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Statistical Relative Approximation on Modular Spaces

Kamil Demirci and Sevda Orhan

Abstract. In the present paper, using the concept of statistical relative convergence, we study the problem of approximation to a function by means of double sequences of positive linear operators defined on a modular space. Also, a non-trivial application is presented.

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1. Introduction and Preliminaries

Let $C(X)$ be the space of all continuous real valued functions on a compact subset X of the real numbers and (L_n) be the sequence of positive linear operators on $C(X)$ say $(L_n(f;x))$. Korovkin [15] established the sufficient conditions for the uniform convergence of (L_n) to a function f by using the test functions $1, x, x^2$. Many researchers have investigated these conditions for various operators defined on different spaces (see for instance [2,13,25]). Recently, Demirci and Orhan [9] introduced statistical relative uniform convergence of single sequences by using the notions of the natural density and the relative uniform convergence. Then, Yılmaz et al. [25] defined a new type of modular convergence by using the notion of the relative uniform convergence. More recently, Demirci and Kolay [10] studied statistical relative modular convergence of single sequences. In this paper, we investigate the problem of statistical relative approximation to a function f by means of double sequences of positive linear operators defined on a modular space.

Let us first remind of the concept of statistical convergence for double sequences.

A double sequence $x = \{x_{m,n}\}$ is said to be convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$, the set of all natural numbers, such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n > N$, where L is called the Pringsheim limit of x and denoted by $P - \lim_{m,n} x_{m,n} = L$ (see [23]). We shall call such an x, briefly, "P-convergent". A double sequence is called bounded if there exists a positive number M such that $|x_{m,n}| \leq M$ for all $(m, n) \in \mathbb{N}^2$. Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded.

Statistical convergence of single sequences was introduced by Steinhaus [24] and studied by many authors [11,12,14]. Recently, this concept was extended to the double sequences. If $E \subset \mathbb{N}^2$ is a two-dimensional subset of positive integers, then $E_{j,k}$ denotes the set $\{(m, n) \in E : m \leq j, n \leq k\}$ and $|E_{j,k}|$ denotes the cardinality of $E_{j,k}$. The double natural density of E [19] is given by

$$\delta_2(E) := P - \lim_{j,k} \frac{1}{jk} |E_{j,k}|,$$

if it exists. For example, let $E = \left\{\left(j^2, k^2\right) : j, k \in \mathbb{N}\right\}$ then $\delta_2(E) = 0$. The number sequence $x = \{x_{m,n}\}$ is statistically convergent to L provided that for every $\varepsilon > 0$, the set

$$E := E_{j,k}(\varepsilon) := \{m \leq j, n \leq k : |x_{m,n} - L| \geq \varepsilon\}$$

has natural density zero; in that case we write $st_2 - \lim_{m,n} x_{m,n} = L$.

Clearly, a P-convergent double sequence is statistically convergent to the same value but its converse is not always true. Also, note that a statistically convergent double sequence need not to be bounded. For example, consider the double sequence $x = \{x_{m,n}\}$ given by

$$x_{m,n} = \begin{cases} mn, & m = k^2 \text{ and } n = l^2 \\ 1, & \text{otherwise.} \end{cases}, \quad k, l = 1, 2, \ldots$$

Then, clearly $st_2 - \lim_{m,n} x_{m,n} = 1$. Nevertheless, x is neither convergent nor bounded.

Also, the statistical convergence for double sequences was characterized in [19] as given below:

A double sequence $x = \{x_{m,n}\}$ is statistically convergent to L if and only if there exists a set $S \subset \mathbb{N}^2$ such that the natural density of S is 1 and

$$P - \lim_{m,n \to \infty \text{ and } (m,n) \in S} x_{m,n} = L.$$

The concepts of *statistical superior limit* and *inferior limit* for double sequences have been introduced by Çakan and Altay [5]. For any real double sequence $x = \{x_{m,n}\}$, the statistical superior limit of x is

$$st_2 - \limsup_{m,n} x_{m,n} = \begin{cases} \sup G_x, & \text{if } G_x \neq \varnothing, \\ -\infty, & \text{if } G_x = \varnothing, \end{cases}$$

where $G_x := \{C \in \mathbb{R} : \delta_2(\{(m,n) : x_{m,n} > C\}) \neq 0\}$ and $\varnothing$ denotes the empty set. We note that, in general, by $\delta_2(K) \neq 0$ we mean either $\delta_2(K) > 0$ or K fails to have the double natural density. Similarly, the statistical inferior limit of x is

$$st_2 - \liminf_{m,n} x_{m,n} = \begin{cases} \inf F_x, & \text{if } F_x \neq \varnothing, \\ \infty, & \text{if } F_x = \varnothing, \end{cases}$$

where $F_x := \{D \in \mathbb{R} : \delta_2(\{(m,n) : x_{m,n} < D\}) \neq 0\}$. As in the ordinary superior or inferior limit, it was proved that

$$st_2 - \liminf_{m,n} x_{m,n} \leq st_2 - \limsup_{m,n} x_{m,n}$$

and also that, for any double sequence $x = \{x_{m,n}\}$ satisfying
$\delta_2(\{(m,n) : |x_{m,n}| > M\}) = 0$ for some $M > 0$,

$$st_2 - \lim_{m,n} x_{m,n} = L \text{ iff } st_2 - \liminf_{m,n} x_{m,n} = st_2 - \limsup_{m,n} x_{m,n} = L.$$

Now, we focus on modular spaces.

Let $I = [a,b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, by $X(I^2)$ we denote the space of all real-valued measurable functions on $I^2 = [a,b] \times [a,b]$ provided with equality a.e. As usual, let $C(I^2)$ denote the space of all continuous real-valued functions, and $C^\infty(I^2)$ denote the space of all infinitely differentiable functions on I^2. In this case, we say that a functional $\rho : X(I^2) \to [0,+\infty]$ is a *modular* on $X(I^2)$ provided that the following conditions hold:

(i) $\rho(f) = 0$ if and only if $f = 0$ a.e. in I^2,
(ii) $\rho(-f) = \rho(f)$ for every $f \in X(I^2)$,
(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ for every $f, g \in X(I^2)$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Recall that a modular ρ is called *N-quasi convex* if the following is satisfied:

- there exists a constant $N \geq 1$ such that

$$\rho(\alpha f + \beta g) \leq N\alpha\rho(Nf) + N\beta\rho(Ng)$$

holds for every $f, g \in X(I^2)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. In particular, if $N = 1$, then ρ is called *convex*.

Furthermore, a modular ρ is called *N-quasi semiconvex* if it holds:

- there exists a constant $N \geq 1$ such that

$$\rho(af) \leq Na\rho(Nf)$$

holds for every $f \in X(I^2)$ and $a \in (0,1]$.

It is clear that every N-quasi semiconvex modular is N-quasi convex. We should recall that the above two concepts were introduced and discussed in details by Bardaro et al. [3,4].

We now consider some appropriate vector subspaces of $X(I^2)$ by means of a modular ρ as follows:

$$L^{\rho}\left(I^2\right) := \left\{ f \in X\left(I^2\right) : \lim_{\lambda \to 0^+} \rho\left(\lambda f\right) = 0 \right\}$$

and

$$E^{\rho}\left(I^2\right) := \left\{ f \in L^{\rho}\left(I^2\right) : \rho\left(\lambda f\right) < +\infty \text{ for all } \lambda > 0 \right\}.$$

Here, $L^{\rho}\left(I^2\right)$ is called the *modular space* generated by ρ; and $E^{\rho}\left(I^2\right)$ is called the space of the finite elements of $L^{\rho}\left(I^2\right)$. Observe that if ρ is N-quasi semiconvex, then the space

$$\left\{ f \in X\left(I^2\right) : \rho\left(\lambda f\right) < +\infty \text{ for some } \lambda > 0 \right\}$$

coincides with $L^{\rho}\left(I^2\right)$. The notions about modulars are introduced in [20] and widely discussed in [4] (see also [16, 21]).

Moore [18] introduced the notion of uniform convergence of a sequence of functions relative to a scale function. Then, Chittenden [7] gave the following definition of relative uniform convergence is equivalent to the definition given by Moore:

A sequence (f_n) of functions, defined on an interval $I \equiv (a \leq x \leq b)$, converges *relatively uniformly to a limit function* f if there exists a function $\sigma(x)$, called a scale function $\sigma(x)$ such that for every $\varepsilon > 0$ there is an integer n_{ε} such that for every $n > n_{\varepsilon}$ the inequality $|f_n(x) - f(x)| < \varepsilon |\sigma(x)|$ holds uniformly in x on the interval I. The sequence (f_n) is said to converge *uniformly relatively to the scale function* σ or more simply, *relatively uniformly.* It will be observed that uniform convergence is the special case of relative uniform convergence in which the scale function is a non-zero constant (for more properties and details, see also [6–8]).

Now we introduce the notions of the *relative modular (or strong) convergence* and *statistical relative modular (or strong) convergence* for double sequences as follows:

Definition 1. Let $\{f_{m,n}\}$ be a double function sequence whose terms belong to $L^{\rho}\left(I^2\right)$. Then, $\{f_{m,n}\}$ is *relatively modularly convergent* to a function $f \in L^{\rho}\left(I^2\right)$ iff there exists a function $\sigma(x, y)$, called a scale function $\sigma \in X\left(I^2\right), |\sigma(x, y)| \neq 0$ such that

$$P - \lim_{m,n} \rho\left(\lambda_0 \left(\frac{f_{m,n} - f}{\sigma}\right)\right) = 0 \quad \text{for some } \lambda_0 > 0. \tag{1}$$

And also, $\{f_{m,n}\}$ is *relatively F-norm convergent* (or, *relatively strongly convergent*) to f iff

$$P - \lim_{m,n} \rho\left(\lambda \left(\frac{f_{m,n} - f}{\sigma}\right)\right) = 0 \quad \text{for every } \lambda > 0. \tag{2}$$

It can be immediately seen that (1) and (2) are equivalent if and only if the modular ρ satisfies the Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\rho(2f) \leq M\rho(f)$ for every $f \in X\left(I^2\right)$. Indeed, relative strong convergence of the double sequence $\{f_{m,n}\}$ to f is equivalent to the condition $P - \lim_{m,n} \rho\left(2^N \lambda \left(\frac{f_{m,n}-f}{\sigma}\right)\right) = 0$, for all $N = 1, 2, \ldots$ and some $\lambda > 0$. Let $\{f_{m,n}\}$ be relatively modularly convergent to f, hence there exists a $\lambda > 0$ such that $P - \lim_{m,n} \rho\left(\lambda \left(\frac{f_{m,n}-f}{\sigma}\right)\right) = 0$.$\Delta_2$-condition implies by induction that $\rho\left(2^N \lambda \left(\frac{f_{m,n}-f}{\sigma}\right)\right) \leq M^N \rho\left(\lambda \left(\frac{f_{m,n}-f}{\sigma}\right)\right)$, then we get

$$P - \lim_{m,n} \rho\left(2^N \lambda \left(\frac{f_{m,n}-f}{\sigma}\right)\right) = 0.$$

Definition 2. Let $\{f_{m,n}\}$ be a function sequence whose terms belong to $L^\rho\left(I^2\right)$. Then, $\{f_{m,n}\}$ is said to be *statistically relatively modularly convergent* to a function $f \in L^\rho\left(I^2\right)$ if there exists a function $\sigma(x, y)$, called a scale function $\sigma \in X\left(I^2\right), |\sigma(x, y)| \neq 0$ such that

$$st_2 - \lim_{m,n} \rho\left(\lambda_0 \left(\frac{f_{m,n}-f}{\sigma}\right)\right) = 0 \quad \text{for some } \lambda_0 > 0.$$

Also, $\{f_{m,n}\}$ *is statistically relatively F-norm convergent* (or, *statistically relatively strongly convergent*) to f iff

$$st_2 - \lim_{m,n} \rho\left(\lambda \left(\frac{f_{m,n}-f}{\sigma}\right)\right) = 0 \quad \text{for every } \lambda > 0.$$

It will be observed that statistical modular convergence is the special case of statistical relative modular convergence in which the scale function is a non-zero constant (cf. [22]). Moreover, if $\sigma(x, y)$ is bounded, statistical relative modular convergence implies statistical modular convergence. However, statistical relative modular convergence does not imply statistical modular convergence, when $\sigma(x, y)$ is unbounded. This is illustrated by the following example:

Example 1. Take $I = [0, 1]$ and let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function for which the following conditions hold:

- φ is convex
- $\varphi(0) = 0, \varphi(u) > 0$ for $u > 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$.

Hence, consider the functional ρ^φ on $X(I^2)$ defined by

$$\rho^\varphi(f) := \int_0^1 \int_0^1 \varphi(|f(x, y)|)\, dxdy \qquad \text{for } f \in X\left(I^2\right).$$

In this case, ρ^φ is a convex modular on $X\left(I^2\right)$, which satisfies all assumptions listed in this section. Consider the Orlicz space generated by φ as follows:

$$L_{\varphi}^{\rho}(I^2) := \left\{ f \in X\left(I^2\right) : \rho^{\varphi}(\lambda f) < +\infty \text{ for some } \lambda > 0 \right\}.$$

For each $m, n \in N$, define $g_{m,n} : [0,1] \times [0,1] \to \mathbb{R}$ by

$$g_{m,n}(x,y) = \begin{cases} 1, & m = k^2 \text{ and } n = l^2 \\ m^2 n(1 - mnxy), & (x,y) \in \left(0, \frac{1}{m}\right) \times \left(0, \frac{1}{n}\right); \\ & m \neq k^2 \text{ and } n \neq l^2 \\ 0, & (x,y) = (0,0) \text{ or } (x,y) \in \left[\frac{1}{m}, 1\right] \times \left[\frac{1}{n}, 1\right]; \\ & m \neq k^2 \text{ and } n \neq l^2 \end{cases},$$
$$k, l = 1, 2, \ldots \tag{3}$$

If $\varphi(x) = x^p$ for $1 \leq p < \infty$, $x \geq 0$, then $L_{\varphi}^{\rho}\left(I^2\right) = L_p\left(I^2\right)$. Moreover we have for any function $f \in L_{\varphi}^{\rho}\left(I^2\right)$

$$\rho^{\varphi}(f) = \|f\|_{L_p}^p.$$

It is clear that $\{g_{m,n}\}$ does not converge statistically modularly but converges to $g = 0$ statistically modularly relative to a scale function.
$\sigma(x,y) = \begin{cases} 1, & (x,y) = (0,0) \\ \frac{1}{x^2 y}, & (x,y) \in (0,1] \times (0,1] \end{cases}$ on $L_1([0,1] \times [0,1])$. Indeed, for some $\lambda_0 > 0$, with the choice of $p = 1$ we have $\rho^{\varphi}(g) = \|g\|_{L_1}$,

$$\begin{aligned} \rho\left(\lambda_0\left(g_{m,n} - g\right)\right) &= \left\|\lambda_0\left(g_{m,n} - g\right)\right\|_{L_1} \\ &= \begin{cases} 1, & m = k^2 \text{ and } n = l^2 \\ \frac{3m}{4}, & m \neq k^2 \text{ and } n \neq l^2 \end{cases}, k, l = 1, 2, \ldots, \end{aligned} \tag{4}$$

then we have

$$st_2 - \lim_{m,n} \left\|\lambda_0\left(g_{m,n} - g\right)\right\|_{L_1} \neq 0.$$

Using the scale function σ,

$$\rho\left(\lambda_0\left(\frac{g_{m,n} - g}{\sigma}\right)\right) = \begin{cases} \frac{1}{6}, & m = k^2 \text{ and } n = l^2 \\ \frac{1}{12mn}, & m \neq k^2 \text{ and } n \neq l^2 \end{cases}, k, l = 1, 2, \ldots,$$

we get

$$st_2 - \lim_{m,n} \left\|\lambda_0\left(\frac{g_{m,n} - g}{\sigma}\right)\right\|_{L_1} = 0.$$

On the other hand, we can easily see that $\{g_{m,n}\}$ does not converge to $g = 0$ modularly relatively to a scale function $\sigma(x,y) = \begin{cases} 1, & (x,y) = (0,0) \\ \frac{1}{x^2 y}, & (x,y) \in (0,1] \times (0,1] \end{cases}$ on $L_1([0,1] \times [0,1])$. Indeed from (4), we observed that the sequence $\left\{\rho\left(\lambda_0\left(\frac{g_{m,n} - g}{\sigma}\right)\right)\right\}$ has two subsequences with different limit points so $\{g_{m,n}\}$ is not relatively modularly convergent to $g = 0$.

In this paper, we will need the following assumptions on a modular ρ :

- ρ is *monotone* if $\rho(f) \leq \rho(g)$ for $|f| \leq |g|$,
- ρ is *finite* if $\chi_A \in L^\rho\left(I^2\right)$ whenever A is measurable subset of I^2 such that $\mu(A) < \infty$,
- ρ is *absolutely finite* if ρ is finite and, for every $\varepsilon > 0, \lambda > 0$, there exists a $\delta > 0$ such that $\rho(\lambda \chi_B) < \varepsilon$ for any measurable subset $B \subset I^2$ with $\mu(B) < \delta$,
- ρ is *strongly finite* if $\chi_{I^2} \in E^\rho\left(I^2\right)$,
- ρ is *absolutely continuous* provided that there exists an $\alpha > 0$ such that, for every $f \in X\left(I^2\right)$ with $\rho(f) < +\infty$, the following condition holds: for every $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(\alpha f \chi_B) < \varepsilon$ whenever B is any measurable subset of I^2 with $\mu(B) < \delta$.

Observe now that (see [2,3]) if a modular ρ is monotone and finite, then we have $C(I^2) \subset L^\rho\left(I^2\right)$. In a similar manner, if ρ is monotone and strongly finite, then $C(I^2) \subset E^\rho\left(I^2\right)$. Also, if ρ is monotone, absolutely finite and absolutely continuous, then $\overline{C^\infty(I^2)} = L^\rho\left(I^2\right)$. Some important relations between the above properties may be found in [1,3,4,17,21].

2. Statistical Relative Korovkin Theorems in Modular Spaces

In this section, we apply the notion of statistical relative modular convergence of a double sequences of positive linear operators defined on a modular space to prove a Korovkin type approximation theorem.

Let ρ be a monotone and finite modular on $X\left(I^2\right)$. Assume that D is a set satisfying $C^\infty\left(I^2\right) \subset D \subset L^\rho\left(I^2\right)$. We can construct such a subset D when ρ is monotone and finite (see [2]). Assume further that $\mathbb{T} := \{T_{m,n}\}$ is a sequence of positive linear operators from D into $X\left(I^2\right)$ for which there exists a subset $X_{\mathbb{T}} \subset D$ containing $C^\infty\left(I^2\right)$ and $\sigma \in X\left(I^2\right)$ is an unbounded function satisfying $\sigma(x,y) \neq 0$ such that

$$st_2 - \limsup_{m,n} \rho\left(\lambda\left(\frac{T_{m,n}h}{\sigma}\right)\right) \leq R\rho(\lambda h) \tag{5}$$

holds for every $h \in X_{\mathbb{T}}, \lambda > 0$ and for an absolute positive constant R.

We denote the value of $T_{m,n}f$ at a point $(x,y) \in I^2$ by $T_{m,n}(f(u,v);x,y)$ or briefly, $T_{m,n}(f;x,y)$.

Theorem 1. *Let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X\left(I^2\right)$. Let $\mathbb{T} := \{T_{m,n}\}$ be a double sequence of positive linear operators from D into $X\left(I^2\right)$ satisfying (5) and suppose that $\sigma_i(x,y)$ is an unbounded function satisfying $|\sigma_i(x,y)| \geq a_i > 0 (i = 0,1,2,3)$. Assume that*

$$st_2 - \lim_{m,n} \rho\left(\lambda\left(\frac{T_{m,n}(e_i) - e_i}{\sigma_i}\right)\right) = 0 \quad \textit{for every } \lambda > 0 \textit{ and } i = 0,1,2,3 \tag{6}$$

where $e_0(x,y)=1, e_1(x,y)=x, e_2(x,y)=y$ *and* $e_3(x,y)=x^2+y^2$*. Now let* f *be any function belonging to* $L^\rho(I^2)$ *such that* $f-g\in X_{\mathbb{T}}$ *for every* $g\in C^\infty(I^2)$*. Then, we have*

$$st_2-\lim_{m,n}\rho\left(\lambda_0\left(\frac{T_{m,n}f-f}{\sigma}\right)\right)=0 \ \ \textit{for some } \lambda_0>0. \tag{7}$$

where $\sigma(x,y)=\max\{|\sigma_i(x,y)|;i=0,1,2,3\}$.

Proof. We first claim that

$$st_2-\lim_{m,n}\rho\left(\eta\left(\frac{T_{m,n}g-g}{\sigma}\right)\right)=0 \quad \text{for every } g\in C(I^2)\cap D \text{ and every } \eta>0. \tag{8}$$

To see this assume that g belongs to $C(I^2)\cap D$. By the continuity of g on I^2, given $\varepsilon>0$, there exists a number $\delta>0$ such that for all $(u,v),(x,y)\in I^2$ satisfying $|u-x|<\delta$ and $|v-y|<\delta$ we have

$$|g(u,v)-g(x,y)|<\varepsilon. \tag{9}$$

Also we get for all $(u,v),(x,y)\in I^2$ satisfying $|u-x|>\delta$ and $|v-y|>\delta$ that

$$|g(u,v)-g(x,y)|\le\frac{2M}{\delta^2}\left\{(u-x)^2+(v-y)^2\right\} \tag{10}$$

where $M:=\sup_{(x,y)\in I^2}|g(x,y)|$. Combining (9) and (10) we have for $(u,v),(x,y)\in I^2$ that

$$|g(u,v)-g(x,y)|<\varepsilon+\frac{2M}{\delta^2}\left\{(u-x)^2+(v-y)^2\right\}.$$

namely,

$$-\varepsilon-\frac{2M}{\delta^2}\left\{(u-x)^2+(v-y)^2\right\} < g(u,v)-g(x,y)<\varepsilon+\frac{2M}{\delta^2}\left\{(u-x)^2+(v-y)^2\right\}. \tag{11}$$

Since $T_{m,n}$ is linear and positive, by applying $T_{m,n}$ to (11) for every $m,n\in\mathbb{N}$ we get

$$\begin{aligned}&-\varepsilon T_{m,n}(e_0;x,y)-\frac{2M}{\delta^2}T_{m,n}\left((u-x)^2+(v-y)^2;x,y\right)\\ &\quad< T_{m,n}(g;x,y)-g(x,y)T_{m,n}(e_0;x,y)\\ &\quad<\varepsilon T_{m,n}(e_0;x,y)+\frac{2M}{\delta^2}T_{m,n}\left((u-x)^2+(v-y)^2;x,y\right)\end{aligned}$$

and hence,

$$\begin{aligned}|T_{m,n}(g;x,y)-g(x,y)|&\le|T_{m,n}(g;x,y)-g(x,y)T_{m,n}(e_0;x,y)|\\ &\quad+|g(x,y)T_{m,n}(e_0;x,y)-g(x,y)|\end{aligned}$$

$$\leq \varepsilon T_{m,n}(e_0;x,y) + M\left|T_{m,n}\left(e_0;x,y\right) - e_0(x,y)\right| + \frac{2M}{\delta^2}T_{m,n}\left(\left(u-x\right)^2 + \left(v-y\right)^2;x,y\right)$$

holds for every $x,y \in I$ and $m,n \in \mathbb{N}$. The above inequality implies that

$$\begin{aligned}\left|T_{m,n}(g;x,y) - g(x,y)\right| \leq \varepsilon + \left\{\varepsilon + M + \frac{4M}{\delta^2}E^2\right\}\left|T_{m,n}\left(e_0;x,y\right) - e_0(x,y)\right| \\ + \frac{4M}{\delta^2}E\left|T_{m,n}\left(e_1;x,y\right) - e_1(x,y)\right| \\ + \frac{4M}{\delta^2}E\left|T_{m,n}\left(e_2;x,y\right) - e_2(x,y)\right| \\ + \frac{2M}{\delta^2}\left|T_{m,n}\left(e_3;x,y\right) - e_3(x,y)\right|.\end{aligned}$$

where $E := \max\{|x|,|y|\}$. Now we multiply the both-sides of the above inequality by $\frac{1}{|\sigma(x,y)|}$ and for any $\eta > 0$, the last inequality gives that

$$\begin{aligned}\eta\left|\frac{T_{m,n}(g;x,y) - g(x,y)}{\sigma(x,y)}\right| \leq \frac{\eta\varepsilon}{|\sigma(x,y)|} + K\eta\left\{\left|\frac{T_{m,n}\left(e_0;x,y\right) - e_0(x,y)}{\sigma(x,y)}\right|\right. \\ + \left|\frac{T_{m,n}\left(e_1;x,y\right) - e_1(x,y)}{\sigma(x,y)}\right| \\ + \left|\frac{T_{m,n}\left(e_2;x,y\right) - e_2(x,y)}{\sigma(x,y)}\right| \\ \left. + \left|\frac{T_{m,n}\left(e_3;x,y\right) - e_3(x,y)}{\sigma(x,y)}\right|\right\},\end{aligned}$$

where $K := \max\left\{\varepsilon + M + \frac{4M}{\delta^2}E^2, \frac{4M}{\delta^2}E, \frac{2M}{\delta^2}\right\}$. Now, applying the modular ρ to both-sides of the above inequality, since ρ is monotone and $\sigma(x,y) = \max\{|\sigma_i(x,y)|; i = 0,1,2,3\}$, we have

$$\begin{aligned}\rho\left(\eta\left(\frac{T_{m,n}g - g}{\sigma}\right)\right) \leq \rho\left(\eta\frac{\varepsilon}{|\sigma|} + \eta K\left|\frac{T_{m,n}e_0 - e_0}{\sigma_0}\right| + \eta K\left|\frac{T_{m,n}e_1 - e_1}{\sigma_1}\right|\right. \\ \left. + \eta K\left|\frac{T_{m,n}e_2 - e_2}{\sigma_2}\right| + \eta K\left|\frac{T_{m,n}e_3 - e_3}{\sigma_3}\right|\right).\end{aligned}$$

Hence, we may write that

$$\begin{aligned}\rho\left(\eta\left(\frac{T_{m,n}g - g}{\sigma}\right)\right) \leq \rho\left(\frac{5\eta\varepsilon}{\sigma}\right) + \rho\left(5\eta K\left(\frac{T_{m,n}e_0 - e_0}{\sigma_0}\right)\right) \\ + \rho\left(5\eta K\left(\frac{T_{m,n}e_1 - e_1}{\sigma_1}\right)\right) \\ + \rho\left(5\eta K\left(\frac{T_{m,n}e_2 - e_2}{\sigma_2}\right)\right)\end{aligned}$$

$$+\rho\left(5\eta K\left(\frac{T_{m,n}e_3-e_3}{\sigma_3}\right)\right)$$

Since ρ is N-quasi semiconvex and strongly finite, we have, assuming $0<\varepsilon\leq 1$,

$$\begin{aligned}\rho\left(\eta\left(\frac{T_{m,n}g-g}{\sigma}\right)\right)\leq N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right)&+\rho\left(5\eta K\left(\frac{T_{m,n}e_0-e_0}{\sigma_0}\right)\right)\\&+\rho\left(5\eta K\left(\frac{T_{m,n}e_1-e_1}{\sigma_1}\right)\right)\\&+\rho\left(5\eta K\left(\frac{T_{m,n}e_2-e_2}{\sigma_2}\right)\right)\\&+\rho\left(5\eta K\left(\frac{T_{m,n}e_3-e_3}{\sigma_3}\right)\right).\end{aligned}$$

For a given $r>0$, choose an $\varepsilon\in(0,1]$ such that $N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right)<r$. Now define the following sets:

$$G_\eta:=\left\{(m,n):\rho\left(\eta\left(\frac{T_{m,n}g-g}{\sigma}\right)\right)\geq r\right\}$$

$$G_{\eta,i}:=\left\{(m,n):\rho\left(5\eta K\left(\frac{T_{m,n}e_i-e_i}{\sigma_i}\right)\right)\geq\frac{r-N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right)}{4}\right\},$$

where $i=0,1,2,3$. Then, it is easy to see that $G_\eta\subseteq\bigcup_{i=0}^{3}G_{\eta,i}$. So, we can write that

$$\delta_2\left(G_\eta\right)\leq\sum_{i=0}^{3}\delta_2\left(G_{\eta,i}\right).$$

Using the hypothesis (6), we get

$$\delta_2\left(G_\eta\right)=0,$$

which proves our claim (8). Obviously (8) also holds for every $g\in C^\infty(I^2)$. Now let $f\in L^\rho\left(I^2\right)$ satisfying $f-g\in X_{\mathbb{T}}$ for every $g\in C^\infty\left(I^2\right)$. Since $\mu\left(I^2\right)<\infty$ and ρ is strongly finite and absolutely continuous, we can see that ρ is also absolutely finite on $X(I^2)$. Using these properties of the modular ρ, it is known from [4,17] that the space $C^\infty(I^2)$ is modularly dense in $L^\rho\left(I^2\right)$, i.e., there exists a sequence $\{g_{k,j}\}\subset C^\infty\left(I^2\right)$ such that

$$P-\lim_{k,j}\rho\left(3\lambda_0^*\left(g_{k,j}-f\right)\right)=0\quad\text{for some }\lambda_0^*>0.$$

This means that, for every $\varepsilon>0$, there is a positive number $k_0=k_0(\varepsilon)$ so that

$$\rho\left(3\lambda_0^*\left(g_{k,j}-f\right)\right)<\varepsilon\quad\text{for every }k,j\geq k_0. \tag{12}$$

On the other hand, by the linearity and positivity of the operators $T_{m,n}$, we may write that

$$\lambda_0^* \left|T_{m,n}(f;x,y)-f(x,y)\right| \leq \lambda_0^* \left|T_{m,n}(f-g_{k_0,k_0};x,y)\right| + \lambda_0^* \left|T_{m,n}(g_{k_0,k_0};x,y)-g_{k_0k_0}(x,y)\right| + \lambda_0^* \left|g_{k_0,k_0}(x,y)-f(x,y)\right|$$

holds for every $x,y \in I$ and $m,n \in \mathbb{N}$. Applying the modular ρ in the last inequality and using the monotonicity of ρ and moreover multiplying the both-sides of the above inequality by $\frac{1}{|\sigma(x,y)|}$, we have

$$\rho\left(\lambda_0^*\left(\frac{T_{m,n}f-f}{\sigma}\right)\right) \leq \rho\left(3\lambda_0^*\frac{T_{m,n}\left(f-g_{k_0,k_0}\right)}{\sigma}\right) + \rho\left(3\lambda_0^*\left(\frac{T_{m,n}g_{k_0,k_0}-g_{k_0,k_0}}{\sigma}\right)\right) + \rho\left(3\lambda_0^*\left(\frac{g_{k_0,k_0}-f}{\sigma}\right)\right).$$

Hence, observing $|\sigma| \geq a > 0 (a = \max\{a_i : i = 0,1,2,3\})$ we may write that

$$\rho\left(\lambda_0^*\left(\frac{T_{m,n}f-f}{\sigma}\right)\right) \leq \rho\left(3\lambda_0^*\frac{T_{m,n}\left(f-g_{k_0,k_0}\right)}{\sigma}\right) + \rho\left(3\lambda_0^*\left(\frac{T_{m,n}g_{k_0,k_0}-g_{k_0,k_0}}{\sigma}\right)\right) + \rho\left(\frac{3\lambda_0^*}{a}\left(g_{k_0,k_0}-f\right)\right). \tag{13}$$

Then, it follows from (12) and (13) that

$$\rho\left(\lambda_0^*\left(\frac{T_{m,n}f-f}{\sigma}\right)\right) \leq \varepsilon + \rho\left(3\lambda_0^*\frac{T_{m,n}\left(f-g_{k_0,k_0}\right)}{\sigma}\right) + \rho\left(3\lambda_0^*\left(\frac{T_{m,n}g_{k_0,k_0}-g_{k_0,k_0}}{\sigma}\right)\right). \tag{14}$$

So, taking statistical limit superior as $m,n \to \infty$ in the both-sides of (14) and also using the facts that $g_{k_0,k_0} \in C^{\infty}(I^2)$ and $f - g_{k_0,k_0} \in X_{\mathbb{T}}$, we obtained from (5) that

$$st_2 - \limsup_{m,n} \rho\left(\lambda_0^*\left(\frac{T_{m,n}f-f}{\sigma}\right)\right) \leq \varepsilon + R\rho\left(3\lambda_0^*(f-g_{k_0,k_0})\right) + st_2 - \limsup_{m,n} \rho\left(3\lambda_0^*\left(\frac{T_{m,n}g_{k_0,k_0}-g_{k_0,k_0}}{\sigma}\right)\right),$$

which gives

$$st_2 - \limsup_{m,n} \rho \left(\lambda_0^* \left(\frac{T_{m,n} f - f}{\sigma} \right) \right)$$
$$\leq \varepsilon(R+1) + st_2 - \limsup_{m,n} \rho \left(3\lambda_0^* \left(\frac{T_{m,n} g_{k_0,k_0} - g_{k_0,k_0}}{\sigma} \right) \right). \quad (15)$$

By (8), since

$$st_2 - \lim_{m,n} \rho \left(3\lambda_0^* \left(\frac{T_{m,n} g_{k_0,k_0} - g_{k_0,k_0}}{\sigma} \right) \right) = 0,$$

we get

$$st_2 - \limsup_{m,n} \rho \left(3\lambda_0^* \left(\frac{T_{m,n} g_{k_0,k_0} - g_{k_0,k_0}}{\sigma} \right) \right) = 0. \quad (16)$$

Combining (15) with (16), we conclude that

$$st_2 - \limsup_{m,n} \rho \left(\lambda_0^* \left(\frac{T_{m,n} f - f}{\sigma} \right) \right) \leq \varepsilon(R+1).$$

Since $\varepsilon > 0$ was arbitrary, we find

$$st_2 - \limsup_{m,n} \rho \left(\lambda_0^* \left(\frac{T_{m,n} f - f}{\sigma} \right) \right) = 0.$$

Furthermore, since $\rho \left(\lambda_0^* \left(\frac{T_{m,n} f - f}{\sigma} \right) \right)$ is non-negative for all $m, n \in \mathbb{N}$, we can easily show that

$$st_2 - \lim_{m,n} \rho \left(\lambda_0^* \left(\frac{T_{m,n} f - f}{\sigma} \right) \right) = 0,$$

which completes the proof. □

If the modular ρ satisfies the Δ_2-condition, then one can get immediately the following result from Theorem 1.

Theorem 2. *Let $\mathbb{T} := \{T_{m,n}\}$, ρ and σ be the same as in Theorem 1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

(a) $st_2 - \lim_{m,n} \rho \left(\lambda \left(\frac{T_{m,n} e_i - e_i}{\sigma_i} \right) \right) = 0$ *for every $\lambda > 0$ and $i = 0, 1, 2, 3$,*

(b) $st_2 - \lim_{m,n} \rho \left(\lambda \left(\frac{T_{m,n} f - f}{\sigma} \right) \right) = 0$ *for every $\lambda > 0$ provided that f is any function belonging to $L^\rho(I^2)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^\infty\left(I^2\right)$.*

If one replaces the scale function by a nonzero constant, then the condition (5) reduces to

$$st_2 - \limsup_{m,n} \rho\left(\lambda\left(T_{m,n} h\right)\right) \leq R \rho\left(\lambda h\right) \quad (17)$$

for every $h \in X_{\mathbb{T}}, \lambda > 0$ and for an absolute positive constant R. In this case, the next results which were obtained in [22] follows from our main theorems, Theorems 1 and 2.

Corollary 1 [22]. *Let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X\left(I^2\right)$. Let $\mathbb{T} := \{T_{m,n}\}$ be a double sequence of positive linear operators from D into $X\left(I^2\right)$ satisfying* (17). *If $\{T_{m,n}e_i\}$ is statistically strongly convergent to e_i for each $i = 0, 1, 2, 3$, then $\{T_{m,n}f\}$ is statistically modularly convergent to f provided that f is any function belonging to $L^\rho\left(I^2\right)$ such that $f - g \in X_\mathbb{T}$ for every $g \in C^\infty\left(I^2\right)$.*

Corollary 2 [22]. *$\mathbb{T} := \{T_{m,n}\}$ and ρ be the same as in Corollary* 1. *If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

(a) $\{T_{m,n}e_i\}$ is statistically strongly convergent to e_i for each $i = 0, 1, 2, 3$,
(b) $\{T_{m,n}f\}$ is statistically strongly convergent to f provided that f is any function belonging to $L^\rho(I^2)$ such that $f - g \in X_\mathbb{T}$ for every $g \in C^\infty\left(I^2\right)$.

If one replaces the statistical limit by the Pringsheim limit, then the condition (5) reduces to

$$P - \limsup_{m,n} \rho\left(\lambda\left(\frac{T_{m,n}h}{\sigma}\right)\right) \leq R\rho\left(\lambda h\right) \tag{18}$$

for every $h \in X_\mathbb{T}, \lambda > 0$ and for an absolute positive constant R. In this case, the following results immediately follows from our Theorems 1 and 2.

Corollary 3. *Let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X\left(I^2\right)$. Let $\mathbb{T} := \{T_{m,n}\}$ be a double sequence of positive linear operators from D into $X\left(I^2\right)$ satisfying* (18). *Moreover suppose that $\sigma_i\left(x, y\right)$ is an unbounded function satisfying $|\sigma_i\left(x, y\right)| \geq a_i > 0 (i = 0, 1, 2, 3)$. If $\{T_{m,n}e_i\}$ is relatively strongly convergent to e_i for each $i = 0, 1, 2, 3$, then $\{T_{m,n}f\}$ is relatively modularly convergent to f provided that f is any function belonging to $L^\rho\left(I^2\right)$ such that $f - g \in X_\mathbb{T}$ for every $g \in C^\infty\left(I^2\right)$.*

Corollary 4. *$\mathbb{T} := \{T_{m,n}\}, \rho$ and $\sigma_i(i = 0, 1, 2, 3)$ be the same as in Corollary* 3. *If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

(a) $\{T_{m,n}e_i\}$ is relatively strongly convergent to e_i for each $i = 0, 1, 2, 3$,
(b) $\{T_{m,n}f\}$ is relatively strongly convergent to f provided that f is any function belonging to $L^\rho(I^2)$ such that $f - g \in X_\mathbb{T}$ for every $g \in C^\infty\left(I^2\right)$.

3. Concluding Remarks

In this section, we display an example such that our Korovkin-type statistical approximation results in modular spaces are stronger than the Corollaries 1 and 3.

Example 2. Take $I = [0, 1]$ *and* φ, ρ^φ and $L^\rho_\varphi\left(I^2\right)$ be the same as in Example 1. Then consider the following bivariate Bernstein-Kantorovich operator$\mathbb{U} := \{U_{m,n}\}$ on the space $L^\rho_\varphi\left(I^2\right)$ which is defined by:

$$U_{m,n}(f;x,y)$$
$$= \sum_{i=0}^{m}\sum_{k=0}^{n} p_{i,k}^{(m,n)}(x,y)(m+1)(n+1) \times \int_{i/(m+1)}^{(i+1)/(m+1)} \int_{k/(n+1)}^{(k+1)/(n+1)} f(t,s)\,dsdt \tag{19}$$

for $x, y \in I$, where $p_{i,k}^{(m,n)}(x,y)$ defined by

$$p_{i,k}^{(m,n)}(x,y) = \binom{m}{i}\binom{n}{k} x^i y^k (1-x)^{m-i}(1-y)^{n-k}.$$

Also, it is clear that,

$$\sum_{i=0}^{m}\sum_{k=0}^{n} p_{i,k}^{(m,n)}(x,y) = 1. \tag{20}$$

Observe that the operator $U_{m,n}$ maps $L_{\varphi}^{\rho}(I^2)$ into itself. Because of (20), as in the proof of [2] Lemma 5.1 and similar to Example 1 [22], we can use the Jensen inequality and obtain that for every $f \in L_{\varphi}^{\rho}(I^2)$ and $m, n \in \mathbb{N}$ there is an absolute constant $M > 0$ such that

$$\rho^{\varphi}\left(\frac{U_{m,n}f}{\sigma}\right) \leq M\rho^{\varphi}(f).$$

Moreover, the property (18) is satisfied with the choice of $X_{\mathbf{U}} := L_{\varphi}^{\rho}(I^2)$. Then, by Corollary 3, we know that, for any function $f \in L_{\varphi}^{\rho}(I^2)$ such that $f - g \in X_{\mathbf{U}}$ for every $g \in C^{\infty}(I^2)$, $\{U_{m,n}f\}$ is relatively modularly convergent to f.

If $\varphi(x) = x^p$ for $1 \leq p < \infty, x \geq 0$, then $L_{\varphi}^{\rho}(I^2) = L_p(I^2)$. Moreover we have $\rho^{\varphi}(f) = \|f\|_{L_p}^{p}$. For $p = 1$, we have $\rho^{\varphi}(f) = \|f\|_{L_1}$.Then, using the operators $U_{m,n}$, we define the sequence of positive linear operators $V := \{V_{m,n}\}$ on $L_1(I^2)$ as follows:

$$V_{m,n}(f;x,y) = (1 + g_{m,n}(x,y))U_{m,n}(f;x,y)$$
$$\text{for } f \in L_1(I^2), (x,y) \in I^2 \text{ and } m,n \in \mathbb{N} \tag{21}$$

where $\{g_{m,n}\}$ is the same as in (3) and we choose $\sigma_i(x,y) = \sigma(x,y)\,(i = 0,1,2,3)$, where $\sigma(x,y) = \begin{cases} 1, & (x,y) = (0,0) \\ \frac{1}{x^2 y}, & (x,y) \in (0,1] \times (0,1] \end{cases}$. As in the proof of Lemma 5.1 [2] and similar to Example 1 [22], we get, for every $f \in L_1(I^2)$, $\lambda > 0$ and for positive constant C, that

$$st_2 - \limsup_{m,n} \left\| \lambda\left(\frac{V_{m,n}f}{\sigma}\right)\right\|_{L_1} \leq C\|\lambda f\|_{L_1}. \tag{22}$$

We now claim that

$$st_2 - \lim_{m,n} \left\| \lambda\left(\frac{V_{m,n}e_i - e_i}{\sigma}\right)\right\|_{L_1} = 0, \quad i = 0,1,2,3. \tag{23}$$

Indeed, first observe that

$$V_{m,n}(e_0;x,y) = 1 + g_{m,n}(x,y),$$
$$V_{m,n}(e_1;x,y) = (1 + g_{m,n}(x,y))\left(\frac{mx}{m+1} + \frac{1}{2(m+1)}\right),$$
$$V_{m,n}(e_2;x,y) = (1 + g_{m,n}(x,y))\left(\frac{ny}{n+1} + \frac{1}{2(n+1)}\right),$$
$$V_{m,n}(e_3;x,y) = (1 + g_{m,n}(x,y))\left(\frac{m(m-1)x^2}{(m+1)^2} + \frac{2mx}{(m+1)^2} + \frac{1}{3(m+1)^2} + \frac{n(n-1)y^2}{(n+1)^2} + \frac{2ny}{(n+1)^2} + \frac{1}{3(n+1)^2}\right).$$

So, we can see, for any $\lambda > 0$, that

$$\left\|\lambda\left(\frac{V_{m,n}(e_0;x,y) - e_0(x,y)}{\sigma(x,y)}\right)\right\|_{L_1} = \lambda\begin{cases}\int_0^1\int_0^1 x^2y\,dxdy, & m = k^2 \text{ and } n = l^2\\ \int_0^{\frac{1}{n}}\int_0^{\frac{1}{m}} m^2n(x^2y - mnx^3y^2)dxdy, & m \neq k^2 \text{ and } n \neq l^2\end{cases}, \quad k,l = 1,2,\ldots,$$
$$= \lambda\begin{cases}\frac{1}{6} & m = k^2 \text{ and } n = l^2\\ \frac{1}{12mn} & m \neq k^2 \text{ and } n \neq l^2\end{cases}, k,l = 1,2,\ldots, \tag{24}$$

Now, since $P - \lim_{m,n\to\infty} \frac{1}{12mn} = 0$, we get

$$st_2 - \lim_{m,n}\left\|\lambda\left(\frac{V_{m,n}e_0 - e_0}{\sigma}\right)\right\|_{L_1} = 0.$$

which guarantees that (23) holds true for $i = 0$. Also, we have

$$\left\|\lambda\left(\frac{V_{m,n}(e_1;x,y) - e_1(x,y)}{\sigma(x,y)}\right)\right\|_{L_1} \leq \left\|\lambda\frac{g_{m,n}(x,y)}{\sigma(x,y)}\left(\frac{mx}{m+1} + \frac{1}{2(m+1)}\right)\right\|_{L_1} + \left\|\lambda\frac{x^2y - 2x^3y}{2(m+1)}\right\|_{L_1}$$
$$< \left\|\lambda\frac{g_{m,n}(x,y)}{\sigma(x,y)}\right\|_{L_1} + \frac{\lambda}{24(m+1)},$$

because of $st_2 - \lim_{m,n}\left\|\lambda\frac{g_{m,n}(x,y)}{\sigma(x,y)}\right\|_{L_1} = 0$ and $P - \lim_{m,n}\frac{\lambda}{24(m+1)} = 0$, we get

$$st_2 - \lim_{m,n}\left\|\lambda\left(\frac{V_{m,n}(e_1;x,y) - e_1(x,y)}{\sigma(x,y)}\right)\right\|_{L_1} = 0.$$

Hence (23) is valid for $i=1$. Similarly, we have

$$st_2-\lim_{m,n}\left\|\lambda\left(\frac{V_{m,n}\left(e_2;x,y\right)-e_2(x,y)}{\sigma\left(x,y\right)}\right)\right\|_{L_1}=0.$$

Finally, since

$$\begin{aligned}&\left\|\lambda\left(\frac{V_{m,n}\left(e_3;x,y\right)-e_3(x,y)}{\sigma\left(x,y\right)}\right)\right\|_{L_1}\\ &\quad\leq\left\|\lambda\frac{g_{m,n}\left(x,y\right)}{\sigma\left(x,y\right)}\left(\frac{m\left(m-1\right)x^2}{\left(m+1\right)^2}+\frac{2mx}{\left(m+1\right)^2}\right.\right.\\ &\qquad\left.\left.+\frac{1}{3\left(m+1\right)^2}\frac{n\left(n-1\right)y^2}{\left(n+1\right)^2}+\frac{2ny}{\left(n+1\right)^2}+\frac{1}{3\left(n+1\right)^2}\right)\right\|_{L_1}\\ &\qquad+\left\|\lambda\left(\frac{\left(3m+1\right)x^4y}{\left(m+1\right)^2}+\frac{\left(3n+1\right)x^2y^3}{\left(n+1\right)^2}+\frac{2mx^3y}{\left(m+1\right)^2}+\frac{2nx^2y^2}{\left(n+1\right)^2}\right.\right.\\ &\qquad\left.\left.+x^2y\left(\frac{1}{3\left(m+1\right)^2}+\frac{1}{3\left(n+1\right)^2}\right)\right)\right\|_{L_1}\\ &\quad<6\left\|\lambda\frac{g_{m,n}\left(x,y\right)}{\sigma\left(x,y\right)}\right\|_{L_1}+\frac{\lambda\left(3m+1\right)}{10\left(m+1\right)^2}+\frac{\lambda\left(3n+1\right)}{12\left(n+1\right)^2}+\frac{\lambda m}{4\left(m+1\right)^2}\\ &\qquad+\frac{2\lambda n}{9\left(n+1\right)^2}+\frac{\lambda}{6}\left(\frac{1}{3\left(m+1\right)^2}+\frac{1}{3\left(n+1\right)^2}\right),\end{aligned}$$

then we have,

$$st_2-\lim_{m,n}\left\|\lambda\left(\frac{V_{m,n}\left(e_3;x,y\right)-e_3(x,y)}{\sigma\left(x,y\right)}\right)\right\|_{L_1}=0.$$

So, our claim (23) holds true for each $i=0,1,2,3$ and for any $\lambda>0$. Now, from (22) and (23), we can say that our sequence $V:=\{V_{m,n}\}$ defined by (21) satisfy all assumptions of Theorem 1. Therefore, we conclude that

$$st_2-\lim_{m,n}\left\|\lambda\left(\frac{V_{m,n}\left(f;x,y\right)-f(x,y)}{\sigma\left(x,y\right)}\right)\right\|_{L_1}=0\quad \textit{for some } \lambda_0>0$$

holds for any $f\in L_1\left(I^2\right)$ such that $f-g\in X_{\mathbb{V}}=L_1(I^2)$ for every $g\in C^{\infty}\left(I^2\right)$.

However, from (24) it can be seen that the sequence $\|\lambda(\frac{V_{m,n}(e_0;x,y)-e_0(x,y)}{\sigma(x,y)})\|_{L_1}$ has two subsequences with different limit points. Since $P-\lim\limits_{m,n}\|\lambda(\frac{V_{m,n}e_0-e_0}{\sigma})\|_{L_1}\neq 0$, Corollary 3 does not work for the sequence $V:=\{V_{m,n}\}$. Also, since $st_2-\lim\limits_{m,n}\left\|\lambda\left(V_{m,n}e_0-e_0\right)\right\|_{L_1}\neq 0$, Corollary 1 does not work for the sequence $V:=\{V_{m,n}\}$ as well.

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Kamil Demirci and Sevda Orhan
Department of Mathematics
Sinop University
Sinop
Turkey
e-mail: kamild@sinop.edu.tr;
sevdaorhan@sinop.edu.tr

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