1}{u-x}\int_x^u\left|\psi^{3/2}(v)D^3f(v)\right|dv\right|du \\ &\leq \frac{n-1}{2}\left\{\sum_{k=0}^{\infty}\psi^{-3/2}(x)V_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}(u-x)^3du\right\}M(\psi^{3/2}D^3f,x) \\ &\leq C\psi^{-3/2}(x)M(\psi^{3/2}D^3f,x)\sum_{k=0}^{\infty}V_{n,k}(x)\left(\left|x-\frac{k}{n}\right|+\frac{1}{n}\right)^3. \end{aligned}$$

By using Cauchy's inequality, (9) and (15) we obtain for $m = 1, 2, 3$

$$\begin{aligned} &\sum_{k=0}^{\infty}V_{n,k}(x)\left|x-\frac{k}{n}\right|^m \\ &\quad\leq\left\{\sum_{k=0}^{\infty}V_{n,k}(x)\left(x-\frac{k}{n}\right)^2\right\}^{1/2}\left\{\sum_{k=0}^{\infty}V_{n,k}(x)\left(x-\frac{k}{n}\right)^{2(m-1)}\right\}^{1/2} \\ &\quad\leq C\left(\frac{\psi(x)}{n}\right)^{m/2}. \end{aligned}$$

From all this it follows

$$I_n \leq \frac{C}{n^{3/2}}M(\psi^{3/2}D^3f,x).$$

(b) $u \leq x$.

$$\begin{aligned} I_n &\leq \frac{n-1}{2}\sum_{k=0}^{\infty} V_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\left(\int_x^u \frac{(u-v)^2}{\psi^{3/2}(v)}\left|\psi^{3/2}(v)D^3f(v)\right|dv\right)du \\ &\leq \frac{n-1}{2}\sum_{k=0}^{\infty} V_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\frac{1}{(1+u)^{3/2}}\int_u^x \frac{(u-v)^2}{v^{3/2}}\left|\psi^{3/2}(v)D^3f(v)\right|dvdu \\ &\leq \frac{n-1}{2}\sum_{k=0}^{\infty} V_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\frac{(u-x)^2}{x^{3/2}(1+u)^{3/2}}\int_u^x\left|\psi^{3/2}(v)D^3f(v)\right|dvdu \\ &\leq \frac{n-1}{2}x^{-3/2}M(\psi^{3/2}f''',x)\sum_{k=0}^{\infty}V_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\frac{(x-u)^3}{(1+u)^{3/2}}du \\ &\leq Cx^{-3/2}M(\psi^{3/2}D^3f,x)\sum_{k=0}^{\infty}V_{n,k}(x)\left(1+\frac{k}{n}\right)^{-3/2}\left(\left|x-\frac{k}{n}\right|+\frac{1}{n}\right)^3. \end{aligned}$$

Again, by using Cauchy's inequality, (9), (14) and (15) we have

$$\begin{aligned} & x^{-3/2}M(\psi^{3/2}D^3f,x)\sum_{k=0}^{\infty}V_{n,k}(x)\left(1+\frac{k}{n}\right)^{-3/2}\left(\left|x-\frac{k}{n}\right|+\frac{1}{n}\right)^3 \\ & \le x^{-3/2}M(\psi^{3/2}D^3f,x)\left\{\sum_{k=0}^{\infty}V_{n,k}(x)\left(1+\frac{k}{n}\right)^{-3}\right\}^{1/2} \\ & \times\left\{\sum_{k=0}^{\infty}V_{n,k}(x)\left(\left|x-\frac{k}{n}\right|+\frac{1}{n}\right)^6\right\}^{1/2} \\ & \le Cn^{-3/2}M(\psi^{3/2}D^3f,x). \end{aligned}$$

By using the Hardy's inequality about maximal function (for $p>1$) we complete the proof of the lemma. □

We also need the next Bernstein type of inequality for $\psi^{3/2}D^3\tilde{V}_nf$.

Lemma 5. *For* $1\le p\le\infty$ *and* $f\in W_p^2(\psi)[0,\infty)$

$$\left\|\psi^{3/2}D^3\tilde{V}_nf\right\|_p\le n^{1/2}\left\|\psi D^2f\right\|_p. \tag{30}$$

Proof. From (16) we have

$$\begin{aligned} \psi^{3/2}(x)D^3\tilde{V}_n(f,x) &= n(n+1)\sum_{k=0}^{\infty}\Delta^2a_k(n-1)DV_{n+2,k}(x) \\ &= n(n+1)(n+2)\psi^{1/2}(x)\sum_{k=0}^{\infty}\Delta^2a_k(n-1)\left(\frac{k}{n+2}-x\right)V_{n+2,k}(x). \end{aligned}$$

We will prove (30) for $p=1$ and $p=\infty$ and apply the Riesz-Thorin theorem.

1. $p=1$.

From the above representation of $\psi^{3/2}(x)D^3\tilde{V}_n(f,x)$ we obtain

$$\begin{aligned} & \left\|\psi^{3/2}D^3\tilde{V}_nf\right\|_1 \\ & \le Cn^3\sum_{k=0}^{\infty}\psi\left(\frac{k+1}{n-1}\right)\left|\Delta^2a_k(n-1)\right|\int_0^{\infty}\frac{\psi^{1/2}(x)}{\psi\left(\frac{k+1}{n-1}\right)}\left(\frac{k}{n+2}-x\right)V_{n+2,k}(x)dx \\ & = Cn^3\sum_{k=0}^{\infty}\psi\left(\frac{k+1}{n-1}\right)\left|\Delta^2a_k(n-1)\right|I_k. \end{aligned}$$

Now, by Cauchy's inequality

$$\begin{aligned} I_k &\le \psi^{-1}\left(\frac{k+1}{n-1}\right)\left\{\int_0^{\infty}\psi(x)V_{n+2,k}(x)dx\right\}^{1/2} \\ &\times\left\{\int_0^{\infty}\left(\frac{k}{n+2}-x\right)^2V_{n+2,k}(x)dx\right\}^{1/2}. \end{aligned}$$

For the first factor, using (17) we have

$$\begin{aligned}\int_0^\infty \psi(x)V_{n+2,k}(x)dx &= \frac{(n+k+1)(k+1)}{n(n+1)}\int_0^\infty V_{n,k+1}(x)dx\\ &= \frac{k+1}{n+1}\left(1+\frac{k+1}{n}\right)\frac{1}{n-1} \le \frac{C}{n}\psi\left(\frac{k+1}{n}\right).\end{aligned}$$

For the second one we use (20) to get

$$\int_0^\infty \left(\frac{k}{n+2}-x\right)^2 V_{n+2,k}(x)dx \le \frac{C}{n}\psi\left(\frac{k+1}{n}\right).$$

Consequently $I_k \le Cn^{-3/2}$ and

$$\left\|\psi^{3/2}D^3\tilde{V}_n f\right\|_1 \le Cn^{3/2}\sum_{k=0}^{\infty}\psi\left(\frac{k+1}{n}\right)\left|\Delta^2 a_k(n-1)\right|.$$

We consider two cases.

(a) $k=0$.

$$\begin{aligned}&\psi\left(\frac{k+1}{n}\right)\left|\Delta^2 a_k(n-1)\right|\\ &\quad= \frac{n^2-1}{n^2}\int_0^{\frac{1}{n-1}}\left[f\left(t+\frac{2}{n-1}\right)-2f\left(t+\frac{1}{n-1}\right)+f(t)\right]dt.\end{aligned}$$

Similarly to [5] we have

$$\psi\left(\frac{k+1}{n}\right)\left|\Delta^2 a_k(n-1)\right| \le Cn^{-1}\left\|\psi D^2 f\right\|_1.$$

(b) $k \ge 1$.

$$\psi\left(\frac{k+1}{n}\right)\left|\Delta^2 a_k(n-1)\right| = \psi\left(\frac{k+1}{n}\right)(n-1)\left(\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} -2\int_{\frac{k+1}{n-1}}^{\frac{k+2}{n-1}} + \int_{\frac{k+2}{n-1}}^{\frac{k+3}{n-1}}\right).$$

Since for $x \in [\frac{k}{n-1}, \frac{k+3}{n-1}]$ we have $\psi\left(\frac{k+1}{n}\right) \sim \psi(x)$ it follows

$$\psi\left(\frac{k+1}{n}\right)\left|\Delta^2 a_k(n-1)\right| \le Cn^{-1}\int_{\frac{k}{n-1}}^{\frac{k+3}{n-1}}\psi(t)D^2 f(t)dt$$

and

$$\begin{aligned}\sum_{k=1}^{\infty}\psi\left(\frac{k+1}{n}\right)\left|\Delta^2 a_k(n-1)\right| &\le Cn^{-1}\sum_{k=1}^{\infty}\int_{\frac{k}{n-1}}^{\frac{k+3}{n-1}}\psi(t)D^2 f(t)dt\\ &\le Cn^{-1}\left\|\psi D^2 f\right\|_1.\end{aligned}$$

2. $p=\infty$.

$$
\begin{aligned}
\left|\Delta^2 a_k(n-1)\right| &= (n-1)\left(\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)dt - 2\int_{\frac{k+1}{n-1}}^{\frac{k+2}{n-1}} f(t)dt + \int_{\frac{k+2}{n-1}}^{\frac{k+3}{n-1}} f(t)dt\right)\\
&= (n-1)\int_0^{\frac{1}{n-1}}\int_0^{\frac{1}{n-1}}\int_0^{\frac{1}{n-1}} D^2 f\left(\frac{k}{n-1}+t_1+t_2+t_3\right)dt_1dt_2dt_3\\
&\le (n-1)\|\psi D^2 f\|_\infty \int_0^{\frac{1}{n-1}}\int_0^{\frac{1}{n-1}}\int_0^{\frac{1}{n-1}} \psi^{-1}\left(\frac{k}{n-1}+t_1+t_2+t_3\right)dt_1dt_2dt_3\\
&\le \frac{n-1}{n+k-1}\|\psi D^2 f\|_\infty \int_0^{\frac{1}{n-1}}\int_0^{\frac{1}{n-1}}\int_0^{\frac{1}{n-1}} \frac{dt_1dt_2dt_3}{\frac{k}{n-1}+t_1+t_2+t_3}\\
&\le \frac{n-1}{n+k-1}\|\psi D^2 f\|_\infty \left(\int_0^{\frac{1}{n-1}} \frac{dt}{\sqrt[3]{\frac{k}{n-1}+t}}\right)^3\\
&\le Cn^{-3}\psi^{-1}\left(\frac{k+1}{n-1}\right)\|\psi D^2 f\|_\infty .
\end{aligned}
$$

Then

$$
\begin{aligned}
&\psi^{3/2}(x)D^3\tilde{V}_n(f,x)\\
&\quad\le Cn\psi^{1/2}(x)\|\psi D^2 f\|_\infty \sum_{k=0}^{\infty}\left|\frac{k}{n+2}-x\right|\psi^{-1}\left(\frac{k+1}{n-1}\right)V_{n+2,k}(x).
\end{aligned}
$$

By Cauchy's inequality

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\left|\frac{k}{n+2}-x\right|\psi^{-1}\left(\frac{k+1}{n-1}\right)V_{n+2,k}(x)\\
&\quad\le\left\{\sum_{k=0}^{\infty}\left(\frac{k}{n+2}-x\right)^2 V_{n+2,k}(x)\right\}^{1/2}\left\{\sum_{k=0}^{\infty}\psi^{-2}\left(\frac{k+1}{n-1}\right)V_{n+2,k}(x)\right\}^{1/2}.
\end{aligned}
$$

From (9) we have

$$
\sum_{k=0}^{\infty}\left(\frac{k}{n+2}-x\right)^2 V_{n+2,k}(x)=\frac{\psi(x)}{n+2}.
$$

Also

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\psi^{-2}\left(\frac{k+1}{n-1}\right)V_{n+2,k}(x)\\
&\quad=\psi^{-2}(x)\sum_{k=0}^{\infty}V_{n-2,k+2}(x)\frac{n+k+1}{n+k}\,\frac{k+2}{k+1}\,\frac{(n-1)^3}{n(n-2)(n+1)}\le C\psi^{-2}(x).
\end{aligned}
$$

The lemma is proved. □

3. Proof of the Main Result

Proof of the direct inequality of Theorem 1. We follow the approach, used by Berens and Xu in [3, pp. 25–46]. Obviously it is enough to establish the inequalities

$$\|f-\tilde{V}_n f\|_p \le C\|f\|_p \quad \text{for} \quad f\in L_p[0,\infty)$$

and

$$\|f-\tilde{V}_n f\|_p \le \frac{C}{n-1}\|\tilde{D}f\|_p \quad \text{for} \quad f\in \tilde{W}_p[0,\infty).$$

We need to prove only the second one because the first one is evident. We will prove it for $p=1$ and $p=\infty$ and by applying the Riesz-Thorin theorem we obtain it for every $1<p<\infty$.

For $\phi(z)=\ln z-\ln(1+z)$ we have

$$f(t)=f(x)+\psi(x)\left[\phi(t)-\phi(x)\right]Df(x)+\int_x^t\left[\phi(t)-\phi(u)\right]\tilde{D}f(u)du.$$

Applying $\tilde{V}_n$ to both sides and using (10) we obtain

$$\begin{aligned}&\tilde{V}_n(f,x)-f(x)\\&\quad=\psi(x)\left[\tilde{V}_n(\phi,x)-\phi(x)\right]Df(x)+\tilde{V}_n\left(\int_x^{(.)}\left[\phi(.)-\phi(u)\right]\tilde{D}f(u)du\right).\end{aligned}$$

Using arguments similar to [3], we see that we need to estimate L_1 and L_∞ norms of $\tilde{V}_n(\phi,x)-\phi(x)$. We have

$$\phi(x)=-\sum_{k=1}^{\infty}\frac{1}{k(1+x)^k}=-\sum_{k=1}^{n-1}\frac{1}{k(1+x)^k}-\sum_{k=n}^{\infty}\frac{1}{k(1+x)^k}.$$

Since the L_1 norm of the last term is

$$\left\|\sum_{k=n}^{\infty}\frac{1}{k(1+x)^k}\right\|_1=\frac{1}{n-1},$$

we need to estimate only $\|\tilde{V}_n(\phi,x)-h(x)\|_1$ where $h(x)=-\sum_{k=1}^{n-1}\frac{1}{k(1+x)^k}$. From the identity (22) of Lemma 1 we have

$$\begin{aligned}&\tilde{V}_n(\phi,x)-h(x)\\&\quad=\sum_{k=0}^{\infty}V_{n,k}(x)\left[(n-1)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\phi(t)dt+\sum_{i=1}^{n-1}\frac{1}{k+i}\right]=\sum_{k=0}^{\infty}V_{n,k}(x)\\&\qquad\times\left[k\ln\left(1+\frac{1}{k}\right)-(n+k-1)\ln\left(1+\frac{1}{n+k-1}\right)-\ln\frac{n+k}{k+1}+\sum_{i=1}^{n-1}\frac{1}{k+i}\right].\end{aligned}$$

By using (17) it is not difficult to see that $\|\tilde{V}_n(\phi,x)-h(x)\|_1\le\frac{C}{n-1}$.

In the case $p=\infty$

$$\begin{aligned}\psi(x)\left[\tilde{V}_n(\phi,x)-\phi(x)\right]Df(x)&=\left[\tilde{V}_n(\phi,x)-\phi(x)\right]\int_0^x\tilde{D}f(u)du\\&\le x\left[\tilde{V}_n(\phi,x)-\phi(x)\right]\|\tilde{D}f\|_\infty\end{aligned}$$

and

$$\tilde{V}_n\left(\int_x^{(.)}[\phi(.)-\phi(u)]\,\tilde{D}f(u)du\right)\le\|\tilde{D}f\|_\infty\tilde{V}_n\left(\int_x^{(.)}[\phi(.)-\phi(u)]\,du,x\right).$$

Let us denote by h_1 and h_2 the functions: $h_1(z)=z\phi(z)$ and $h_2(z)=\int^z\phi(v)dv$, i.e. $Dh_2(z)=\phi(z)$. Then

$$\tilde{V}_n\left(\int_x^{(.)}[\phi(.)-\phi(u)]\,du,x\right)$$
$$=\left[\tilde{V}_n\left(h_1,x\right)-h_1(x)\right]-x\left[\tilde{V}_n(\phi,x)-\phi(x)\right]-\left[\tilde{V}_n(h_2,x)-h_2(x)\right].$$

Now we consider two cases. For $x<\frac{1}{n}$ we use again the representation (22) of Lemma 1 and proceed as in the case for L_1 norm. For $x\ge\frac{1}{n}$ we can approximate $\tilde{V}_n$ by the classical Baskakov operator V_n and use the main result from [6, Theorem 1.1] (since $\|\psi D^2h_1\|_\infty\le 2$ and $\|\psi D^2h_2\|_\infty=1$). □

Proof of the converse inequality of Theorem 1. We take $g=\tilde{V}_n^2(f,x)=\tilde{V}_n\left(\tilde{V}_n(f,x)\right)$. Then

$$\|f-g\|_p=\|f-\tilde{V}_n^2f\|_p\le 2\|f-\tilde{V}_nf\|_p$$

because $\tilde{V}_n$ is a contraction.

Now we will estimate $\tilde{D}g$. From (29) of Lemma 4 we have

$$\frac{1}{2(l-1)}\|\tilde{D}\tilde{V}_n^2f\|_p\le\left\|\tilde{V}_l\tilde{V}_n^2f-\tilde{V}_n^2f\right\|_p+C\Big[l^{-3/2}\left\|\psi^{3/2}D^3\tilde{V}_n^2f\right\|_p$$
$$+l^{-2}\left\|D^2\tilde{V}_n^2f\right\|_p+l^{-2}\left\|\psi D^2\tilde{V}_n^2f\right\|_p+l^{-3}\left\|D^3\tilde{V}_n^2f\right\|_p\Big].$$

From (30) of Lemma 5:

$$\left\|\psi^{3/2}D^3\tilde{V}_n^2f\right\|_p\le C_1\sqrt{n}\left\|\psi D^2\tilde{V}_nf\right\|_p$$
$$\le C_1\sqrt{n}\left\|\psi D^2\left(\tilde{V}_nf-\tilde{V}_n^2f\right)\right\|_p+C_1\sqrt{n}\left\|\psi D^2\tilde{V}_nf\right\|_p.$$

From (27) of Lemma 3:

$$\left\|D^2\tilde{V}_n^2f\right\|_p\le C_2n\left\|D\tilde{V}_nf\right\|_p\le C_2n\left\|D\left(\tilde{V}_nf-\tilde{V}_n^2f\right)\right\|_p+C_2n\left\|D\tilde{V}_n^2f\right\|_p$$
$$\le C_2n\left\|D\left(\tilde{V}_nf-\tilde{V}_n^2f\right)\right\|_p+C_2n\left\|\tilde{D}\tilde{V}_n^2f\right\|_p.$$

From (25) of Lemma 2:

$$\left\|\psi D^2\tilde{V}_n^2f\right\|_p\le C_3\left\|\tilde{D}\tilde{V}_n^2f\right\|_p.$$

From (28) of Lemma 3 and (23) of Lemma 2:

$$\left\|D^3\tilde{V}_n^2f\right\|_p\le C_4n^2\left\|D\tilde{V}_nf\right\|_p\le C_4n^2\left\|D\left(\tilde{V}_nf-\tilde{V}_n^2f\right)\right\|_p+C_4n^2\left\|D\tilde{V}_n^2f\right\|_p$$
$$\le C_4n^2\left\|D\left(\tilde{V}_nf-\tilde{V}_n^2f\right)\right\|_p+C_4n^2\left\|\tilde{D}\tilde{V}_n^2f\right\|_p.$$

So,

$$\frac{1}{2(l-1)}\|\tilde{D}^2\tilde{V}_n^2 f\|_p \leq \left\|\tilde{V}_l\tilde{V}_n^2 f-\tilde{V}_n^2 f\right\|_p + CC_1 l^{-3/2}\sqrt{n}\left\|\psi D^2\left(\tilde{V}_n f-\tilde{V}_n^2 f\right)\right\|_p$$
$$+\left(CC_2 l^{-2}n+C_4 l^{-3}n^2\right)\left\|D\left(\tilde{V}_n f-\tilde{V}_n^2 f\right)\right\|_p$$
$$+\left(CC_1C_3 l^{-3/2}n^{1/2}+CC_2 l^{-2}n+CC_4 l^{-3}n^2\right)\left\|\tilde{D}\tilde{V}_n^2 f\right\|_p.$$

If we choose $l \geq Rn$ such that

$$CC_1C_3 l^{-3/2}n^{1/2}+CC_2 l^{-2}n+CC_4 l^{-3}n^2 \leq \frac{1}{4(l-1)}$$

we have

$$\frac{1}{4(l-1)}\|\tilde{D}\tilde{V}_n^2 f\|_p \leq \left\|\tilde{V}_l\tilde{V}_n^2 f-\tilde{V}_n^2 f\right\|_p + CC_1 l^{-3/2}\sqrt{n}\left\|\psi D^2\left(\tilde{V}_n f-\tilde{V}_n^2 f\right)\right\|_p$$
$$+\left(CC_2 l^{-2}n+C_4 l^{-3}n^2\right)\left\|D\left(\tilde{V}_n f-\tilde{V}_n^2 f\right)\right\|_p.$$

Since $\tilde{V}_l$ is a contraction we have

$$\left\|\tilde{V}_l\tilde{V}_n^2 f-\tilde{V}_n^2 f\right\|_p \leq 4\left\|\tilde{V}_n f-f\right\|_p+\left\|\tilde{V}_l f-f\right\|_p.$$

Also, from (21) it follows that

$$\left\|\psi D^2\left(\tilde{V}_n f-\tilde{V}_n^2 f\right)\right\|_p \leq C_5 n\left\|\tilde{V}_n f-f\right\|_p$$

and from (26):

$$\left\|D\left(\tilde{V}_n f-\tilde{V}_n^2 f\right)\right\|_p \leq C_6 n\left\|\tilde{V}_n f-f\right\|_p.$$

Then

$$\frac{1}{4(l-1)}\|\tilde{D}\tilde{V}_n^2 f\|_p \leq \left\|\tilde{V}_l f-f\right\|_p + l^{-3/2}\left(CC_1C_5 n^{3/2}\right.$$
$$\left.+CC_2C_6 l^{-1/2}n^2+C_4C_6 l^{-3/2}n^3\right)\left\|\tilde{V}_n f-f\right\|_p.$$

By taking R big enough we complete the proof of Theorem 1. □

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