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Results in Mathematics

A Korovkin-Type Approximation Theorem and Power Series Method

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Abstract. In this paper, using power series methods we give an abstract Korovkin type approximation theorem for a sequence of positive linear operators mapping $C(X, \mathbb{R})$ into itself.

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1. Introduction

A well-known class of summability methods are the power series methods, perhaps the most popular ones in this class being Abel's and Borel's method [12]. In this paper we consider how the concept of power series methods can be applied to approximation theory. We prove an abstract Korovkin theorem using power series methods in summability. That is, this paper is mainly concerned with an abstract version of Korovkin theory on the space $C(X, \mathbb{R})$ using the above methods. Abstract Korovkin theory has been studied in [2,7,10], respectively using summation process, statistical convergence and ordinary convergence. A detailed account of the Korovkin type approximation theory may be found in [1,8]. Also recent generalizations of the Korovkin theorem by using new type of convergence are given in [4,5]. Moreover abstract Korovkin type theorems in modular spaces are studied in [3].

We collect some notation and basic definitions used in this paper.

Let (q_n) be a real sequence with $q_0 > 0$ and $q_n \geq 0$ $(n \in \mathbb{N})$ and such that the corresponding power series $q(y) := \sum_{n=0}^{\infty} q_n y^n$ has radius of convergence R with $0 < R \leq \infty$. Let $x = (x_n)$ be a sequence of real numbers. If, for all $y \in (0, R)$,

$$\lim_{y\to R^-}\frac{1}{q(y)}\sum_{n=0}^{\infty}x_nq_ny^n=L$$

then we say that $x=(x_n)$ is convergent in the sense of power series method [9,11]. Note that the power series method is regular if and only if for each $n\in\mathbb{N}$

$$\lim_{y\to R^-}\frac{q_ny^n}{q(y)}=0 \tag{1.1}$$

holds [6]. Throughout the paper, the methods fulfill condition (1.1).

Let $C(X,\mathbb{R})$ denote the space of all continuous real-valued functions defined on a compact Hausdorff space X with at least two points. It is well-known that $C(X,\mathbb{R})$ is a Banach space with the usual norm

$$\|f\|=\sup_{x\in X}|f(x)|\,,f\in C(X,\mathbb{R})\,.$$

An operator $T:C(X,\mathbb{R})\to C(X,\mathbb{R})$ is called positive if $Tf\geq 0$ for every positive $f\in C(X,\mathbb{R})$. Clearly, T is positive if and only if T is monotone, i.e., f,g in $C(X,\mathbb{R})$ and $f\leq g$ imply that $Tf\leq Tg$. In particular, if T is a positive operator, then $|Tf|\leq T(|f|)$ for $f\in C(X,\mathbb{R})$.

Let $\{T_n\}$ be a sequence of positive linear operators from $C(X,\mathbb{R})$ into itself such that for every $y\in(0,R)$,

$$\frac{1}{q(y)}\sum_{n=0}^{\infty}\|T_ne_0\|\,q_ny^n<\infty,$$

where e_0 is the function defined by $e_0(t)=1$.

Then for all $f\in C(X,\mathbb{R})$ and $y\in(0,R)$ the operator J_y given by

$$J_y(f):=J_y(f;x)=\frac{1}{q(y)}\sum_{n=0}^{\infty}(T_nf)(x)q_ny^n$$

is a well-defined positive linear operator from $C(X,\mathbb{R})$ into itself.

2. An Abstract Version of the Korovkin Theorem Via Power Series Method

In this section, using power series methods we give an abstract Korovkin type approximation theorem for a sequence of positive linear operators mapping $C(X,\mathbb{R})$ into itself.

Assume that $f_1,f_2,\ldots,f_m\in C(X,\mathbb{R})$ have the following properties:

There exist functions $g_1,g_2,\ldots,g_m\in C(X,\mathbb{R})$ such that for every $x,t\in X$,

$$P_x(t):=P(x,t)=\sum_{i=1}^{m}g_i(x)\,f_i(t)\geq 0 \tag{2.1}$$

and

$$P_x(t)=0 \text{ if and only if } t=x. \tag{2.2}$$

Throughout the paper we use P_x in the form of (2.1) and (2.2). To prove our main theorem we need the following lemmas.

Lemma 1. *Let $\{T_n\}$ be a sequence of positive linear operators from $C(X,\mathbb{R})$ into itself. If*

$$\lim_{y\to R^-} \|J_y(f_i)-f_i\| = 0, \quad i=1,2,\ldots,m$$

then, for every function P defined by

$$P(t)=\sum_{i=1}^{m} c_i f_i(t), \tag{2.3}$$

we have

$$\lim_{y\to R^-} \|J_y(P)-P\| = 0,$$

where $c_1,c_2,\ldots,c_m\in\mathbb{R}$ and $t\in X$.

In particular this implies that $\lim_{y\to R^-}\|J_y(P)\|=\|P\|$.

Proof. Using positivity and linearity of T_n we have, for $x\in X$ and $y\in(0,R)$, that

$$\begin{aligned}|J_y(P;x)-P(x)| &= \left|\frac{1}{q(y)}\sum_n T_n\left(\sum_{i=1}^{m} c_i f_i;x\right) q_n y^n-\sum_{i=1}^{m} c_i f_i(x)\right|\\ &\le \sum_{i=1}^{m} |c_i|\,|J_y(f_i;x)-f_i(x)|.\end{aligned}$$

Let $H:=\max_{1\le i\le m}|c_i|$. We obtain

$$\|J_y(P)-P\|\le H\sum_{i=1}^{m}\|J_y(f_i)-f_i\|. \tag{2.4}$$

Letting $y\to R^-$ in both sides of (2.4) we get

$$\lim_{y\to R^-}\|J_y(P)-P\|=0$$

which concludes the proof. □

Lemma 2. *Let $\{T_n\}$ be a sequence of positive linear operators from $C(X,\mathbb{R})$ into itself. If*

$$\lim_{y\to R^-} \|J_y(f_i)-f_i\| = 0, \quad i=1,2,\ldots,m$$

then we have

$$\lim_{y\to R^-}\left(\max_{x\in X} J_y(P_x;x)\right)=0.$$

Proof. Using positivity and linearity of T_n and (2.2) we obtain, for $x \in X$ and $y \in (0, R)$, that

$$J_y(P_x; x) = |J_y(P_x; x)| = \left| \frac{1}{q(y)} \sum_n T_n \left(\sum_{i=1}^{m} g_i(x) f_i; x \right) q_n y^n - \sum_{i=1}^{m} g_i(x) f_i(x) \right|$$
$$\leq \sum_{i=1}^{m} |g_i(x)| \, |J_y(f_i; x) - f_i(x)| \, .$$

By the continuity of every g_i on X, we have

$$\max_{x \in X} J_y(P_x; x) \leq K \sum_{i=1}^{m} \|J_y(f_i) - f_i\| \, ,$$

where $K = \max_{1 \leq i \leq m} \|g_i\| < \infty$. Taking limit as $y \to R^-$ the result follows. □

Lemma 3. *Let $\{T_n\}$ be a sequence of positive linear operators from $C(X, \mathbb{R})$ into itself. If*

$$\lim_{y \to R^-} \|J_y(f_i) - f_i\| = 0, \quad i = 1, 2, \ldots, m$$

then we get

$$\sup_{y \in B_R} \|J_y\| < \infty,$$

where B_R is a left-neighbourhood of R.

Proof. Let u, v be fixed two distinct points of X. Then define a function Q by

$$Q(t) = P_u(t) + P_v(t) \, , \quad t \in X \tag{2.5}$$

where P_u and P_v are the functions given by (2.1). Since $Q(t) > 0$ for all $t \in X$, it is easy to see that $\frac{1}{Q} \in C(X, \mathbb{R})$. Let $c_i := g_i(u) + g_i(v)$, $(i = 1, 2, \ldots, m)$ where each g_i is the function used in (2.1). Hence Q has form (2.3). Since $\frac{1}{Q(t)} \leq \|\frac{1}{Q}\|$ for all $t \in X$, we get

$$1 \leq \left\| \frac{1}{Q} \right\| Q(t) \, , \quad t \in X. \tag{2.6}$$

Now using positivity and linearity of T_n for $x \in X$ and $y \in (0, R)$, we get from (2.6) that

$$|J_y(e_0; x)| = \left| \frac{1}{q(y)} \sum_n T_n(e_0; x) q_n y^n \right|$$
$$\leq \left\| \frac{1}{Q} \right\| |J_y(Q; x)| \, .$$

This yields that

$$\|J_y\| = \|J_y e_0\| \leq \left\| \frac{1}{Q} \right\| \|J_y(Q)\| \, .$$

Then using Lemma 1, we obtain

$$\sup_{y\in B_R} \|J_y\| < \infty.$$

□

Lemma 4. *Let* $\{T_n\}$ *be a sequence of positive linear operators from* $C(X,\mathbb{R})$ *into itself. If*

$$\lim_{y\to R^-} \|J_y(f_i) - f_i\| = 0, \quad i = 1, 2, \ldots, m$$

and $s : X \times X \to \mathbb{R}$ *is continuous function such that* $s(x,x) = 0$ *for all* $x \in X$*, then we have*

$$\lim_{y\to R^-} \left(\max_{x\in X} |J_y(s_x; x)| \right) = 0,$$

where $s_x(t) := s(x,t)$ *on* $X \times X$*.*

Proof. For arbitrary $\varepsilon > 0$, there exists an open neighbourhood U of $\{(x,x) : x \in X\}$ such that $|s_x(t)| < \varepsilon$ for $(x,t) \in U$. Let $m = \min\{P_x(t) : (x,t) \in (X \times X) \setminus U\}$ and $M = \max\{|s_x(t)| : (x,t) \in (X \times X) \setminus U\}$. Since $(X \times X) \setminus U$ is compact, $m > 0$. Now if $(x,t) \in U$, then

$$|s_x(t)| < \varepsilon. \tag{2.7}$$

Otherwise

$$|s_x(t)| \le M \le \frac{M}{m} P_x(t). \tag{2.8}$$

Observe that by (2.7) and (2.8), for all $(x,t) \in (X \times X)$, we have

$$|s_x(t)| \le \varepsilon + \frac{M}{m} P_x(t).$$

Hence using linearity and positivity of T_n we obtain

$$|J_y(s_x; x)| \le \varepsilon J_y(1; x) + \frac{M}{m} J_y(P_x; x)$$

and also

$$\max_{x\in X} |J_y(s_x; x)| \le \varepsilon \|J_y\| + \frac{M}{m} \max_{x\in X} J_y(P_x; x).$$

Then it follows from Lemmas 2 and 3 that

$$\lim_{y\to R^-} \left(\max_{x\in X} |J_y(s_x; x)| \right) = 0.$$

□

Now we present the following abstract version of Korovkin theorem via power series method.

Theorem 1. *Let $\{T_n\}$ be a sequence of positive linear operators from $C(X,\mathbb{R})$ into itself. If*

$$\lim_{y\to R^-} \|J_y(f_i) - f_i\| = 0, \quad i = 1,2,\ldots,m$$

then for all $f \in C(X,\mathbb{R})$,

$$\lim_{y\to R^-} \|J_y(f) - f\| = 0.$$

Proof. Define the function h_x on $X \times X$ by

$$h_x(t) = h(x,t) = f(t) - \frac{f(x)}{Q(x)} Q(t)$$

where Q is the function given by (2.5). It is easy to see that h_x satisfies all conditions of Lemma 4. Then we obtain

$$|J_y(f;x) - f(x)| \leq |J_y(h_x;x)| + \left|\frac{f(x)}{Q(x)}\right| |J_y(Q;x) - Q(x)|.$$

Taking $K := \max\{1, \|\frac{f}{Q}\|\}$, we conclude that

$$\|J_y(f) - f\| \leq K \left(\max_{x\in X} |J_y(h_x;x)| + \|J_y(Q) - Q\| \right).$$

Now letting $y \to R^-$, by Lemmas 1 and 4, for all $f \in C(X,\mathbb{R})$, we have

$$\lim_{y\to R^-} \|J_y(f) - f\| = 0,$$

which completes the proof. □

3. Remarks

Let $X = [a,b]$. In the case of $R = 1$, $q(y) = \frac{1}{1-y}$ and for $n \geq 1$, $q_n = 1$ the power series methods coincide with Abel method which is a sequence-to-function transformation [12]. Now consider $f_1(t) = 1$, $f_2(t) = t$, $f_3(t) = t^2$, $g_1(x) = x^2$, $g_2(x) = -2x$ and $g_3(x) = 1$ in Theorem 1, then the classical Korovkin theorem is obtained for Abel convergence [13]. Note that also getting $f_1(t) = 1$, $f_2(t) = \cos t$, $f_3(t) = \sin t$, $g_1(x) = 1$, $g_2(x) = -\cos x$ and $g_3(x) = \sin x$ the Korovkin theorem is obtained for Abel convergence in the space of all 2π-periodic functions.

Let $X = [a,b]$. In the case of $R = \infty$, $q(y) = e^y$ and for $n \geq 1$, $q_n = \dfrac{1}{n!}$ the power series methods coincide with Borel method. Choose the functions $f_1(t) = 1$, $f_2(t) = t$, $f_3(t) = t^2$, $g_1(x) = x^2$, $g_2(x) = -2x$ and $g_3(x) = 1$. Then Theorem 1 reduces to the classical Korovkin theorem for Borel summability method.

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