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Results in Mathematics

Cellular Covers of Mixed Abelian Groups

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Abstract. In this paper, we answer a question of R. Göbel and L. Fuchs by showing that there exits large classes of non-splitting mixed groups which have no non-trivial cellular covers.

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A *cellular covering sequence* for an Abelian group A is an exact sequence

$$0 \to K \to G \xrightarrow{\gamma} A \to 0$$

for which the induced map $\gamma_* : \mathrm{Hom}(G,G) \to \mathrm{Hom}(G,A)$ is an isomorphism. Every group A admits a cellular covering sequence $0 \to 0 \to A \xrightarrow{\gamma} A \to 0$ with γ an automorphism of A, *called a trivial cellular cover*. Cellular covers of Abelian groups have been investigated by several authors over the last ten years (see [2–5]). While these investigations revealed interesting connections to infinite combinatorial principles, this paper focuses on cellular covers of mixed groups of finite torsion-free rank.

Unfortunately, a satisfactory description of torsion-free groups with non-trivial cellular covers exits only for subgroups of $\mathbb{Q}$: A torsion-free group of rank 1 admits a non-trivial cellular covering sequence if and only if its type is not idempotent [5, Theorem 5.4]. Fuchs and Göbel also showed that no reduced torsion group has a non-trivial cellular cover [5, Theorem 5.4], and asked whether there exist (large classes of) honest, i.e. non-splitting, mixed groups without any non-trivial covering sequences. It is the goal of this paper to give a positive answer to this question (Theorem 2). Often, non-trivial cellular covers $0 \to K \to G \to A \to 0$ of a torsion-free group A are constructed in such a way that $E(A) \cong E(G)$ [5, Lemma 2.2]. Theorem 2 will also show that this approach may fail for mixed groups.

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Lemma 1 [5, Lemma 2.1 and Theorem 4.3]. *Whenever* $0 \to K \to G \xrightarrow{\gamma} A \to 0$ *is a cellular covering sequence of a reduced Abelian group* A*, then*

(i) K *is torsion-free and reduced,*
(ii) $\phi(K) = 0$ *for all* $\phi \in \mathrm{Hom}(G, A)$*, and*
(iii) $\gamma|tG : tG \to tA$ *is an isomorphism.*

Let Γ be the class of reduced mixed groups G such that G_p is bounded for all primes p and G/tG is divisible. It contains the class $\mathcal{G}$ of (honest) self-small mixed groups A of finite torsion-free rank such that A/tA is divisible. The classes Γ and $\mathcal{G}$ play an important role in the theory of mixed torsion-free Abelian groups, and have been investigated in detail by several authors (see [1,6]).

Theorem 2. (a) *No Abelian group* $A \in \Gamma$ *has a non-trivial cellular cover.*
(b) *Let* A *be a mixed Abelian group of finite torsion-free rank such that* A_p *is finite for all primes* p*. If* A/pA *is finite for all primes* p *with* $A_p \neq 0$ *and* $A = pA$ *for all primes* p *with* $A_p = 0$*, then* A *has no non-trivial covering sequence* $0 \to K \to G \to A \to 0$ *with* $tE(G) \cong tE(A)$*.*

Proof. (a) We will show that $K = 0$ whenever $0 \to K \to G \xrightarrow{\gamma} A \to 0$ is a cellular covering sequence of a group $A \in \Gamma$. By Lemma 1, $\gamma|tG$ is an isomorphism, and K is a torsion-free group which fits into the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & tG & \xrightarrow{\gamma|tG} & tA & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{\gamma} & A & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G/tG & \longrightarrow & A/tA & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

The Snake-Lemma yields an exact sequence $0 \to K \to G/tG \to A/tA \to 0$.

For $\alpha \in E(A)$, there is a unique $\beta \in E(G)$ with $\gamma\beta = \alpha\gamma$ since γ_* is an isomorphism. Define a map $\phi : E(A) \to E(G)$ by $\phi(\alpha) = \beta$. To see that ϕ is a ring homomorphism, consider $\alpha_1, \alpha_2 \in E(A)$, and select $\beta_1, \beta_2 \in E(G)$ with $\gamma\beta_i = \alpha_i\gamma$. Since $(\alpha_1\alpha_2)\gamma = \alpha_1(\gamma\beta_2) = \gamma(\beta_1\beta_2)$, we obtain $\phi(\alpha_1\alpha_2) = \beta_1\beta_2 = \phi(\alpha_1)\phi(\alpha_2)$. Moreover, $\phi(1_A) = 1_G$ because of $\gamma 1_G = 1_A\gamma$. Finally, if $\phi(\alpha) = 0$, then $0 = \gamma\beta$ yields $\beta = 0$ since γ_* is an isomorphism. Thus, $\alpha = 0$.

Since $tE(G)$ is a two-sided ideal of $E(G)$, the canonical projection $\pi : E(G) \to S = E(G)/tE(G)$ is a ring homomorphism. The ring-homomorphism $\pi\phi : E(A) \to S$ induces a ring-monomorphism $\lambda : E(A)/tE(A) \to S$ such that $\lambda(1_A + tE(A)) = 1_S$. Therefore, we can view S as a right module over $R = \lambda(E(A)/tE(A))$ whose additive group is torsion-free.

Because of $A \in \Gamma$, the restriction maps induce a pure embedding of $E(A)$ into $\Pi_p E(A_p)$. Since each A_p is bounded, $tE(A) = \oplus_p E(A_p)$ and $[E(A)/tE(A)]^+$ is isomorphic to a pure subgroup of the divisible Abelian group $\Pi_p E(A_p)/\oplus_p E(A_p)$. Thus, R^+ is divisible, and the additive group of S is divisible too. For $0 \neq n \in \mathbb{Z}$, there are $\sigma \in E(G)$ and $\tau \in tE(G)$ such that $1_G - n\sigma = \tau$. Since $\tau(G) \subseteq tG$, we have $g - n\sigma(g) = \tau(g) \in tG$ for all $g \in G$. Thus, G/tG is divisible. Since $0 \to K \to G/tG \to A/tA \to 0$ is exact, K is divisible too. By Lemma 1, this is only possible if $K = 0$.

(b) As in the proof of a), a cellular covering sequence $0 \to K \to G \xrightarrow{\gamma} A \to 0$ of A satisfies $tA \cong tG$, and induces an exact sequence $0 \to K \to G/tG \to A/tA \to 0$. Referring to [5, Proposition 2.6], we may also assume that multiplication by p is an automorphism of G and K whenever $A_p = 0$ since $A = pA$ in this case.

Let p be a prime with $A_p \neq 0$. Since A_p is finite, we can write $A = A_p \oplus B$ where $A/tA \cong B/tB$ and $\mathrm{Hom}(B/tB, A_p) \cong \mathrm{Hom}(B, A_p)$ since $\mathrm{Hom}(tB, A_p) = 0$. If $F_p \cong \oplus_{m_p}\mathbb{Z}$ is a p-basic subgroup of B/tB, then

$$F_p/pF_p \cong (B/tB)/p(B/tB) \cong B/(pB + tB)$$

is finite as an image of the finite group A/pA. Thus,

$$\mathrm{Hom}(A/tA, A_p) \cong \mathrm{Hom}(B/tB, A_p) \cong \oplus_{m_p} A_p.$$

Therefore, $E(A)_p = \mathrm{Hom}(A, A_p) \cong E(A_p) \oplus \mathrm{Hom}(A/tA, A_p)$ is finite since $B_p = 0$. Because of $tE(A) \cong tE(G)$, we obtain that $E(G)_p$ is finite too. However, $G_p \cong A_p$ is finite, and $G = G_p \oplus H$. As before, $H/tH \cong G/tG$ and $\mathrm{Hom}(H, G_p) \cong \mathrm{Hom}(H/tH, G_p)$. We obtain

$$E(G)_p = \mathrm{Hom}(G, G_p) \cong E(G_p) \oplus \mathrm{Hom}(G/tG, G_p)$$

since $H_p = 0$. However, $E(A)_p \cong E(G)_p$ is finite. Hence, $|\,\mathrm{Hom}(G/tG, G_p)| < \infty$. Consequently,

$$\begin{aligned} |\,\mathrm{Hom}(A/tA, A_p)| &= \frac{|E(A)_p|}{|E(A_p)|} \\ &= \frac{|E(G)_p|}{|E(G_p)|} = |\,\mathrm{Hom}(G/tG, A_p)| < \infty \end{aligned}$$

Therefore, the first map in the sequence

$$0 \to \mathrm{Hom}(A/tA, A_p) \to \mathrm{Hom}(G/tG, A_p) \to \mathrm{Hom}(K, A_p) \to \mathrm{Ext}(G/tG, A_p) = 0$$

has to be an isomorphism. Since the Ext-group vanishes because A_p is finite, $\mathrm{Hom}(K, A_p) = 0$. This is only possible if $K = pK$ in view of the fact that K is torsion-free. Thus, K is divisible. By Lemma 1, $K = 0$. □

We want to emphasize two particular classes of mixed groups without non-trivial cellular covers:

Corollary 3. (a) *If $A \in \mathcal{G}$, then A does not have a non-trivial cellular cover.*

(b) *If A_p is bounded for each prime p, then $\Pi_p A_p$ does not have a non-trivial cellular cover.*

Part (a) of the last corollary raises the question whether all self-small mixed groups have only trivial cellular covers. The next result shows that this is not the case:

Theorem 4. *There exist honest self-small mixed group A_1 and A_2 of torsion-free rank $n \geq 2$ with $tA_1 \cong tA_2$ and $E(A_1) \cong E(A_2)$ such that A_1 admits a non-trivial cellular cover $0 \to K \to G \to A_1 \to 0$ with $E(G) \cong E(A_1)$, while A_2 admits no non-trivial cellular covering sequences at all.*

Proof. Divide the set $\mathcal{P}$ of primes into two infinite disjoint subsets $\mathcal{P}_1$ and $\mathcal{P}_2$. Choose a group $B \in \mathcal{G}$ with torsion-free rank $n-1$ such that $B_p \neq 0$ if $p \in \mathcal{P}_1$ and $B_p = 0$ otherwise. Select a rank 1 group B_1 with type $\tau_1 = [k_p]$ where $k_p = 1$ if $p \in \mathcal{P}_2$ and $k_p = \infty$ otherwise. Similarly, choose a rank 1 group B_2 with type $\tau_2 = [k_p]$ where $k_p = 0$ if $p \in \mathcal{P}_2$ and $k_p = \infty$ otherwise. Then, $E(B_1) \cong E(B_2)$. We consider $A_i = B_i \oplus B$ for $i = 1, 2$.

Since $\mathrm{Hom}(B_1, B_p) = \mathrm{Hom}(B_2, B_p) = 0$ for all primes $p \in \mathcal{P}_1$ yields $\mathrm{Hom}(B_i, B) = 0$ for $i = 1, 2$, we obtain that A_1 and A_2 are self-small groups with $E(A_i) \cong \mathbb{Z}_{\mathcal{P}_1} \times E(B)$. [5, Proposition 2.6 and Theorem 5.4] yield that B_1 has a non-trivial cellular cover $0 \to K \to G \to B_1 \to 0$ such that $G = pG$ and $K = pK$ for all $p \in \mathcal{P}_1$ and $E(G) \cong E(B_1)$. Since $B \in \mathcal{G}$, we have $\mathrm{Hom}(B, B_i) = \mathrm{Hom}(B, K) = 0$ for $i = 1, 2$. By [5, Proposition 2.5], $0 \to K \to G \oplus B \to B_1 \oplus B \to 0$ is a non-trivial cellular covering sequence of A_1. As before, $\mathrm{Hom}(B, G) = 0 = \mathrm{Hom}(G, B)$ yields $E(G \oplus B) \cong E(B) \times E(B_1) \cong E(A_1)$.

It remains to show that A_2 does not have a non-trivial cellular cover. If $0 \to K' \to G' \to A_2 \to 0$ were a non-trivial cellular cover, then it would induce an non-trivial cellular cover either for B_2 or for B. However, either by Fuchs' and Göbel's results on rank 1 torsion-free groups in [5] and Part (a) of Theorem 2, this is not possible. □

This result shows that one cannot expect to obtain a classification of the self-small mixed groups which admit non-trivial cellular covers.

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