Results. Math. 69 (2016), 431–441
© 2015 Springer International Publishing
1422-6383/16/030431-11
published online December 28, 2015
DOI 10.1007/s00025-015-0522-7

Results in Mathematics

von Neumann–Schatten Dual Frames and their Perturbations

Ali Akbar Arefijamaal and Ghadir Sadeghi

Abstract. In this paper, we discuss the dual of a von Neumann–Schatten p-frames in separable Banach spaces and obtain some of their characterizations. Moreover, we present a classical perturbation result to von Neumann–Schatten p-frames.

Mathematics Subject Classification. Primary 46C50, Secondary 42C99.

Keywords. von Neumann–Schatten operator, dual frame, perturbation.

1. Introduction and Preliminaries

The concept of a frame in Hilbert space has been introduced by Duffin and Schaeffer [12] in nonharmonic Fourier series. Since then various generalization of frames such as frame of subspaces [4,6], pseudo-frames [15], oblique frames [9], continuous frames [1,3,13], generalized frames [18], time-frequency localization operators [11], 2-frames [2] have been developed by several mathematicians. The concept of a frames in Banach space has been introduced by Christensen and Stoeva [8], Casazza, Han and Larson [7] and Gröchenig [14].

The p-frame and g-frame are two important generalizations of frames in Banach spaces and Hilbert spaces. In [17] the authors unify these two concepts. By utilizing von Neumann–Schatten frames many basic properties of frames can be derived in a more general setting. Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $\{\mathcal{K}_i : i \in \mathbb{N}\} \subset \mathcal{K}$ denotes a sequence of Hilbert spaces. Note that for any sequence $\{\mathcal{K}_i : i \in \mathbb{N}\}$ of Hilbert spaces, we can always find a larger space $\mathcal{K}$ containing all the Hilbert space $\mathcal{K}_i$ by setting $\mathcal{K} = \bigoplus_{i\in\mathbb{N}} \mathcal{K}_i$.

A sequence $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{N}\}$ is called a *generalized frame*, or simply a *g-frame* for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in \mathbb{N}\}$ if there are two positive constants A and B such that

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$$A\|f\|^2 \le \sum_{i\in\mathcal{I}} \|\Lambda_i f\|^2 \le B\|f\|^2 \qquad (f \in \mathcal{H}). \tag{1.1}$$

Let $(\mathcal{X}, \|.\|)$ be a separable Banach space with dual $\mathcal{X}^*$, $1 < p < \infty$ and q be the conjugate exponent to p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. A sequence $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$ is called a *p-frame* $(1 < p < \infty)$ if the norm $\|.\|$ of $\mathcal{X}$ is equivalent to the ℓ^p-norm of the sequence $\{g_i(.)\}$ or there exist constants A and B such that

$$A\|f\| \le \left(\sum_{i=1}^{\infty} |g_i(f)|^p\right)^{\frac{1}{p}} \le B\|f\| \qquad (f \in \mathcal{X}). \tag{1.2}$$

Christensen and Stoeva [8] studied p-frames in Banach spaces and obtained that every element $g \in \mathcal{X}^*$ to be represented as an unconditionally series $g = \sum_{i=1}^{\infty} \alpha_i g_i$ for coefficients $\{\alpha_i\} \in \ell^q$.

In [17] the authors introduce the notion of a *von Neumann–Schatten p-frame* and show that every p-frame for a separable Banach space $\mathcal{X}$ is a von Neumann–Schatten p-frame with respect to $\mathbb{C}$. Also, they give a characterization of *von Neumann–Schatten q-Riesz bases* for $\mathcal{X}^*$.

In the sequel, we introduce some necessary definitions and notations and refer the reader to [10,16]. Suppose $\{\mathcal{X}_i : i \in \mathcal{I}\}$ is a collection of normed spaces. Then $\Pi\{\mathcal{X}_i : i \in \mathcal{I}\}$ is a vector space if the linear operations are defined coordinatewise. For $1 \le p < \infty$, define

$$\bigoplus_p \mathcal{X}_i \equiv \{x \in \Pi_i \mathcal{X}_i : \|x\| = \left(\sum_i \|x_i\|^p\right)^{\frac{1}{p}} < \infty\}.$$

It is known that $\bigoplus_p \mathcal{X}_i$ is a Banach space if and only if so is each $\mathcal{X}_i$.

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$. For a compact operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$, let $s_1(\mathcal{A}) \ge s_2(\mathcal{A}) \ge \cdots \ge 0$ denote the singular values of $\mathcal{A}$, i.e., the eigenvalues of the positive operator $|\mathcal{A}| = (\mathcal{A}^*\mathcal{A})^{\frac{1}{2}}$, arranged in a decreasing order and repeated according to multiplicity. For $1 \le p < \infty$, the von Neumann–Schatten p-class $\mathcal{C}_p$ is defined to be the set all compact operators $\mathcal{A}$ for which $\sum_{i=1}^{\infty} s_i^p(\mathcal{A}) < \infty$.

For $\mathcal{A} \in \mathcal{C}_p$, the von Neumann Schatten p-norm of $\mathcal{A}$ is defined by

$$\|\mathcal{A}\|_p = \left(\sum_{i=1}^{\infty} s_i^p(\mathcal{A})\right)^{\frac{1}{p}} = (\tau|\mathcal{A}|^p)^{\frac{1}{p}} \tag{1.3}$$

where τ is the usual trace functional which is defined as $\tau(\mathcal{A}) = \sum_{e\in\mathcal{E}} \langle \mathcal{A}(e), e\rangle$, where $\mathcal{E}$ is any orthonormal basis of $\mathcal{H}$. It is convenient to let $\mathcal{C}_\infty$ denote the class of compact operators, and in this case $\|\mathcal{A}\|_\infty = s_1(\mathcal{A})$ is the usual operator norm.

If x, y are elements of a Hilbert space $\mathcal{H}$ we define the operator $x \otimes y$ on $\mathcal{H}$ by

$$(x \otimes y)(z) = \langle z, y \rangle x.$$

It is obvious that $\|x \otimes y\| = \|x\| \|y\|$ and the rank of $x \otimes y$ is one if x and y are non-zero. If $x_1, x_2, y_2, y_2 \in \mathcal{H}$, then the following equalities are easily verified:

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = \langle y_1, x_2 \rangle (x_1 \otimes y_2)$$
$$(x_1 \otimes y_1)^* = y_1 \otimes x_1.$$

Note that if $x, y \in \mathcal{H}$, then $\|x \otimes y\|_p = \|x \otimes y\|_q = \|x\| \|y\|$ and $\tau(x \otimes y) = \langle x, y \rangle$ so $x \otimes y$ is in $\mathcal{C}_p$ for all $p \geq 1$. We recall that $\mathcal{C}_p$ is a Banach space with respect to the norm $\|.\|_p$. It is known that the space $\mathcal{C}_2$ with the inner product

$$\langle \mathcal{A}, \mathcal{B} \rangle_\tau = \tau(\mathcal{B}^* \mathcal{A})$$

is a Hilbert space. If $\{\eta_i\}_{i=1}^\infty$ and $\{\zeta_i\}_{i=1}^\infty$ are orthonormal bases in $\mathcal{H}$ and $\nu_{i,j} = \eta_i \otimes \zeta_j$, then $\{\nu_{i,j}\}_{i,j=1}^\infty$ is an orthonormal basis of $\mathcal{C}_2$.

A countable family $\{\mathcal{G}_i\}_{i=1}^\infty$ of bounded linear operators from $\mathcal{X}$ to $\mathcal{C}_p \subseteq \mathcal{B}(\mathcal{H})$ is said to be a *von Neumann–Schatten p-frame* for $\mathcal{X}$ with respect to $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\| \leq \left(\sum_{i=1}^\infty \|\mathcal{G}_i(f)\|_p^p \right)^{\frac{1}{p}} \leq B\|f\| \qquad (f \in \mathcal{X}). \tag{1.4}$$

The sequence $\{\mathcal{G}_i\}_{i=1}^\infty$ is a *von Neumann–Schatten p-Bessel sequence* if at least the upper von Neumann–Schatten p-frame condition is satisfied. In the other words, a countable family $\{\mathcal{G}_i\}_{i=1}^\infty$ of bounded linear operators from $\mathcal{X}$ to $\mathcal{C}_p$ is a von Neumann–Schatten p-frame for $\mathcal{X}$ with respect to $\mathcal{H}$ if the norm $\|.\|$ of $\mathcal{X}$ is equivalent to ℓ^p-norm of the sequence $\{\mathcal{G}_i(.)\}_{i=1}^\infty$.

Now we define von Neumann–Schatten p-frame operators as follows. Let $\{\mathcal{G}_i\}_{i=1}^\infty$ be a von Neumann–Schatten p-frame for $\mathcal{X}$ with respect to $\mathcal{H}$. Define

$$U_\mathcal{G} : \mathcal{X} \to \bigoplus_p \mathcal{C}_P \tag{1.5}$$
$$U_\mathcal{G}(f) = \{\mathcal{G}_i(f)\}_{i=1}^\infty$$

and

$$T_\mathcal{G} : \bigoplus_q \mathcal{C}_q \to \mathcal{X}^* \tag{1.6}$$
$$T_\mathcal{G}(\{\mathcal{A}_i\}_{i=1}^\infty) = \sum_{i=1}^\infty \mathcal{A}_i \mathcal{G}_i.$$

The operator $U_\mathcal{G}$ is frequently called the analysis operator and $T_\mathcal{G}$ is the synthesis operator. It is clear that if $\{\mathcal{G}_i\}_{i=1}^\infty$ is a von Neumann–Schatten p-Bessel sequence then $U_\mathcal{G}$ is bounded. Moreover, $U_\mathcal{G}$ has closed range, and $\mathcal{X}$ is reflexive when $\{\mathcal{G}_i\}_{i=1}^\infty$ is a von Neumann–Schatten p-frame. A sequence

$\{\mathcal{G}_i\}_{i=1}^{\infty} \subseteq \mathcal{B}(\mathcal{X}, \mathcal{C}_p)$ is a von Neumann–Schatten p-Bessel sequence with respect to $\mathcal{H}$ with a bound B if and only if the operator defined by (1.6) is a well defined bounded operator with $\|T\| \leq B$. In addition, if $\mathcal{X}$ is reflexive, then $U_{\mathcal{G}} = T_{\mathcal{G}}^*$ and $\{\mathcal{G}_i\}_{i=1}^{\infty} \subseteq \mathcal{B}(\mathcal{X}, \mathcal{C}_p)$ is a von Neumann–Schatten p-frame for $\mathcal{X}$ if and only if the operator T is well-defined and onto, see [17] for more details.

If $\mathcal{H} = \mathbb{C}$, then $\mathcal{B}(\mathcal{H}) = \mathcal{C}_p = \mathcal{C}_q = \mathbb{C}$, $\bigoplus_p \mathcal{C}_p = \ell^p$ and also $\bigoplus_q \mathcal{C}_q = \ell^q$. Hence, a p-frame for $\mathcal{X}$ can be considered as a von Neumann–Schatten p-frame for $\mathcal{X}$ with respect to $\mathbb{C}$.

2. Dual of von Neumann–Schatten Frames

In this section, we discuss the dual of a von Neumann–Schatten p-frame in separable Banach spaces and obtain some of their characterizations. Moreover, we get a classical perturbation result to von Neumann–Schatten p-frames.

Definition 2.1. Let $\{\mathcal{G}_i\}_{i=1}^{\infty}$ be a von Neumann–Schatten p-Bessel sequence. A von Neumann–Schatten q-Bessel sequence $\{\mathcal{F}_i\}_{i=1}^{\infty} \subseteq \mathcal{B}(\mathcal{X}^*, \mathcal{C}_q)$ where $\frac{1}{p} + \frac{1}{q} = 1$, is called a dual for $\{\mathcal{G}_i\}_{i=1}^{\infty}$ if

$$\sum_{i=1}^{\infty} \mathcal{F}_i(\eta)\mathcal{G}_i = \eta, \quad (\eta \in \mathcal{X}^*). \tag{2.1}$$

The above equation, which is equivalent to the identity $T_{\mathcal{G}}T_{\mathcal{F}}^* = I_{\mathcal{X}^*}$, implies that $\{\mathcal{G}_i\}_{i=1}^{\infty}$ and $\{\mathcal{F}_i\}_{i=1}^{\infty}$ are in fact von Neumann–Schatten p-frame. Indeed, if B is an upper bound of $\{\mathcal{G}_i\}_{i=1}^{\infty}$, then for each $\eta \in \mathcal{X}^*$ we have

$$\begin{aligned}
\|\eta\| &= \left\| \sum_{i=1}^{\infty} \mathcal{F}_i(\eta)\mathcal{G}_i \right\| \\
&= \sup_{f \in \mathcal{X}, \|f\| \leq 1} \left\{ \sum_{i=1}^{\infty} \mathcal{F}_i(\eta)\mathcal{G}_i(f) \right\} \\
&\leq \sup_{f \in \mathcal{X}, \|f\| \leq 1} \left\{ \sum_{i=1}^{\infty} \|\mathcal{F}_i(\eta)\|_q^q \right\}^{\frac{1}{q}} \left\{ \sum_{i=1}^{\infty} \|\mathcal{G}_i(f)\|_p^p \right\}^{\frac{1}{p}} \\
&\leq B \left(\sum_{i=1}^{\infty} \|\mathcal{F}_i(\eta)\|_q^q \right)^{\frac{1}{q}}.
\end{aligned}$$

The argument to obtain the lower bound of $\{\mathcal{G}_i\}_{i=1}^{\infty}$ is similar. Notice that the existence of a dual p-frame and therefore von Neumann–Schatten p-frame is not guaranteed in general [8].

In the following proposition, we present a well-known characterization of dual of discrete frames to von Neumann–Schatten p-frames.

Proposition 2.2. *Let $\{\mathcal{G}_i\}_{i=1}^{\infty}$ be a von Neumann–Schatten p-frame. There exists an one-to-one correspondence between duals of $\{\mathcal{G}_i\}_{i=1}^{\infty}$ and left inverse of $T_{\mathcal{G}}^*$.*

Proof. Let $W : \bigoplus_p \mathcal{C}_p \to \mathcal{X}$ be a left inverse of $T^*_{\mathcal{G}}$. Define

$$\mathcal{F}_i : X^* \to \mathcal{C}_q, \quad \mathcal{F}_i(\eta) = \pi_i(W^*\eta) \tag{2.2}$$

where $\pi_i : \bigoplus \mathcal{C}_q \to \mathcal{C}_q$ is the standard projection on the i-th component. Then

$$\begin{aligned}\sum_{i=1}^{\infty} \|\mathcal{F}_i(\eta)\|_q^q &= \sum_{i=1}^{\infty} \|\pi_i(W^*\eta)\|_q^q \\ &= \|\{W^*\eta\}\|_q^q \leq \|W\|^q \|\eta\|^q.\end{aligned}$$

Hence, $\{\mathcal{F}_i\}_{i=1}^{\infty}$ is a von Neumann–Schatten p-Bessel. Moreover,

$$\begin{aligned}T_{\mathcal{G}} T^*_{\mathcal{F}}(\eta) &= \sum_{i=1}^{\infty} \mathcal{F}_i(\eta)\mathcal{G}_i \\ &= \sum_{i=1}^{\infty} \pi_i(W^*\eta)\mathcal{G}_i = T_{\mathcal{G}} W^*\eta = \eta.\end{aligned}$$

Conversely, assume that $\{\mathcal{F}_i\}_{i=1}^{\infty}$ is a dual for $\{\mathcal{G}_i\}_{i=1}^{\infty}$. Obviously $T_{\mathcal{F}}$ is a left inverses of $T^*_{\mathcal{G}}$. □

In the next theorem, we give a perturbation result on von Neumann–Schatten p-frames.

Theorem 2.3. *Let $\{\mathcal{G}_i\}_{i=1}^{\infty}$ be a von Neumann–Schatten p-frame with bounds A, B and let $\{\mathcal{V}_i\}_{i=1}^{\infty}$ be a sequence in $B(X, \mathcal{C}_p)$. Assume that there exist constants $\lambda_2 \in [0, 1)$ and $\lambda_1, \mu \geq 0$ such that*

$$\left\| \sum_{i=1}^{n} \mathcal{A}_i(\mathcal{G}_i - \mathcal{V}_i) \right\| \leq \lambda_1 \left\| \sum_{i=1}^{n} \mathcal{A}_i\mathcal{G}_i \right\| + \lambda_2 \left\| \sum_{i=1}^{n} \mathcal{A}_i\mathcal{V}_i \right\| + \mu \left(\sum_{i=1}^{n} \|\mathcal{A}_i\|_q^q \right)^{\frac{1}{q}} \tag{2.3}$$

for each finite sequence $\{\mathcal{A}_i\}_{i=1}^{n} \subseteq \oplus_q \mathcal{C}_q$. Then $\{\mathcal{V}_i\}_{i=1}^{\infty}$ is a von Neumann–Schatten p-Bessel for $\mathcal{X}$ with bound

$$\frac{(1 + \lambda_1)\, B + \mu}{1 - \lambda_2}.$$

Moreover, if there exists a dual $\{\mathcal{F}_i\}_{i=1}^{\infty} \subseteq B(X^*, \oplus \mathcal{C}_q)$ for $\{\mathcal{G}_i\}_{i=1}^{\infty}$ with upper bound D such that $\max\{\lambda_1 + \mu D, \lambda_2\} < 1$, then $\{\mathcal{V}_i\}_{i=1}^{\infty}$ is a von Neumann–Schatten p-frame with the lower bound

$$\frac{1 - (\lambda_1 + \mu D)}{D\,(1 + \lambda_2)}.$$

Proof. The synthesis operator $T_{\mathcal{G}}$ is bounded and $\|T_{\mathcal{G}}\| \leq B$ since $\{\mathcal{G}_i\}_{i=1}^{\infty}$ is a von Neumann–Schatten p-frame. Moreover, condition (2.3) implies that

$$\left\| \sum_{i=1}^{n} \mathcal{A}_i\mathcal{V}_i \right\| \leq \frac{1 + \lambda_1}{1 - \lambda_2} \left\| \sum_{i=1}^{n} \mathcal{A}_i\mathcal{G}_i \right\| + \frac{\mu}{1 - \lambda_2} \left(\sum_{i=1}^{n} \|\mathcal{A}_i\|_q^q \right)^{\frac{1}{q}} \tag{2.4}$$

for each finite sequence $\{\mathcal{A}_i\}_{i=1}^{n}$. It terms of the synthesis operators, (2.4) states that

$$\|T_{\mathcal{V}}\{\mathcal{A}_i\}\| \leq \frac{1+\lambda_1}{1-\lambda_2}\left\|\sum_{i=1}^{\infty} T_{\mathcal{G}}\mathcal{A}_i\right\| + \frac{\mu}{1-\lambda_2}\left(\sum_{i=1}^{\infty}\|\mathcal{A}_i\|_q^q\right)^{\frac{1}{q}}$$
$$\leq \frac{(1+\lambda_1)\,B+\mu}{1-\lambda_2}\|\{\mathcal{A}_i\}\|, \quad (\{\mathcal{A}_i\}\in\oplus\mathcal{C}_q).$$

So, $\{\mathcal{V}_i\}_{i=1}^{\infty}$ is a von Neumann–Schatten p-Bessel with the desired upper bound. Now assume that $\{\mathcal{F}_i\}_{i=1}^{\infty}$ is a dual of $\{\mathcal{G}_i\}_{i=1}^{\infty}$, then (2.3) can be expressed in terms of operators as

$$\|T_{\mathcal{G}}\mathcal{A}_i - T_{\mathcal{V}}\mathcal{A}_i\| \leq \lambda_1\|T_{\mathcal{G}}\mathcal{A}_i\| + \lambda_2\|T_{\mathcal{V}}\mathcal{A}_i\| + \mu\|\{\mathcal{A}_i\}\|. \tag{2.5}$$

Using (2.5) on the sequence $\{\mathcal{A}_i\} = T_{\mathcal{F}}^*\eta$ we obtain that

$$\begin{aligned}\|\eta - T_{\mathcal{V}}T_{\mathcal{F}}^*\eta\| &= \|T_{\mathcal{G}}T_{\mathcal{F}}^*\eta - T_{\mathcal{V}}T_{\mathcal{F}}^*\eta\|\\ &\leq \lambda_1\|\eta\| + \lambda_2\|T_{\mathcal{V}}T_{\mathcal{F}}^*\eta\| + \mu\|T_{\mathcal{F}}^*\eta\|\\ &= (\lambda_1+\mu D)\|\eta\| + \lambda_2\|T_{\mathcal{V}}T_{\mathcal{F}}^*\eta\|.\end{aligned}$$

It follows from [5, Lemma 1], that $T_{\mathcal{V}}T_{\mathcal{F}}^*$ is invertible and

$$\|T_{\mathcal{V}}T_{\mathcal{F}}^*\| \leq \frac{1+\lambda_1+\mu D}{1-\lambda_2}, \quad \left\|(T_{\mathcal{V}}T_{\mathcal{F}}^*)^{-1}\right\| \leq \frac{1+\lambda_2}{1-(\lambda_1+\mu D)}.$$

Hence, for any $\eta \in X^*$ we have

$$\|\eta\| = \left\|T_{\mathcal{V}}T_{\mathcal{F}}^*(T_{\mathcal{V}}T_{\mathcal{F}}^*)^{-1}\eta\right\| \leq \|T_{\mathcal{V}}\|\,\frac{D(1+\lambda_2)}{1-(\lambda_1+\mu D)}\|\eta\|.$$

Therefore

$$\frac{1-(\lambda_1+\mu D)}{D(1+\lambda_2)}\|f\| \leq \left(\sum_{i=1}^{\infty}\|\mathcal{V}_i(f)\|_p^p\right)^{\frac{1}{p}}.$$

□

3. Hilbert–Schmidt Frames and their Duals

Hilbert–Schmidt frames as a class of von Neumann–Schatten p-frames were introduced in [17]. In this section, we show that such frames have a dual and try to characterize their duals.

A sequence $\{\mathcal{G}_i\}_{i=1}^{\infty}$ of bounded linear operators from $\mathcal{H}$ into $\mathcal{C}_2 \subseteq \mathcal{B}(\mathcal{K})$ is said to be a Hilbert–Schmidt frame, or simply a HS-frame for $\mathcal{H}$ with respect to $\mathcal{K}$, if there exist two positive numbers A and B such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty}\|\mathcal{G}_i(f)\|_2^2 \leq B\|f\|^2, \qquad (f\in\mathcal{H}). \tag{3.1}$$

In particular, every g-frame can be considered as a Hilbert–Schmidt frame. More precisely, let $\{\Lambda_i : i \in \mathbb{N}\}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in \mathbb{N}\}$. Then $\{\Lambda_i : i \in \mathbb{N}\}$ is HS-frame for $\mathcal{H}$ with respect $\mathcal{K} = \bigoplus_{i\in\mathbb{N}} \mathcal{K}_i$.

Let $\{\mathcal{G}_i\}_{i=1}^{\infty}$ be a HS-frame for $\mathcal{H}$ with respect to $\mathcal{K}$. Define the HS-frame operator $S_{\mathcal{G}}$ as follows,

$$S_{\mathcal{G}} f = \sum_{i=1}^{\infty} \mathcal{G}_i^* \mathcal{G}_i(f) \qquad (f \in \mathcal{H}), \tag{3.2}$$

where $\mathcal{G}_i^*$ is the adjoint operator of $\mathcal{G}_i$. In [17], it is shown that HS-frame operator is bounded, invertible and self-adjoint, and therefore we have the following reconstruction formulas

$$\sum_{i=1}^{\infty} \mathcal{G}_i^* \mathcal{G}_i S^{-1} f = f = \sum_{i=1}^{\infty} S^{-1} \mathcal{G}_i^* \mathcal{G}_i f \quad (f \in \mathcal{H}). \tag{3.3}$$

Here, the synthesis operator $T_{\mathcal{G}} : \oplus\mathcal{C}_2 \to \mathcal{H}$ is given by

$$T_{\mathcal{G}}(\{\mathcal{A}_i\}) = \sum_{i=1}^{\infty} \mathcal{A}_i \mathcal{G}_i$$

where $(\sum_{i=1}^{\infty} \mathcal{A}_i \mathcal{G}_i) f = \sum_{i=1}^{\infty} \mathcal{A}_i \mathcal{G}_i(f)$ defines an element of $\mathcal{H}$.

According to (2.1) a HS-frame $\{\mathcal{F}_i\}$ is called a dual of $\{\mathcal{G}_i\}_{i=1}^{\infty}$ if $T_{\mathcal{F}} T_{\mathcal{G}}^* = I_{\mathcal{H}}$, or equivalently,

$$\sum \mathcal{G}_i^* \mathcal{F}_i(f) = f, \quad (f \in \mathcal{H}). \tag{3.4}$$

Due to (3.3) the sequence $\{\mathcal{G}_i S_{\mathcal{G}}^{-1}\}$, which is a HS-frame, is a dual of $\{\mathcal{G}_i\}_{i=1}^{\infty}$; so called the canonical dual. We now present a characterization of dual HS-frames.

Theorem 3.1. *Let $\{\mathcal{G}_i\}_{i=1}^{\infty}$ be a HS-frame with an upper bound B. There exists a one-to-one correspondence between the duals of $\{\mathcal{G}_i\}_{i=1}^{\infty}$ and the bounded operators $\Psi \in B(\mathcal{H}, \oplus\mathcal{C}_2)$ such that $T_{\mathcal{G}}\Psi = 0$*

Proof. Let $\{\mathcal{F}_i\}$ be a dual of HS-frame $\{\mathcal{G}_i\}_{i=1}^{\infty}$. Define $\Psi : \mathcal{H} \to \oplus\mathcal{C}_2$ by

$$\Psi f = \{\mathcal{F}_i f - \mathcal{G}_i S_{\mathcal{G}}^{-1} f\}, \quad (f \in \mathcal{H}) \tag{3.5}$$

Obviously, Ψ is bounded and

$$\begin{aligned} T_{\mathcal{G}} \Psi f &= \sum_{i=1}^{\infty} \left(\mathcal{F}_i(f) - \mathcal{G}_i S_{\mathcal{G}}^{-1} f\right) \mathcal{G}_i \\ &= \sum_{i=1}^{\infty} \mathcal{G}_i^* \left(\mathcal{F}_i(f) - \mathcal{G}_i S_{\mathcal{G}}^{-1} f\right) \\ &= \sum_{i=1}^{\infty} T_{\mathcal{G}} T_{\mathcal{F}}^* f - \sum_{i=1}^{\infty} \mathcal{G}_i^* \mathcal{G}_i S_{\mathcal{G}}^{-1} f = 0. \end{aligned}$$

Conversely, let $T_{\mathcal{G}}\Psi = 0$ for some $\Psi \in B(\mathcal{H}, \oplus\mathcal{C}_2)$. Letting $\mathcal{F}_i : \mathcal{H} \to \mathcal{C}_2$ by

$$\mathcal{F}_i f = \mathcal{G}_i S_{\mathcal{G}}^{-1} f + \pi_i(\Psi f), \quad (f \in \mathcal{H}). \tag{3.6}$$

Then

$$\begin{aligned}\sum_{i=1}^{\infty} \|\mathcal{F}_i f\|^2 &\leq \sum_{i=1}^{\infty} \|\mathcal{G}_i S_{\mathcal{G}}^{-1} f\|^2 + \sum_{i=1}^{\infty} \|\pi_i(\Psi f)\|^2 + 2\sum_{i=1}^{\infty} \|\mathcal{G}_i S_{\mathcal{G}}^{-1} f\| \|\pi_i(\Psi f)\| \\ &\leq \left(B\|S_{\mathcal{G}}^{-1}\| + \|\Psi\|^2 + 2\sqrt{B\|S_{\mathcal{G}}^{-1}\|}\|\Psi\| \right) \|f\|^2.\end{aligned}$$

i.e. $\{\mathcal{F}_i\}$ is a HS-Bessel. Moreover,

$$\begin{aligned}\sum_{i=1}^{\infty} \mathcal{G}_i^* \mathcal{F}_i f &= \sum_{i=1}^{\infty} \mathcal{G}_i^* \mathcal{G}_i S_{\mathcal{G}}^{-1} f + \sum_{i=1}^{\infty} \mathcal{G}_i^* \pi_i(\Psi f) \\ &= f + T_{\mathcal{G}}\Psi f = f.\end{aligned}$$

The proof is complete. □

The largest lower frame bound and the smallest upper frame bound in (3.1) are called the optimal frame bounds. Let $\{\mathcal{G}_i\}_{i=1}^{\infty}$ be a HS-frame with the optimal frame bounds A and B. It is not difficult to see that $B = \|S_{\mathcal{G}}\|$ and $A = \|S_{\mathcal{G}}^{-1}\|^{-1}$, moreover, $\frac{1}{B}$ and $\frac{1}{A}$ are the optimal frame bounds of $\{\mathcal{G}_i S_{\mathcal{G}}^{-1}\}_{i=1}^{\infty}$. The relation between optimal bounds of a frame and its duals is discussed in the following theorem.

Theorem 3.2. *Let $\{\mathcal{G}_i\}_{i=1}^{\infty}$ be a HS-frame with the optimal frame bounds A and B, and $\{\mathcal{F}_i\}_{i=1}^{\infty}$ be its dual with the optimal frame bounds C and D. Then $AD \geq 1$ and $BC \geq 1$. Furthermore, the following are equivalent:*

1. $D = \frac{1}{A}$,
2. $C = \frac{1}{B}$,
3. $\{\mathcal{F}_i\}_{i=1}^{\infty}$ *is the canonical dual.*

Proof. By using Theorem 3.1 there exists a bounded operator Ψ such that $T_{\mathcal{G}}\Psi = 0$ and

$$\mathcal{F}_i = \mathcal{G}_i S_{\mathcal{G}}^{-1} + \pi_i \Psi. \tag{3.7}$$

Hence,

$$\begin{aligned}\langle S_{\mathcal{F}} f, f\rangle &= \sum_{i=1}^{\infty} \langle \mathcal{F}_i f, \mathcal{F}_i f\rangle \\ &= \sum_{i=1}^{\infty} \left\langle \mathcal{G}_i S_{\mathcal{G}}^{-1} f, \mathcal{G}_i S_{\mathcal{G}}^{-1} f\right\rangle + \left\langle \mathcal{G}_i S_{\mathcal{G}}^{-1} f, \pi_i \Psi f\right\rangle \\ &\quad + \left\langle \pi_i \Psi f, \mathcal{G}_i S_{\mathcal{G}}^{-1} f\right\rangle + \|\pi_i \Psi f\|^2 \\ &= \left\langle f, S_{\mathcal{G}}^{-1} f\right\rangle + \left\langle T_{\mathcal{G}}^* S_{\mathcal{G}}^{-1} f, \Psi f\right\rangle + \left\langle \Psi f, T_{\mathcal{G}}^* S_{\mathcal{G}}^{-1} f\right\rangle + \langle \Psi f, \Psi f\rangle \\ &= \left\langle f, S_{\mathcal{G}}^{-1} f\right\rangle + \|\Psi f\|^2.\end{aligned}$$

for each $f \in \mathcal{H}$. On the other hand, since D is the optimal upper bound of $\{\mathcal{F}_i\}_{i=1}^{\infty}$ we obtain

$$\begin{aligned} D = \|S_{\mathcal{F}}\| &= \sup_{\|f\|=1} \{\langle S_{\mathcal{F}} f, f\rangle\} \\ &= \|S_{\mathcal{F}}^{-1}\| + \|\Psi\|^2 \\ &= \frac{1}{A} + \|\Psi\|^2 \geq \frac{1}{A}. \end{aligned}$$

In particular, $D = \frac{1}{A}$ if and only if $\Psi = 0$ or equivalently $\{\mathcal{F}_i\}_{i=1}^{\infty}$ is the canonical dual. Now (3.7) immediately shows (1) and (3) are equivalent. If we interchange HS-frames $\{\mathcal{G}_i\}_{i=1}^{\infty}$ and $\{\mathcal{F}_i\}_{i=1}^{\infty}$, we obtain $C \geq \frac{1}{B}$. Similarly one can show that (2) and (3) are equivalent. $\square$

Let $\{\mathcal{G}_i\}_{i=1}^{\infty}$ be a HS-frame for $\mathcal{H}$ with respect to $\mathcal{K}$ and $\{\nu_{i,j} : i, j \in \mathbb{N}\}$ be a orthonormal basis of $\mathcal{C}_2$. Define a bounded linear functional on $\mathcal{H}$ as follows

$$f \mapsto \langle \mathcal{G}_i f, \nu_{i,j}\rangle_\tau \quad (f \in \mathcal{H}).$$

By Riesz representation theorem, there exists $g_{j,l,k} \in \mathcal{H}$ such that

$$\langle \mathcal{G}_i f, \nu_{i,j}\rangle_\tau = \langle f, g_{j,l,k}\rangle \quad (f \in \mathcal{H}).$$

Hence

$$\mathcal{G}_i f = \sum_{l,k} \langle f, g_{j,l,k}\rangle \nu_{l,k} \quad (f \in \mathcal{H}).$$

The sequence $\{g_{j,l,k}\}$ is a Bessel sequence since

$$\sum_{l,k} |\langle f, g_{j,l,k}\rangle|^2 = \|\mathcal{G}_i f\|^2 \leq \|\mathcal{G}_i\|^2 \|f\|^2 \quad (f \in \mathcal{H}).$$

Now, for any $f \in \mathcal{H}$ and $\mathcal{A} \in \mathcal{C}_2$, we get

$$\langle f, \mathcal{G}_i^* \mathcal{A}\rangle = \langle \mathcal{G}_i f, \mathcal{A}\rangle_\tau = \left\langle \sum_{l,k} \langle f, g_{j,l,k}\rangle \nu_{l,k}, \mathcal{A} \right\rangle_\tau = \left\langle f, \sum_{l,k} \langle \mathcal{A}, \nu_{l,k}\rangle_\tau g_{j,l,k} \right\rangle.$$

Therefore

$$\mathcal{G}_i^* \mathcal{A} = \sum_{l,k} \langle \mathcal{A}, \nu_{l,k}\rangle \tau g_{j,l,k} \quad (\mathcal{A} \in \mathcal{C}_2).$$

In particular, if $\mathcal{A} = \nu_{n,m} = e_n \otimes e_m$, then

$$\mathcal{G}_i^*(e_n \otimes e_m) = g_{j,n,m}. \tag{3.8}$$

Based on the above discussion, we get a characterization of HS-frames, Riesz bases, and orthonormal bases.

Theorem 3.3. *Let $\{\mathcal{G}_i\}_{i=1}^{\infty}$ be a HS-frame and $g_{j,l,k}$ be defined as in (3.8). The sequence $\{\mathcal{G}_i\}_{i=1}^{\infty}$ is a HS-frame (resp. HS-Bessel sequence, tight HS-frame, HS-Riesz basis, HS-rthonormal basis) for $\mathcal{H}$ if and only if $\{g_{j,l,k} : j, k, l \in \mathbb{N}\}$ is a frame (resp. Bessel sequence, tight frame, Riesz basis orthonormal basis) for $\mathcal{H}$.*

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Ali Akbar Arefijamaal and Ghadir Sadeghi
Department of Mathematics and Computer Sciences
Hakim Sabzevari University
Sabzevar, Iran
e-mail: Arefijamaal@gmail.com;
Arefijamaal@hsu.ac.ir

Ghadir Sadeghi
e-mail: ghadir54@gmail.com;
g.sadeghi@hsu.ac.ir

Received: October 15, 2015.
Accepted: December 14, 2015.