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Results in Mathematics

Orthogonal Polynomials for Modified Chebyshev Measure of the First Kind

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Abstract. Given numbers $n, s \in \mathbb{N}$, $n \geq 2$, and the nth-degree monic Chebyshev polynomial of the first kind $\widehat{T}_n(x)$, the polynomial system "induced" by $\widehat{T}_n(x)$ is the system of orthogonal polynomials $\{p_k^{n,s}\}$ corresponding to the modified measure $\mathrm{d}\sigma^{n,s}(x) = \widehat{T}_n^{2s}(x)\,\mathrm{d}\sigma(x)$, where $\mathrm{d}\sigma(x) = 1/\sqrt{1-x^2}\,\mathrm{d}x$ is the Chebyshev measure of the first kind. Here we are concerned with the problem of determining the coefficients in the three-term recurrence relation for the polynomials $p_k^{n,s}$. The desired coefficients are obtained analytically in a closed form.

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1. Introduction

Let $\mathrm{d}\sigma(x)$ be a positive measure on $\mathbb{R}$, with finite or unbounded support, having finite moments of all orders, and let $\{p_k\}$, $k \in \mathbb{N}_0$, be the corresponding (monic) orthogonal polynomials,

$$p_k(x) = p_k(x; \mathrm{d}\sigma), \quad k \in \mathbb{N}_0,$$

which satisfy the following three-term recurrence relation (cf. [5, p. 97])

$$p_{k+1}(x) = (x - \alpha_k)p_k(x) - \beta_k p_{k-1}(x), \quad k \in \mathbb{N}_0,$$
$$p_0(x) = 1, \quad p_{-1}(x) = 0,$$

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where $\alpha_k = \alpha_k(\mathrm{d}\sigma) \in \mathbb{R}$, $\beta_k = \beta_k(\mathrm{d}\sigma) > 0$, and by convention, $\beta_0 = \beta_0(\mathrm{d}\sigma) = \mathrm{d}\sigma(\mathbb{R})$. Orthogonal polynomials can be characterized in terms of extremal problems (cf. [5, pp. 89–93]), as unique (monic) polynomials which minimize the $L^2(\mathrm{d}\sigma)$-norm, i.e.,

$$\min_{p \in \widehat{\mathcal{P}}_k} \int_{\mathbb{R}} |p(x)|^2 \, \mathrm{d}\sigma(x) = \int_{\mathbb{R}} |p_k(x)|^2 \, \mathrm{d}\sigma(x), \tag{1.1}$$

where $\widehat{\mathcal{P}}_k$ is the set of all monic polynomials of degree k. It can be extended to $L^{2s+2}(\mathrm{d}\sigma)$-norm ($s \in \mathbb{N}_0$) and it then leads to a case of the power orthogonality, precisely to the so-called s-orthogonal polynomials $\{p_{k,s}\}$, $k \in \mathbb{N}_0$, as unique (monic) polynomials which minimize

$$\min_{p \in \widehat{\mathcal{P}}_k} \int_{\mathbb{R}} |p(x)|^{2s+2} \, \mathrm{d}\sigma(x) = \int_{\mathbb{R}} |p_{k,s}(x)|^{2s+2} \, \mathrm{d}\sigma(x).$$

These (monic) polynomials must satisfy the "orthogonality conditions" (cf. Ghizzetti and Ossicini [4])

$$\int_{\mathbb{R}} (p_{k,s}(x))^{2s+1} x^{\nu} \, \mathrm{d}\sigma(x) = 0, \quad \nu = 0, 1, \ldots, k-1. \tag{1.2}$$

In the case $s = 0$, the s-orthogonal polynomials reduce to the standard orthogonal polynomials, $p_{k,0} = p_k$.

For given n and s, putting $\mathrm{d}\sigma^{n,s}(x) = (p_{n,s}(x))^{2s} \, \mathrm{d}\sigma(x)$, Milovanović [6] reinterpreted (1.2) in terms of conditions for standard orthogonal polynomials, i.e., as

$$\int_{\mathbb{R}} p_k^{n,s}(x) x^k \, \mathrm{d}\sigma^{(n,s)}(x) = 0, \quad k = 0, 1, \ldots, n-1, \tag{1.3}$$

where $\{p_k^{n,s}\}$ is a sequence of monic orthogonal polynomials with respect to the new measure $\mathrm{d}\sigma^{n,s}(x) = (p_{n,s}(x))^{2s} \, \mathrm{d}\sigma(x)$. Notice that $p_{n,s}(x) \equiv p_n^{n,s}(x)$. As we can see, the polynomials $p_k^{n,s}$, $k = 0, 1, \ldots$, are implicitly defined, because the new measure $\mathrm{d}\sigma^{n,s}(x)$ depends of $p_{n,s}(x)$.

According the previous fact, in order to find s-orthogonal polynomials $p_{n,s}$, $n = 0, 1, \ldots, N$, we need to construct the standard polynomials $p_k^{n,s}(x)$, $k \le n$ (orthogonal w.r.t. $\mathrm{d}\sigma^{n,s}(x) = (p_{n,s}(x))^{2s} \, \mathrm{d}\sigma(x)$), for each $n \le N$, and take $p_{n,s} = p_n^{n,s}$, $n = 0, 1, \ldots, N$. A survey on power orthogonality, quadratures with multiple nodes, and moment-preserving spline approximation was given by Milovanović [7].

In this paper we consider a special case with the Chebyshev measure on the first kind, $\mathrm{d}\sigma(x) = (1-x^2)^{-1/2} \, \mathrm{d}x$ on $(-1, 1)$, and for fixed $n, s \in \mathbb{N}$, $n \ge 2$, we define a new measure $\mathrm{d}\sigma^{n,s}$ by

$$\mathrm{d}\sigma^{n,s}(x) := w^{n,s}(x) \, \mathrm{d}x = \frac{\widehat{T}_n^{2s}(x)}{\sqrt{1-x^2}} \, \mathrm{d}x, \tag{1.4}$$

where $\widehat{T}_n(x) = T_n(x)/2^{n-1} = \cos(n \arccos x)/2^{n-1}$ is the nth-degree monic Chebyshev polynomial of the first kind. In 1930, Bernstein [1] showed that the monic Chebyshev polynomial $T_n(t)$ minimizes all integrals of the form

$$\int_{-1}^{1} \frac{|\pi_n(x)|^{k+1}}{\sqrt{1-x^2}}\,\mathrm{d}x \quad (k \geq 0),$$

where π_n is an arbitrary monic polynomial of degree n. It means that the monic Chebyshev polynomials $\widehat{T}_n$ are s-orthogonal on $[-1, 1]$ for each $s \geq 0$.

The set of orthogonal polynomials we wish to study is

$$p_k^{n,s}(x) = p_k^{n,s}(x;\, \mathrm{d}\sigma^{n,s}), \quad k \in \mathbb{N}_0,$$

for which we know that $p_n^{n,s} = p_{n,s} = \widehat{T}_n$. Their existence is assured, since $\mathrm{d}\sigma^{n,s}(x)$ is a positive measure.

In [3], Gautschi and Li considered the special case $s = 1$ and proved the following result, which will be used in the proof of our main result as a base of induction.

Theorem 1.1. *For any $n \geq 2$, the polynomials $p_k^n(x) = p_k^n(x;\, \mathrm{d}\sigma^n)$, $k \in \mathbb{N}_0$, where $\mathrm{d}\sigma^n(x) = \widehat{T}_n^2(x)/\sqrt{1-x^2}\,\mathrm{d}x$, satisfy*

$$p_{k+1}^n(x) = xp_k^n(x) - \beta_k^n p_{k-1}^n(x), \quad k \in \mathbb{N}_0,$$

$$p_0^n(x) = 1, \quad p_{-1}^n(x) = 0,$$

where

$$\beta_k^n = \begin{cases} \dfrac{\pi}{2^{2n-1}}, & \text{if } k = 0, \\[2ex] \dfrac{1}{4}\left(1 - \dfrac{(-1)^{k/n}}{1+k/n}\right), & \text{if } k \equiv 0 \pmod{n} \quad (k \neq 0), \\[2ex] \dfrac{1}{4}\left(1 + \dfrac{(-1)^{(k-1)/n}}{1+(k-1)/n}\right), & \text{if } k \equiv 1 \pmod{n}, \\[2ex] \dfrac{1}{4}, & \text{otherwise.} \end{cases}$$

In the next section we give recurrence coefficients for the modified Chebyshev measure $\mathrm{d}\sigma^{n,s}(x)$.

2. Recurrence Coefficients for the Modified Chebyshev Measure of the First Kind

Our main result is given in the following statement:

Theorem 2.1. *For any $n, s \in \mathbb{N}$, $n \geq 2$, the polynomials $p_k^{n,s}(x) = p_k^{n,s}(x;\, \mathrm{d}\sigma^{n,s})$, $k \in \mathbb{N}_0$, satisfy the three-term recurrence relation*

$$
\begin{aligned}
p_{k+1}^{n,s}(x) = x p_k^{n,s}(x) - \beta_k^{n,s} p_{k-1}^{n,s}(x), \quad k \in \mathbb{N}_0, \\
p_0^{n,s}(x) = 1, \quad p_{-1}^{n,s}(x) = 0,
\end{aligned} \tag{2.1}
$$

where $\beta_0^{n,s} = \dfrac{\pi}{2^{2ns}}\dbinom{2s}{s}$ *and for* $k \in \mathbb{N}$

$$
\beta_k^{n,s} = \begin{cases}
\dfrac{k}{4(k+ns)}, & \textit{if } k \equiv 0 \pmod{2n}, \\[2ex]
\dfrac{k+2ns-1}{4(k+ns-1)}, & \textit{if } k \equiv 1 \pmod{2n}, \\[2ex]
\dfrac{k+2ns}{4(k+ns)}, & \textit{if } k \equiv n \pmod{2n}, \\[2ex]
\dfrac{k-1}{4(k+ns-1)}, & \textit{if } k \equiv n+1 \pmod{2n}, \\[2ex]
\dfrac{1}{4}, & \textit{otherwise.}
\end{cases} \tag{2.2}
$$

Proof. Suppose for each $n, s \in \mathbb{N}$, $n \geq 2$, the sequence $\{p_k^{n,s}\}_{k\in\mathbb{N}_0}$ is given by (2.1), with $\beta_k^{n,s}$ as in (2.2). Evidently, every $p_k^{n,s}$ is a monic polynomial.

First we compute the coefficient $\beta_0^{n,s}$. By definition, we have

$$
\beta_0^{n,s} = \mu_0^{n,s} = \frac{1}{2^{2(n-1)s}} \int_\pi^0 \frac{\cos^{2s} nt}{\sin t}(-\sin t)\,\mathrm{d}t = \frac{1}{2^{2(n-1)s}} \int_0^\pi \cos^{2s} nt\,\mathrm{d}t.
$$

Further we get

$$
\begin{aligned}
\beta_0^{n,s} &= \frac{1}{2^{2(n-1)s}} \int_0^\pi \left(\frac{\mathrm{e}^{\mathrm{i}nt} + \mathrm{e}^{-\mathrm{i}nt}}{2}\right)^{2s} \mathrm{d}t \\
&= \frac{1}{2^{2(n-1)s+2s}} \int_0^\pi \sum_{j=0}^{2s} \binom{2s}{j} \mathrm{e}^{\mathrm{i}jnt} \mathrm{e}^{-\mathrm{i}(2s-j)nt}\,\mathrm{d}t \\
&= \frac{1}{2^{2ns}} \int_0^\pi \sum_{j=0}^{2s} \binom{2s}{j} \mathrm{e}^{-2\mathrm{i}(s-j)nt}\,\mathrm{d}t \\
&= \frac{1}{2^{2ns}} \sum_{j=0}^{2s} \binom{2s}{j} \int_0^\pi \mathrm{e}^{-2\mathrm{i}(s-j)nt}\,\mathrm{d}t \\
&= \frac{1}{2^{2ns}} \sum_{j=0}^{2s} \binom{2s}{j} \frac{1-\mathrm{e}^{-2\mathrm{i}(s-j)n\pi}}{2\mathrm{i}(s-j)n} \\
&= \frac{\pi}{2^{2ns}} \binom{2s}{s}.
\end{aligned}
$$

In the sequel, in order to short notations, we will omit arguments of polynomials, so that we will put, for example, only $\widehat{T}_n^2 p_{2nk}^{n,s}$ instead of $\widehat{T}_n(x)^2 p_{2nk}^{n,s}(x)$.

Now, we show that for each $k \in \mathbb{N}_0$ the following relations hold true:

$$\left.\begin{aligned}
\widehat{T}_n^2 p_{2nk}^{n,s} &= p_{2n(k+1)}^{n,s-1} + \frac{(2k+2s-2)(2k+2s-1)}{2^{2n}(2k+s-1)(2k+s)} p_{2nk}^{n,s-1},\\
\widehat{T}_n^2 p_{2nk+j}^{n,s} &= p_{2n(k+1)+j}^{n,s-1} + \frac{k+s}{2^{2j-1}(2k+s)(2k+s+1)} p_{2n(k+1)-j}^{n,s-1}\\
&\quad + \frac{(2k+2s-1)(2k+2s)}{2^{2n}(2k+s)^2} p_{2nk+j}^{n,s-1}, \quad j=1,\ldots,n-1,\\
\widehat{T}_n^2 p_{2nk+n}^{n,s} &= p_{2n(k+1)+n}^{n,s-1} + \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s)(2k+s+1)} p_{2nk+n}^{n,s-1},\\
\widehat{T}_n^2 p_{2nk+n+j}^{n,s} &= p_{2n(k+1)+n+j}^{n,s-1}\\
&\quad - \frac{2k+2s+1}{2^{2j}(2k+s+1)(2k+s+2)} p_{2n(k+1)+n-j}^{n,s-1}\\
&\quad + \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)^2} p_{2nk+n+j}^{n,s-1}, \quad j=1,\ldots,n-1.
\end{aligned}\right\} \tag{2.3}$$

Since the sequence of polynomials given by the left hand side in (2.3) obviously satisfies the relation

$$\widehat{T}_n^2 p_{k+1}^{n,s} - x\widehat{T}_n^2 p_k^{n,s} + \beta_k^{n,s}\widehat{T}_n^2 p_{k-1}^{n,s} = \widehat{T}_n^2\left(p_{k+1}^{n,s} - xp_k^{n,s} + \beta_k^{n,s} p_{k-1}^{n,s}\right) = 0$$

for each $k \in \mathbb{N}_0$, it suffices to show that sequence given by the right hand side, say $\{q_i\}_{i\in\mathbb{N}_0}$, in (2.3) satisfies the same relation and that $\widehat{T}_n^2 p_0^{n,s} = q_0$ and $\widehat{T}_n^2 p_1^{n,s} = q_1$ as well.

To prove that $q_{i+1} - xq_i + \beta_i^{n,s} q_{i-1} = 0$ for all $i \in \mathbb{N}$, we distinguish eight cases.

Case $i \equiv 1 \pmod{2n}$.

Here, we have to prove that the following term

$$\begin{aligned}
&p_{2n(k+1)+2}^{n,s-1} + \frac{k+s}{2^3(2k+s)(2k+s+1)} p_{2n(k+1)-2}^{n,s-1}\\
&\quad + \frac{(2k+2s-1)(2k+2s)}{2^{2n}(2k+s)^2} p_{2nk+2}^{n,s-1}\\
&\quad - x\left(p_{2n(k+1)+1}^{n,s-1} + \frac{k+s}{2(2k+s)(2k+s+1)} p_{2n(k+1)-1}^{n,s-1}\right.\\
&\quad \left. + \frac{(2k+2s-1)(2k+2s)}{2^{2n}(2k+s)^2} p_{2nk+1}^{n,s-1}\right)\\
&\quad + \frac{2k+2s}{2^2(2k+s)}\left(p_{2n(k+1)}^{n,s-1} + \frac{(2k+2s-2)(2k+2s-1)}{2^{2n}(2k+s-1)(2k+s)} p_{2nk}^{n,s-1}\right)
\end{aligned}$$

is equal to zero. We will use the fact that we can transform the expression into

$$\begin{aligned}
&p^{n,s-1}_{2n(k+1)+2} - xp^{n,s-1}_{2n(k+1)+1} + \beta^{n,s-1}_{2n(k+1)+1}p^{n,s-1}_{2n(k+1)}\\
&\quad - \left(\frac{2k+2s}{2^2(2k+s)} - \beta^{n,s-1}_{2n(k+1)+1}\right)\left(p^{n,s-1}_{2n(k+1)} + xp^{n,s-1}_{2n(k+1)-1} - \beta^{n,s-1}_{2n(k+1)-1}p^{n,s-1}_{2n(k+1)-2}\right)\\
&\quad + \frac{(2k+2s-1)(2k+2s)}{2^{2n}(2k+s)^2}\left(p^{n,s-1}_{2nk+2} - xp^{n,s-1}_{2nk+1} + \beta^{n,s-1}_{2nk+1}p^{n,s-1}_{2nk}\right),
\end{aligned}$$

which is shown to be true after expansion of the β-coefficients. We can clearly see that every row in the prevous equation equals zero.

Case $i \equiv j \pmod{2n}$, $j = 2, \ldots, n-2$.

In this case we have to consider the following term

$$\begin{aligned}
&p^{n,s-1}_{2n(k+1)+j+1} + \frac{k+s}{2^{2j+1}(2k+s)(2k+s+1)}p^{n,s-1}_{2n(k+1)-j-1}\\
&\quad + \frac{(2k+2s-1)(2k+2s)}{2^{2n}(2k+s)^2}p^{n,s-1}_{2nk+j+1}\\
&\quad - x\left(p^{n,s-1}_{2n(k+1)+j} + \frac{k+s}{2^{2j-1}(2k+s)(2k+s+1)}p^{n,s-1}_{2n(k+1)-j}\right.\\
&\quad \left.+\frac{(2k+2s-1)(2k+2s)}{2^{2n}(2k+s)^2}p^{n,s-1}_{2nk+j}\right)\\
&\quad + \frac{1}{4}\left(p^{n,s-1}_{2n(k+1)+j-1} + \frac{k+s}{2^{2j-3}(2k+s)(2k+s+1)}p^{n,s-1}_{2n(k+1)-j+1}\right.\\
&\quad \left.+\frac{(2k+2s-1)(2k+2s)}{2^{2n}(2k+s)^2}p^{n,s-1}_{2nk+j-1}\right).
\end{aligned}$$

This term can be written in the form

$$\begin{aligned}
&p^{n,s-1}_{2n(k+1)+j+1} - xp^{n,s-1}_{2n(k+1)+j} + \beta^{n,s-1}_{2n(k+1)+1}p_{2n(k+1)+j-1}\\
&\quad + u^{(j)}_{k,s}\left(p^{n,s-1}_{2n(k+1)-j+1} - xp^{n,s-1}_{2n(k+1)-j} + \beta^{n,s-1}_{2n(k+1)-j}p^{n,s-1}_{2n(k+1)-j-1}\right)\\
&\quad + v^{(n)}_{k,s}\left(p^{n,s-1}_{2nk+j+1} - xp^{n,s-1}_{2nk+j} + \beta^{n,s-1}_{2nk+j}p^{n,s-1}_{2nk+j-1}\right) = 0,
\end{aligned}$$

where

$$u^{(j)}_{k,s} = \frac{k+s}{2^{2j-1}(2k+s)(2k+s+1)} \quad \text{and} \quad v^{(n)}_{k,s} = \frac{(2k+2s-1)(2k+2s)}{2^{2n}(2k+s)^2}.$$

Clearly, its value is zero.

Case $i \equiv n-1 \pmod{2n}$.

In this case we obtain

$$p^{n,s-1}_{2n(k+1)+n} + \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s)(2k+s+1)} p^{n,s-1}_{2nk+n}$$

$$- x\left(p^{n,s-1}_{2n(k+1)+n-1} + \frac{k+s}{2^{2n-3}(2k+s)(2k+s+1)} p^{n,s-1}_{2n(k+1)-n+1}\right.$$

$$\left. + \frac{(2k+2s-1)(2k+2s)}{2^{2n}(2k+s)^2} p^{n,s-1}_{2nk+n-1}\right)$$

$$+ \frac{1}{4}\left(p^{n,s-1}_{2n(k+1)+n-2} + \frac{k+s}{2^{2n-5}(2k+s)(2k+s+1)} p^{n,s-1}_{2n(k+1)-n+2}\right.$$

$$\left. + \frac{(2k+2s-1)(2k+2s)}{2^{2n}(2k+s)^2} p^{n,s-1}_{2nk+n-2}\right),$$

which can be written as

$$p^{n,s-1}_{2n(k+1)+n} - x p^{n,s-1}_{2n(k+1)+n-1} + \beta^{n,s-1}_{2n(k+1)+n-1} p^{n,s-1}_{2n(k+1)+n-2}$$

$$+ u^{(n)}_{k,s}\left(p_{2n(k+1)-n+2} - x p^{n,s-1}_{2n(k+1)-n+1} + \beta^{n,s-1}_{2n(k+1)-n+1} p^{n,s-1}_{2n(k+1)-n}\right)$$

$$+ v^{(n)}_{k,s}\left(p^{n,s-1}_{2nk+n} - x p^{n,s-1}_{2nk+n-1} + \beta^{n,s-1}_{2nk+n-1} p_{2nk+n-2}\right),$$

where again every row equals zero. Here

$$u^{(n)}_{k,s} = \frac{k+s}{2^{2n-3}(2k+s)(2k+s+1)} \quad \text{and} \quad v^{(n)}_{k,s} = \frac{(2k+2s-1)(2k+2s)}{2^{2n}(2k+s)^2}.$$

Case $i \equiv n \pmod{2n}$.

In this case the corresponding term becomes

$$p^{n,s-1}_{2n(k+1)+n+1} - \frac{2k+2s+1}{4(2k+s+1)(2k+s+2)} p^{n,s-1}_{2n(k+1)+n-1}$$

$$+ \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)^2} p^{n,s-1}_{2nk+n+1}$$

$$- x\left(p^{n,s-1}_{2n(k+1)+n} + \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s)(2k+s+1)} p^{n,s-1}_{2nk+n}\right)$$

$$+ \frac{2k+2s+1}{4(2k+s+1)}\left(p^{n,s-1}_{2n(k+1)+n-1} + \frac{k+s}{2^{2n-3}(2k+s)(2k+s+1)} p^{n,s-1}_{2n(k+1)-n+1}\right.$$

$$\left. + \frac{(2k+2s-1)(2k+2s)}{2^{2n}(2k+s)^2} p^{n,s-1}_{2nk+n-1}\right),$$

which can be written in the form

$$p^{n,s-1}_{2n(k+1)+n+1} - xp^{n,s-1}_{2n(k+1)+n} + \beta^{n,s-1}_{2n(k+1)+n} p^{n,s-1}_{2n(k+1)+n-1}$$

$$+ \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s)(2k+s+1)} \left(p^{n,s-1}_{2nk+n+1} - xp^{n,s-1}_{2nk+n} + \beta^{n,s-1}_{2nk+n} p^{n,s-1}_{2nk+n-1} \right) = 0.$$

Again it is equal to zero since every row equals to zero.

Case $i \equiv n+1 \pmod{2n}$.

The corresponding term in this case is

$$p^{n,s-1}_{2n(k+1)+n+2} - \frac{2k+2s+1}{2^4(2k+s+1)(2k+s+2)} p^{n,s-1}_{2n(k+1)+n-2}$$

$$+ \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)^2} p^{n,s-1}_{2nk+n+2}$$

$$- x \left(p^{n,s-1}_{2n(k+1)+n+1} - \frac{2k+2s+1}{2^2(2k+s+1)(2k+s+2)} p^{n,s-1}_{2n(k+1)+n-1} \right.$$

$$\left. + \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)^2} p^{n,s-1}_{2nk+n+1} \right)$$

$$+ \frac{2k+1}{2^2(2k+s+1)} \left(p^{n,s-1}_{2n(k+1)+n} + \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s)(2k+s+1)} p^{n,s-1}_{2nk+n} \right)$$

$$= p^{n,s-1}_{2n(k+1)+n+2} - xp^{n,s-1}_{2n(k+1)+n+1} + \beta^{n,s-1}_{2n(k+1)+n+1} p^{n,s-1}_{2n(k+1)+n}$$

$$- u_{k,s} \left(p^{n,s-1}_{2n(k+1)+n} - xp^{n,s-1}_{2n(k+1)+n-1} + \beta^{n,s-1}_{2n(k+1)+n-1} p^{n,s-1}_{2n(k+1)+n-2} \right)$$

$$+ v^{(n)}_{k,s} \left(p^{n,s-1}_{2nk+n+2} - xp^{n,s-1}_{2nk+n+1} + \beta^{n,s-1}_{2nk+n+1} p^{n,s-1}_{2nk+n} \right) = 0,$$

where

$$u_{k,s} = \frac{2k+2s+1}{(2k+s+1)(2k+s+2)} \quad \text{and} \quad v^{(n)}_{k,s} = \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)^2}.$$

Again it is equal to zero since every row equals to zero.

Case $i \equiv n+j+1 \pmod{2n}$, $j = 1, \ldots, n-3$.

$$p^{n,s-1}_{2n(k+1)+n+j+2} - \frac{2k+2s+1}{2^{2(j+2)}(2k+s+1)(2k+s+2)} p^{n,s-1}_{2n(k+1)+n-j-2}$$

$$+ \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)^2} p^{n,s-1}_{2nk+n+j+2}$$

$$-x \left(p^{n,s-1}_{2n(k+1)+n+j+1} - \frac{2k+2s+1}{2^{2(j+1)}(2k+s+1)(2k+s+2)} p^{n,s-1}_{2n(k+1)+n-j-1} \right.$$

$$+\frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)^2}p^{n,s-1}_{2nk+n+j+1}\Bigg)$$

$$+\frac{1}{4}\Bigg(p^{n,s-1}_{2n(k+1)+n+j}-\frac{2k+2s+1}{2^{2j}(2k+s+1)(2k+s+2)}p^{n,s-1}_{2n(k+1)+n-j}$$

$$+\frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)^2}p^{n,s-1}_{2nk+n+j}\Bigg),$$

which reduces to

$$p_{2n(k+1)+n+j+2}-xp_{2n(k+1)+n+j+1}+\beta^{n,s-1}_{2n(k+1)+n+j+1}p_{2n(k+1)+n+j}$$

$$-u^{(j)}_{k,s}\left(p^{n,-1}_{2n(k+1)+n-j}-xp^{n,s-1}_{2n(k+1)+n-j-1}+\beta^{n,s-1}_{2n(k+1)+n-j-1}p^{n,s-1}_{2n(k+1)+n-j-2}\right)$$

$$+v^{(n)}_{k,s}\left(p^{n,s-1}_{2nk+n+j+2}-xp^{n,s-1}_{2nk+n+j+1}+\beta^{n,s-1}_{2nk+n+j+1}p^{n,s-1}_{2nk+n+j}\right),$$

which is equal to zero, since, every row equals zero. Here,

$$u^{(j)}_{k,s}=\frac{2k+2s+1}{2^{2j+2}(2k+s+1)(2k+s+2)}\quad\text{and}\quad v^{(n)}_{k,s}=\frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)^2}.$$

Case $i\equiv 2n-1 \pmod{2n}$.

In this case we have

$$p^{n,s-1}_{2n(k+2)}+\frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)(2k+s+2)}p^{n,s-1}_{2n(k+1)}$$

$$-x\Bigg(p^{n,s-1}_{2n(k+2)-1}-\frac{2k+2s+1}{2^{2n-1}(2k+s+1)(2k+s+2)}p^{n,s-1}_{2n(k+1)+1}$$

$$+\frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)^2}p^{n,s-1}_{2n(k+1)-1}\Bigg)$$

$$+\frac{1}{4}\Bigg(p^{n,s-1}_{2n(k+2)-2}-\frac{2k+2s+1}{2^{2n-2}(2k+s+1)(2k+s+2)}p^{n,s-1}_{2n(k+1)+2}$$

$$+\frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)^2}p^{n,s-1}_{2n(k+1)-2}\Bigg),$$

which reduces to

$$p^{n,s-1}_{2n(k+2)}-xp^{n,s-1}_{2n(k+2)-1}+\beta^{n,s-1}_{2n(k+2)-1}p^{n,s-1}_{2n(k+2)-2}$$

$$-u^{(n)}_{k,s}\left(p^{n,s-1}_{2n(k+1)+2}-xp^{n,s-1}_{2n(k+1)+1}+\beta^{n,s-1}_{2n(k+1)+1}p^{n,s-1}_{2n(k+1)}\right)$$

$$+v^{(n)}_{k,s}\left(p^{n,s-1}_{2n(k+1)}-xp^{n,s-1}_{2n(k+1)-1}+\beta^{n,s-1}_{2n(k+1)-1}p^{n,s-1}_{2n(k+1)-2}\right),$$

which is equal to zero since every row equals zero. Here,

$$u_{k,s}^{(n)} = \frac{2k+2s+1}{2^{2n-1}(2k+s+1)(2k+s+2)} \quad \text{and} \quad v_{k,s}^{(n)} = \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)^2}.$$

Case $i \equiv 2n \pmod{2n}$.

In this case we have

$$p_{2n(k+2)+1}^{n,s-1} + \frac{k+s+1}{2(2k+s+2)(2k+s+3)} p_{2n(k+2)-1}^{n,s-1}$$

$$+ \frac{(2k+2s+1)(2k+2s+2)}{2^{2n}(2k+s+2)^2} p_{2n(k+1)+1}^{n,s-1}$$

$$- x\left(p_{2n(k+2)}^{n,s-1} + \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)(2k+s+2)} p_{2n(k+1)}^{n,s-1}\right)$$

$$+ \frac{2(k+1)}{4(2k+s+2)}\left(p_{2n(k+2)-1}^{n,s-1} - \frac{2k+2s+1}{2^{2n-1}(2k+s+1)(2k+s+2)} p_{2n(k+1)+1}^{n,s-1}\right.$$

$$\left. + \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)^2} p_{2n(k+1)-1}^{n,s-1}\right)$$

which reduces to

$$p_{2n(k+2)+1}^{n,s-1} - x p_{2n(k+2)}^{n,s-1} + \beta_{2n(k+2)}^{n,s-1} p_{2n(k+2)-1}^{n,s-1}$$

$$+ \frac{(2k+2s)(2k+2s+1)}{2^{2n}(2k+s+1)(2k+s+2)} \left(p_{2n(k+1)+1}^{n,s-1} - x p_{2n(k+1)}^{n,s-1} + \beta_{2n(k+1)}^{n,s-1} p_{2n(k+1)-1}^{n,s-1}\right),$$

which is again equal to zero since every row equals zero.

Now we should check the following equality

$$\widehat{T}_n^2 = p_{2n}^{n,s-1} + \frac{(2s-2)(2s-1)}{2^{2n}(s-1)s}. \tag{2.4}$$

Notice that for the monic Chebyshev polynomial of the first kind $\widehat{T}_n$ we have $\widehat{T}_n^2 = 1/2^{2n-1} + \widehat{T}_{2n}$. Using the recurrence relation for Chebyshev polynomials $\widehat{T}_{n+1} = x\widehat{T}_n - \widehat{T}_{n-1}/4$ and the fact that

$$\beta_n^{n,s} = \frac{1+2s}{4(1+s)} = \frac{1}{4} + \frac{s}{4(1+s)},$$

we get

$$p_{n+1}^{n,s} = x p_n^{n,s} - \beta_n^{n,s} p_{n-1}^{n,s} = \widehat{T}_{n+1} - \frac{s}{4(s+1)}\widehat{T}_{n-1},$$

$$p_{n+2}^{n,s} = x p_{n+1}^{n,s} - \beta_{n+1}^{n,s} p_n^{n,s} = \widehat{T}_{n+2} - \frac{sx}{4(s+1)}\widehat{T}_{n-1} + \frac{s}{4(s+1)}\widehat{T}_n.$$

If we denote by

$$u_0 = -\frac{s}{4(s+1)}\widehat{T}_{n-1}, \quad u_1 = -\frac{sx}{4(s+1)}\widehat{T}_{n-1} + \frac{s}{4(s+1)}\widehat{T}_n,$$

we can write

$$p_{n+k}^{n,s} = \widehat{T}_{n+k} + u_{k-1}$$

where sequence of the polynomials u_k, $k = 0, \dots, n-1$, satisfy the relation

$$u_{k+1} = xu_k - \frac{1}{4}u_{k-1}.$$

We can prove easily that

$$u_k = -\frac{s}{4(s+1)}(\widehat{U}_k\widehat{T}_{n-1} - \widehat{U}_{k-1}\widehat{T}_n),$$

using induction, where $\widehat{U}_k$ is the kth-degree monic Chebyshev polynomial of the second kind.

Now we have

$$\begin{aligned} p_{2n}^{n,s} = \widehat{T}_{2n} + u_{n-1} &= \widehat{T}_{2n} - \frac{s}{4(s+1)}(\widehat{U}_{n-1}\widehat{T}_{n-1} - \widehat{U}_{n-2}\widehat{T}_n) \\ &= \widehat{T}_{2n} - \frac{s}{4(s+1)}\frac{1}{2^{2n-3}}, \end{aligned}$$

wherefrom it follows

$$p_{2n}^{n,s} = \widehat{T}_n^2 - \frac{1}{2^{2n-1}} - \frac{s}{s+1}\frac{1}{2^{2n-1}} = \widehat{T}_n^2 - \frac{1}{2^{2n-1}}\frac{2s+1}{s+1},$$

which is exactly (2.4) for $s := s+1$.

It remains to show that $\widehat{T}_n^2 p_1^{n,s} = q_1$. Thus, we have

$$\begin{aligned} q_1 &= p_{2n+1}^{n,s-1} + \frac{1}{2(s+1)}p_{2n-1}^{n,s-1} + \frac{2s-1}{2^{2n-1}s}p_1^{n,s-1} \\ &= xp_{2n}^{n,s-1} - \beta_{2n}^{n,s-1}p_{2n-1}^{n,s-1} + \frac{1}{2(s+1)}p_{2n-1}^{n,s-1} + \frac{2s-1}{2^{2n-1}s}x \\ &= xp_{2n}^{n,s-1} - \frac{1}{2(s+1)}p_{2n-1}^{n,s-1} + \frac{1}{2(s+1)}p_{2n-1}^{n,s-1} + \frac{2s-1}{2^{2n-1}s}x \\ &= \left(p_{2n}^{n,s-1} + \frac{2s-1}{2^{2n-1}s}\right)x = \widehat{T}_n^2 p_1^{n,s}. \end{aligned}$$

Having established (2.3) we now turn to verifying orthogonality of the sequences $\{p_k^{n,s}\}_{k\in\mathbb{N}_0}$ for $n, s \in \mathbb{N}$, $n \geq 2$.

Here we use an induction on s. The case $s = 1$ is just Theorem 1.1.

If we suppose that $n, s \in \mathbb{N}$, $n \geq 2$, are such that $\{p_k^{n,s-1}\}_{k\in\mathbb{N}_0}$ is a sequence of monic polynomials orthogonal with respect to the weight function $w^{n,s-1}$, then the orthogonality of the sequence $\{p_k^{n,s}\}_{k\in\mathbb{N}_0}$ easily follows from the relation $w^{n,s} = \widehat{T}_n^2 w^{n,s-1}$. After multiplying (2.3) by an appropriate $p_j^{n,s-1}$ and taking the integral of the both sides we get

$$\int_{-1}^{1} p_m^{n,s} p_j^{n,s-1} w^{n,s}\,\mathrm{d}x = \int_{-1}^{1} \widehat{T}_n^2 p_m^{n,s} p_j^{n,s-1} w^{n,s-1}\,\mathrm{d}x = 0.$$

Thus, the proof is finished. □

We used symbolic computations in MATHEMATICA, with the software package `OrthogonalPolynomials`, described in [2], in order to verify all given formulas.

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