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Comparison Results for Proper Nonnegative Splittings of Matrices

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Abstract. The theory of splittings of matrices is a useful tool in the analysis of iterative methods for solving systems of linear equations. When two splittings are given, it is of interest to compare the spectral radii of the corresponding iteration matrices. The aim of this paper is to bring out a few more comparison results for the recent matrix splitting called proper nonnegative splitting introduced by Mishra (Comput Math Appl 67:136–144, 2014). Comparison results for double proper nonnegative splittings are also discussed.

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1. Introduction

A real $n \times n$ matrix A is called *monotone* (or a matrix of "*monotone kind*") if $Ax \geq 0 \ \Rightarrow x \geq 0$. Here, $y \geq 0$ for $(y_1, y_2, \ldots, y_n)^T = y \in \mathbb{R}^n$ means that $y_i \geq 0$ (or y_i is non-negative) for all $i = 1, 2, \ldots, n$. This notion was introduced by Collatz, who showed that A is monotone if and only if A^{-1} exists and $A^{-1} \geq 0$, where the latter denotes that all the entries of A^{-1} are non-negative. The book by Collatz [8] has details of how monotone matrices arise naturally in the study of finite difference approximation methods for certain elliptic partial differential equations. The problem of characterizing monotone (also referred as *inverse positive*) matrices has been extensively dealt with in the literature. Motivated by Collatz's result, Mangasarian [17] extended the concept of monotone matrices to the rectangular case, and proved that a rectangular matrix is monotone if and only if it has a non-negative left inverse.

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The books by Berman and Plemmons [3] and Varga [28] give an excellent account of many of these characterizations. The former also presents several extensions of the notion monotonicity to rectangular and singular matrices in terms of different non-negative generalized inverses. One of the important generalized inverse is called Moore–Penrose inverse, and the definition is as follows. For a real $m \times n$ matrix A, the matrix G satisfying the four equations known as Penrose equations: $AGA = A,\ GAG = G,\ (AG)^T = AG$ and $(GA)^T = GA$ is called the *Moore–Penrose inverse* of A where B^T denotes the transpose of B. It always exists and unique, and is denoted by $A^\dagger$. In case of a nonsingular matrix A, we have $A^\dagger = A^{-1}$. If $A^\dagger \geq 0$, then A is called as a *semi-monotone* matrix.

Much effort also has been devoted to characterize semi-monotonicity in terms of matrix splitting.[1] Splitting matrix A into $A = U - V$, we have the following iterative method:

$$x^{(i+1)} = U^\dagger V x^{(i)} + U^\dagger b \tag{1}$$

for solving the linear system $Ax = b$, with $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. The scheme (1) is said to be *convergent* if the spectral radius of $U^\dagger V$ is less than 1, and $U^\dagger V$ is called the *iteration matrix*. A splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper splitting* [2] if $R(U) = R(A)$ and $N(U) = N(A)$, where $R(A)$ and $N(A)$ denote the range space and the null space of A, respectively. Berman and Plemmons [2] showed that the scheme (1) converges to $A^\dagger b$, the least square solution of minimum norm for any initial vector x^0 if and only if the spectral radius of $U^\dagger V$ is less than 1 (see [2], Corollary 1). A characterization of semi-monotonicity using proper splitting can be found in [2], Theorem 3. Nevertheless, before Berman and Plemmons [2], Keller [13] considered the problem of finding solution of a consistent singular and semidefinite linear system, iteratively. Further extension of such a problem was studied by Joshi [12] where he obtained necessary and sufficient conditions for finding solution of a consistent rectangular linear system, iteratively. Further studies on the problem of finding the solution of consistent singular system of linear equations by iterative method can be found in [20,25] and [18].

Now, we are going to recall the notion of regular, weak regular and nonnegative splittings for rectangular matrices. Note that earlier, Climent and Perea [6] also proposed the extension of regular splitting but they simply call this as regular splitting even for rectangular case (see [6], Definition 1). Again, Climent et al. [4] introduced extensions of weak regular and nonnegative splittings for rectangular matrices in [4], Definition 2, however they call as weak nonnegative proper splitting of the first type and weak proper splitting of the first type, respectively. We remark that all the authors in the literature did not

[1] A splitting of a real rectangular matrix A is an expression of the form $A = U - V$, where U and V are matrices of the same order as in A.

use the same classification even for nonsingular matrices (see [5,7,9,14,16,18–21,26–28] and [30]). In this article, we will follow the classifications as mentioned below. A proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is

(i) *proper regular splitting* if $U^\dagger \geq 0$ and $V \geq 0$ [11],
(ii) *proper weak regular splitting* if $U^\dagger \geq 0$ and $U^\dagger V \geq 0$ [11],
(iii) *proper nonnegative* if $U^\dagger V \geq 0$ [23].

In case of nonsingular matrices, the above definitions coincide with regular, weak regular [28] and nonnegative [27] splitting. Jena et al. [11] and Mishra [23] have proved several convergence and comparison results for the above splittings. Not only that, the above authors also introduced the notion of double proper regular and weak regular splittings [11] and double proper nonnegative splitting [23]. They again obtained convergence and comparison results for these double splittings. In this note, we are again going to prove a few more comparison results for proper nonnegative splittings and double proper nonnegative splittings. More specifically, we provide comparison theorems for two proper nonnegative splittings $A = U_1 - V_1 = U_2 - V_2$ of the same matrix A. Besides these, we prove comparison results for two proper nonnegative splittings $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ arising out of two real rectangular linear systems $A_1x = b$ and $A_2x = b$. The same problem is again studied using theory of double splitting.

The structure of the paper is as follows. The next section contains notation, definitions and preliminary tools. Section 3 discusses a few main results which compare convergence rate of proper nonnegative splittings. While Sect. 4 deals with the case of double proper nonnegative splittings. Finally, we end up with a section named Conclusions which addresses a scope of this work along with the summary of this work.

2. Preliminaries

In this section, we gather some notation, definitions and preliminary results which will be used later. $A \in \mathbb{R}^{m\times n}$ is *non-negative* if $A \geq 0$, and $B \geq C$ if $B - C \geq 0$. Let L and M be complementary subspaces of $\mathbb{R}^n$. Let also $P_{L,M}$ be a projector on L along M. Then $P_{L,M}A = A$ if and only if $R(A) \subseteq L$ and $AP_{L,M} = A$ if and only if $N(A) \supseteq M$. If $L \perp M$, then $P_{L,M}$ will be denoted by P_L. The following properties of $A^\dagger$ [1] will be used in the proofs of the next section: $R(A^\dagger) = R(A^T)$; $N(A^\dagger) = N(A^T)$; $AA^\dagger = P_{R(A)}$; $A^\dagger A = P_{R(A^T)}$.

Berman and Plemmons [2] showed that if $A = U - V$ is a proper splitting of $A \in \mathbb{R}^{m\times n}$, then $I - U^\dagger V$ is invertible and $A^\dagger = (I - U^\dagger V)^{-1}U^\dagger$. Similarly, the fact $U = A + V$ is a proper splitting implies that $I + A^\dagger V$ and $I + VA^\dagger$ are invertible, and $U^\dagger = (I + A^\dagger V)^{-1}A^\dagger = A^\dagger(I + VA^\dagger)^{-1}$. (See [2], Theorem 1 and [24], Theorem 3.1 for the respective proofs.) Also, 0 is not an eigenvalue of $I - U^\dagger V$ as $I - U^\dagger V$ is invertible. Hence 1 does not lie in the spectra of

$U^{\dagger}V$. Similarly, -1 does not lie in the spectra of $A^{\dagger}V$. The next lemma shows a relation between the eigenvalues of $U^{\dagger}V$ and $A^{\dagger}V$.

Lemma 2.1. ([24], Lemma 3.6) *Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m\times n}$. Let μ_i, $1 \le i \le s$ and λ_j, $1 \le j \le s$ be the eigenvalues of the matrices $U^{\dagger}V$ and $A^{\dagger}V$, respectively. Then for every j, we have $1 + \lambda_j \neq 0$. Also, for every i, there exists j such that $\mu_i = \frac{\lambda_j}{1+\lambda_j}$ and for every j, there exists i such that $\lambda_j = \frac{\mu_i}{1-\mu_i}$.*

Note that if $A = U - V$ is a proper splitting of $A \in \mathbb{R}^{m\times n}$, then $U^{\dagger}VA^{\dagger} = U^{\dagger}(U - A)A^{\dagger} = A^{\dagger} - U^{\dagger} = A^{\dagger}UU^{\dagger} - A^{\dagger}AU^{\dagger} = A^{\dagger}VU^{\dagger}$. This fact will be used in the next section. The spectral radius of $A \in \mathbb{R}^{n\times n}$, denoted by $\rho(A)$ is defined by $\rho(A) = \max\limits_{1\le i\le n} |\lambda_i|$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. It is known that for any two rectangular matrices B and C such that BC and CB are defined, $\rho(BC) = \rho(CB)$. The next results deal with non-negativity and spectral radius which will used in next two sections.

Theorem 2.2. ([28], Theorem 2.20) *Let A be a real square non-negative matrix. Then*

(i) *A has a non-negative real eigenvalue equal to its spectral radius.*
(ii) *There exists a non-negative eigenvector for its spectral radius.*

Theorem 2.3. ([28], Theorem 2.7) *Let $B \in \mathbb{R}^{n\times n}$ be an irreducible matrix and $B \ge 0$. Then*

(i) *B has a positive real eigenvalue equal to its spectral radius.*
(ii) *To $\rho(B)$ there corresponds an eigenvector $x > 0$.*
(iii) *$\rho(B)$ increases when any entry of B increases.*
(iv) *$\rho(B)$ is a simple eigenvalue of B.*

Theorem 2.4. ([28], Theorem 2.21) *If $A, B \in \mathbb{R}^{n\times n}$ and $A \ge B \ge 0$, then $\rho(A) \ge \rho(B)$.*

Lemma 2.5. ([19], Corollary 3.2) *If $B \in \mathbb{R}^{n\times n}$, $B \ge 0$ and $x \ge 0$ is such that $Bx - \alpha x \ge 0$, then $\alpha \le \rho(B)$.*

Theorem 2.6. ([28], Theorem 3.16) *Let $X \in \mathbb{R}^{n\times n}$ and $X \ge 0$. Then $\rho(X) < 1$ if and only if $(I - X)^{-1}$ exists and $(I - X)^{-1} = \sum\limits_{k=0}^{\infty} X^k \ge 0$.*

Lemma 2.7. ([26], Lemma 2.2) *Let $B, C \in \mathbb{R}^{n\times n}$, and I and O are the identity and null matrices, respectively, of order n. Suppose that $X = \begin{pmatrix} B & C \\ I & O \end{pmatrix} \ge 0$ and $\rho(B + C) < 1$. Then $\rho(X) < 1$.*

3. Comparison of Proper Nonnegative Splittings

In this section, we first recall some convergence and comparison results which are proved in [2,4] and [23]. We then present four new comparison results for proper nonnegative splittings, and also obtain new results for nonnegative splittings as corollaries. The next two are convergence results for proper nonnegative splitting of a matrix $A \in \mathbb{R}^{m\times n}$. The first one is also proved in [4], Theorem 2, however for a complete rank matrix (or left invertible matrix). While the author of [23] proved for any rectangular matrix, and is given below.

Lemma 3.1. ([23], Lemma 3.4) *Let $A = U - V$ be a proper nonnegative splitting of $A \in \mathbb{R}^{m\times n}$. If $A^\dagger U \geq 0$, then $\rho(U^\dagger V) = \frac{\rho(A^\dagger U)-1}{\rho(A^\dagger U)} < 1$.*

The proofs of the next result given in [2] and [23] are different. However, [2], Theorem 2 and [4], Theorem 2 are more general than the Lemma given below.

Lemma 3.2. ([2], Theorem 2 and [23], Lemma 3.5) *Let $A = U - V$ be a proper nonnegative splitting of $A \in \mathbb{R}^{m\times n}$. If $A^\dagger V \geq 0$, then $\rho(VU^\dagger) = \rho(U^\dagger V) = \frac{\rho(A^\dagger V)}{1+\rho(A^\dagger V)} < 1$.*

Comparison theorems between the spectral radii of matrices are useful tools in analysis of rate of convergence of iterative methods or for judging the efficiency of pre-conditioners. A matrix may have different matrix splittings $A = U_1 - V_1 = U_2 - V_2$. In practice, we seek such an U which not only makes the computation x^{i+1}(given x^i) simpler but also yields spectral radius of $U^\dagger V$ (which is of course less than 1) as small as possible for better convergence rate of the scheme (1). An accepted rule for preferring one iteration scheme to another is to choose the scheme having the smaller spectral radius. In this context, Jena et al. [11], Mishra and Sivakumar [24] and Mishra [23] have introduced various comparison results for different matrix splittings of rectangular matrices. We next recall a comparison result which appeared in [23].

Theorem 3.3. ([23],Theorem 3.10) *Let $A = U_1 - V_1 = U_2 - V_2$ be two proper nonnegative splittings of $A \in \mathbb{R}^{m\times n}$. If $A^\dagger V_2 \geq A^\dagger V_1 \geq 0$, then $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$.*

Our first result in this direction is given below.

Theorem 3.4. *Let $A = U_1 - V_1 = U_2 - V_2$ be two proper nonnegative splittings of $A \in \mathbb{R}^{m\times n}$. If $A^\dagger U_2 \geq A^\dagger U_1 \geq 0$, then $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$.*

Proof. By Lemma 3.1, the conditions $A^\dagger U_i \geq 0$ for $i = 1, 2$ imply $\rho(U_i^\dagger V_i) = \frac{\rho(A^\dagger U_i)-1}{\rho(A^\dagger U_i)} < 1$. The condition $A^\dagger U_1 \leq A^\dagger U_2$ and Theorem 2.4 together yield $\rho(A^\dagger U_1) \leq \rho(A^\dagger U_2)$. Let λ_i be the modulus of eigenvalue of $A^\dagger U_i$ for $i = 1, 2$. Since $\frac{\lambda_i - 1}{\lambda_i}$ is a strictly increasing function for $\lambda_i > 0$ and $\rho(A^\dagger U_2) \geq \rho(A^\dagger U_1)$, so we have $\frac{\rho(A^\dagger U_2)-1}{\rho(A^\dagger U_2)} \geq \frac{\rho(A^\dagger U_1)-1}{\rho(A^\dagger U_1)}$. Hence $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$. □

The next example shows that the converse is not true.

Example 3.5. Let $A = \begin{bmatrix} 8 & -5 & 0 \\ -7 & 9 & 0 \end{bmatrix} = \begin{bmatrix} 8 & -1 & 0 \\ -7 & 7 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 4 & 0 \\ 0 & -2 & 0 \end{bmatrix} = U_1 - V_1$. Then $U_1^\dagger V_1 = \begin{bmatrix} 0 & 26/49 & 0 \\ 0 & 12/49 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0$ and $A^\dagger U_1 = \begin{bmatrix} 1 & 26/37 & 0 \\ 0 & 49/37 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0$. Again $A = U_2 - V_2 = \begin{bmatrix} 8 & -2 & 0 \\ -7 & 8 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 3 & 0 \\ 0 & -1 & 0 \end{bmatrix}$. Then $R(U_2) = R(A)$, $N(U_2) = N(A)$, $U_2^\dagger V_2 = \begin{bmatrix} 0 & 22/50 & 0 \\ 0 & 13/50 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0$ and $A^\dagger U_2 = \begin{bmatrix} 1 & 22/37 & 0 \\ 0 & 50/37 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0$. So $A = U_1 - V_1 = U_2 - V_2$ are two proper nonnegative splittings with $\rho(U_1^\dagger V_1) = .2449 \leq .2600 = \rho(U_2^\dagger V_2) < 1$. But $A^\dagger U_1 \nleq A^\dagger U_2$.

When A is nonsingular, we have the following result for nonnegative splitting.

Corollary 3.6. *Let $A = U_1 - V_1 = U_2 - V_2$ be two nonnegative splittings of $A \in \mathbb{R}^{n\times n}$. If $A^{-1}U_2 \geq A^{-1}U_1 \geq 0$, then $\rho(U_1^{-1}V_1) \leq \rho(U_2^{-1}V_2) < 1$.*

The next result is true only for square matrices.

Theorem 3.7. *Let $A = U_1 - V_1 = U_2 - V_2$ be two proper nonnegative splittings of a semi-monotone matrix $A \in \mathbb{R}^{n\times n}$. If $V_2U_2^\dagger \geq U_1^\dagger V_1$ and $V_i \geq 0$ for $i = 1, 2$, then*

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1.$$

Proof. We have $\rho(U_i^\dagger V_i) = \rho(V_iU_i^\dagger)$ for $i = 1, 2$. Then, Lemma 3.2, $\rho(U_i^\dagger V_i) < 1$ as $A^\dagger V_i \geq 0$ for $i = 1, 2$. Also, we have $(I + A^\dagger V_1)^{-1}A^\dagger = U_1^\dagger$ and $U_2^\dagger = A^\dagger(I + V_2A^\dagger)^{-1}$. Now $V_2U_2^\dagger \geq U_1^\dagger V_1$ implies $V_2A^\dagger(I + V_2A^\dagger)^{-1} \geq (I + A^\dagger V_1)^{-1}A^\dagger V_1$. Then pre-multiplying $I + A^\dagger V_1$ and post-multiplying $I + V_2A^\dagger$ (as $I + A^\dagger V_1 \geq 0$ and $I + V_2A^\dagger \geq 0$), we obtain $V_2A^\dagger \geq A^\dagger V_1$. By Lemma 2.4, we get $\rho(V_2A^\dagger) = \rho(A^\dagger V_2) \geq \rho(A^\dagger V_1)$, i.e., $\rho(V_2A^\dagger) \geq \rho(A^\dagger V_1)$. Since $\frac{\lambda}{\lambda+1}$ is a strictly increasing function for $\lambda \geq 0$, we have $\frac{\rho(A^\dagger V_2)}{1+\rho(A^\dagger V_2)} \geq \frac{\rho(A^\dagger V_1)}{1+\rho(A^\dagger V_1)}$. Hence $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$. □

The example given below demonstrates that the converse of the above result is not true.

Example 3.8. Let $A = \begin{bmatrix} 5 & -4 \\ -7 & 6 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -7 & 5 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = U_1 - V_1$. Then $U_1^\dagger V_1 = \begin{bmatrix} 0 & 1/2 \\ 0 & 1/2 \end{bmatrix} \geq 0$. Consider $A = \begin{bmatrix} 5 & -2 \\ -7 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} = U_2 - V_2$.

Then $U_2^\dagger V_2 = \begin{bmatrix} 0 & 4/6 \\ 0 & 4/6 \end{bmatrix} \geq 0$ and $V_2 U_2^\dagger = \begin{bmatrix} 14/6 & 10/6 \\ -14/6 & -10/6 \end{bmatrix}$. So $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$. But $V_i \ngeq 0$ and $U_1^\dagger V_1 \nleq V_2 U_2^\dagger$.

Theorem 3.7 admits the following corollary.

Corollary 3.9. *Let $A = U_1 - V_1 = U_2 - V_2$ be two nonnegative splittings of a monotone matrix $A \in \mathbb{R}^{n\times n}$. If $V_2 U_2^{-1} \geq U_1^{-1} V_1$ and $V_i \geq 0$ for $i = 1, 2$, then*

$$\rho(U_1^{-1} V_1) \leq \rho(U_2^{-1} V_2) < 1.$$

We remark that Theorem 3.7 can also be restated as following for proper regular splittings.

Remark 3.10. Let $A = U_1 - V_1 = U_2 - V_2$ be two proper regular splittings of a semi-monotone matrix $A \in \mathbb{R}^{n\times n}$. If $V_2 U_2^\dagger \geq U_1^\dagger V_1$, then

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1.$$

We now proceed to discuss comparison results for two different linear system of equations. One of the motivation for comparing two different linear system comes from theory of pre-conditioning. Suppose that we have two pre-conditioners P_1 and P_2 such that $A_1 = P_1 A$ and $A_2 = P_2 A$, where P_1 and P_2 two are real square matrices of order m. It is of then interest to know which system will converge faster. This will also help us to choose a better pre-conditioner for solving a linear system. This query is addressed in the next two results.

Theorem 3.11. *Let $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be two proper nonnegative splittings of semi-monotone matrices A_1 and A_2, respectively. If $A_2^\dagger V_2 \geq A_1^\dagger V_1 \geq 0$, then $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$.*

Proof. Since $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ are two proper nonnegative splitting matrices. We then have $U_1^\dagger V_1 \geq 0$, $U_2^\dagger V_2 \geq 0$ and $A_2^\dagger V_2 \geq A_1^\dagger V_1 \geq 0$. As $A_i^\dagger V_i \geq 0$, so $\rho(U_i^\dagger V_i) < 1$ for $i = 1, 2$. We have to now show that $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2)$. Let λ_i and μ_i be the modulus of eigenvalues of $A_i^\dagger V_i$ and $U_i^\dagger V_i$, respectively. By Lemma 2.1, $\mu_i = \frac{\lambda_i}{1+\lambda_i}$. Hence μ_i attains its maximum if λ_i is maximum. But λ_i is maximum when $\lambda_i = \rho(A_i^\dagger V_i)$, as a result μ_i is maximum when $\mu_i = \rho(U_i^\dagger V_i)$. Let $f(\lambda_i) = \frac{\lambda_i}{1+\lambda_i}$ be the increasing function for $\lambda_i \geq 0$ for $i = 1,\ 2$. Then $\rho(U_i^\dagger V_i) = \frac{\rho(A_i^\dagger V_i)}{1+\rho(A_i^\dagger V_i)}$ is an increasing function for $A_i^\dagger V_i \geq 0$. The condition $A_2^\dagger V_2 \geq A_1^\dagger V_1$ yields $\frac{\rho(A_2^\dagger V_2)}{1+\rho(A_2^\dagger V_2)} \geq \frac{\rho(A_1^\dagger V_1)}{1+\rho(A_1^\dagger V_1)}$. Hence $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$. □

In case of nonsingular matrices, we have the following Corollary which is also a part of [7], Theorem 15 whether the authors proved the same result in a more general setting.

Corollary 3.12. *Let $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be two nonnegative splittings of monotone matrices A_1 and A_2, respectively. If $A_2^{-1}V_2 \geq A_1^{-1}V_1 \geq 0$, then $\rho(U_1^{-1}V_1) \leq \rho(U_2^{-1}V_2) < 1$.*

Next result deals with comparison results of different types of proper nonnegative splittings. Note that Climent et al. [4] proposed these definitions first and call them weak proper splittings of different types (see [4], Definition 2). However, we call them here proper nonnegative of different types. The definitions are recalled next.

Definition 3.13. ([4], *Definition* 2) A proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a proper nonnegative splitting of type I if $U^\dagger V \geq 0$.

Hence proper nonnegative splitting of type I is same as proper nonnegative splitting. However, proper nonnegative splitting of type II is slightly different than this, and is presented next.

Definition 3.14. ([4], *Definition* 2) A proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a proper nonnegative splitting of type II if $VU^\dagger \geq 0$.

[4], Remark 2 says that Lemma 3.2 is also true for proper nonnegative splitting of type II. The next result compares rate of convergence of two different types of proper nonnegative splittings.

Theorem 3.15. *Let $A_1 = U_1 - V_1$ be a proper nonnegative splitting of type II and $0 \neq A_2 = U_2 - V_2$ be a proper nonnegative splitting of type I of semi-monotone matrices A_1 and A_2, respectively. Suppose that $A_1^\dagger - A_2^\dagger \geq 0$ and $A_2^\dagger V_2 \geq 0$. If $U_1^\dagger - U_2^\dagger \geq A_1^\dagger - A_2^\dagger$, then $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$.*

Proof. Since $A_2 = U_2 - V_2$ is a proper nonnegative splitting of type I of a semi-monotone matrix A_2 and $A_2^\dagger V_2 \geq 0$, then $\rho(U_2^\dagger V_2) < 1$, by Lemma 3.2. Hence it suffices to show that $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2)$. Since the splittings $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ are proper splittings, so we obtain $U_i^\dagger U_i A_i^\dagger = A_i^\dagger$ and $U_i^\dagger A_i A_i^\dagger = U_i^\dagger$ for $i = 1,\ 2$, and we also have $A_1^\dagger A_1 U_1^\dagger = U_1^\dagger$ and $A_1^\dagger U_1 U_1^\dagger = A_1^\dagger$. Using $U_1^\dagger - U_2^\dagger \geq A_1^\dagger - A_2^\dagger$ and properties of proper nonnegative splitting, we obtain

$$\begin{aligned} U_2^\dagger V_2 A_2^\dagger &= U_2^\dagger (U_2 - A_2) A_2^\dagger \\ &= A_2^\dagger - U_2^\dagger \\ &\geq A_1^\dagger - U_1^\dagger \\ &= U_1^\dagger (U_1 - A_1) A_1^\dagger \\ &= U_1^\dagger V_1 A_1^\dagger \\ &= A_1^\dagger V_1 U_1^\dagger \\ &\geq 0. \end{aligned}$$

Also, we have $V_1U_1^\dagger \geq 0$ and $U_2^\dagger V_2 \geq 0$. By (ii) of Theorem 2.2, there exist two non-negative vectors x and y such that

$$V_1U_1^\dagger x = \rho(U_1^\dagger V_1)x,\ y^TU_2^\dagger V_2 = y^T\rho(U_2^\dagger V_2).$$

Thus

$$\rho(U_2^\dagger V_2)y^TA_2^\dagger x = y^TU_2^\dagger V_2A_2^\dagger x \geq y^TA_1^\dagger V_1U_1^\dagger x = \rho(U_1^\dagger V_1)y^TA_1^\dagger x.$$

Since $A_1^\dagger \geq A_2^\dagger$, so we have

$$\rho(U_2^\dagger V_2)y^TA_2^\dagger x \geq \rho(U_1^\dagger V_1)y^TA_2^\dagger x.$$

Therefore

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2)$$

in case of $y^TA_2^\dagger x > 0$. The case $y^TA_2^\dagger x = 0$ is discussed below. Let J be real matrix of same order as in A_1 and A_2 such that $J > 0$. Then

$$(\epsilon JA_2^\dagger + V_1U_1^\dagger)\widehat{x} = \widehat{\lambda_1}\widehat{x} \tag{2}$$

and

$$\widehat{y}^T(\epsilon A_1^\dagger J + U_2^\dagger V_2) = \widehat{\lambda_2}\widehat{y}^T, \tag{3}$$

by Theorem 2.2, where $\epsilon > 0$, $\widehat{\lambda_1} = \rho(\epsilon JA_2^\dagger + V_1U_1^\dagger)$ and $\widehat{\lambda_2} = \rho(\epsilon A_1^\dagger J + U_2^\dagger V_2)$. The condition $\epsilon > 0$ yields that both $\epsilon JA_2^\dagger + V_1U_1^\dagger$ and $\epsilon A_1^\dagger J + U_2^\dagger V_2$ are irreducible. By Theorem 2.3, $\widehat{\lambda_1}$ and $\widehat{\lambda_2}$ are increasing functions of $\epsilon \geq 0$, and $\widehat{\lambda_1} = \rho(V_1U_1^\dagger)$ and $\widehat{\lambda_2} = \rho(U_2^\dagger V_2)$. Pre-multiplying $A_1^\dagger$ to Eq. (2) and post-multiplying $A_2^\dagger$ to Eq. (3), we have

$$(\epsilon A_1^\dagger JA_2^\dagger + A_1^\dagger V_1U_1^\dagger)\widehat{x} = \widehat{\lambda_1}A_1^\dagger\widehat{x}$$

and

$$\widehat{y}^T(\epsilon A_1^\dagger JA_2^\dagger + U_2^\dagger V_2A_2^\dagger) = \widehat{\lambda_2}\widehat{y}^TA_2^\dagger.$$

Using the condition $U_2^\dagger V_2A_2^\dagger \geq A_1^\dagger V_1U_1^\dagger$, we have

$$(\epsilon A_1^\dagger JA_2^\dagger + U_2^\dagger V_2A_2^\dagger)\widehat{x} \geq (\epsilon A_1^\dagger JA_2^\dagger + A_1^\dagger V_1U_1^\dagger)\widehat{x} = \widehat{\lambda_1}A_1^\dagger\widehat{x}.$$

Now, pre-multiplying $\widehat{y}^T$ to the above equation, we get

$$\widehat{\lambda_2}\widehat{y}^TA_2^\dagger\widehat{x} \geq \widehat{\lambda_1}\widehat{y}^TA_1^\dagger\widehat{x} \geq \widehat{\lambda_1}\widehat{y}^TA_2^\dagger\widehat{x}.$$

Since $\epsilon > 0$, both vectors $\widehat{x}$ and $\widehat{y}$ are positive, so $\widehat{y}^TA_2^\dagger\widehat{x} > 0$. Hence $\widehat{\lambda_1} \leq \widehat{\lambda_2}$. Taking the limit for $\epsilon \to 0$, we have $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$. □

Note that the condition $A_2^\dagger V_2 \geq 0$ in the above result can also be replaced by $A_2^\dagger U_2 \geq 0$. One easy choice of J is a matrix whose entries are 1. The idea of the proof for the case $y^TA_2^\dagger x = 0$ is borrowed from the remark given in page 310 of [30]. As a corollary, we have the following result for nonsingular matrices.

Corollary 3.16. *Let* $A_1 = U_1 - V_1$ *be a nonnegative splitting of type II and* $0 \neq A_2 = U_2 - V_2$ *be a nonnegative splitting of type I of monotone matrices* A_1 *and* A_2*, respectively. Suppose that* $A_1^{-1} - A_2^{-1} \geq 0$ *and* $A_2^{-1}V_2 \geq 0$*. If* $U_1^{-1} - U_2^{-1} \geq A_1^{-1} - A_2^{-1}$*, then* $\rho(U_1^{-1}V_1) \leq \rho(U_2^{-1}V_2) < 1$.

We remark that Theorem 3.15 extends [22], Theorem 2.4, where the authors have proved a similar result for proper weak regular splittings of different types. *Not only that the proof of* [22]*, Theorem 2.4 did not contain the case* $y^T A_2^{\dagger} x = 0$. Converse of Theorem 3.15 is also not true, and is shown by the following example.

Example 3.17. Let $A_1 = \begin{bmatrix} 4 & -5 & 0 \\ -3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 0 \\ -3 & 5 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \end{bmatrix} =$ $U_1 - V_1$. We have $A_1^{\dagger} = \begin{bmatrix} 6/9 & 5/9 \\ 3/9 & 4/9 \\ 0 & 0 \end{bmatrix}$ and $U_1^{\dagger} = \begin{bmatrix} 5/11 & 3/11 \\ 3/11 & 4/11 \\ 0 & 0 \end{bmatrix}$. Let $A_2 =$ $\begin{bmatrix} 7 & -3 & 0 \\ -8 & 7 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -2 & 0 \\ -8 & 9 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} = U_2 - V_2$. Then $A_2^{\dagger} = \begin{bmatrix} 7/25 & 3/25 \\ 8/25 & 7/25 \\ 0 & 0 \end{bmatrix}$, $A_1^{\dagger} - A_2^{\dagger} = \begin{bmatrix} 0.3866 & 0.43555 \\ 0.01333 & 0.16444 \\ 0 & 0 \end{bmatrix}$ and $U_1^{\dagger} - U_2^{\dagger} = \begin{bmatrix} 0.2630 & 0.2301 \\ 0.10251 & 0.21470 \\ 0 & 0 \end{bmatrix}$. Here, we have $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ are proper nonnegative splittings of type II and type I, respectively, and $\rho(U_1^{\dagger}V_1) = .1818 \leq .4681 = \rho(U_2^{\dagger}V_2) < 1$, but $U_1^{\dagger} - U_2^{\dagger} \ngeq A_1^{\dagger} - A_2^{\dagger}$.

Finally, we conclude this section with the remark that Theorem 3.15 can also be proved in a similar way for the case $A_1 = U_1 - V_1$ is a proper nonnegative splitting of type I and $A_2 = U_2 - V_2$ is a proper nonnegative splitting of II.

4. Comparison of Double Proper Nonnegative Splittings

One of the necessity to study theory of double splitting is motivated by the fact that we can not ensure convergence of all proper nonnegative splittings using the known results. This issue can be partially settled by studying convergence theory of double proper nonnegative splitting because of [23], Theorem 4.3 which says that a convergent double proper nonnegative splitting is also a convergent proper nonnegative splitting. Study of convergence theory of double proper splitting further extends the case of double splitting for nonsingular matrix introduced by Woźnicki [29]. Standard iterative methods like Jacobi, Gauss-Seidel, SOR etc. can also be obtained by choosing particular matrices in the double splitting of A, and is shown in [29]. This section focuses on comparison results for double proper nonnegative splittings. Some results even add new theory to the existing theory for nonsingular matrices. As an application, theory of double proper splitting is also helpful in choosing pre-conditioners (see [21], Sect. 4).

Jena et al. [11] and Mishra [23] proposed different types of double splittings for real rectangular matrices and studied their convergence theory. We now recall the same theory first and then obtain a few comparison results. A double splitting $A = P - R - S$ of $A \in \mathbb{R}^{m\times n}$ is called *double proper* if $R(P) = R(A)$ and $N(P) = N(A)$. A double proper splitting $A = P - R - S$ of A (to $Ax = b$) leads to the following iterative scheme spanned by three iterates:

$$x^{i+1} = P^{\dagger}Rx^{i} + P^{\dagger}Sx^{i-1} + P^{\dagger}b, \quad i > 0. \tag{4}$$

Then

$$\begin{pmatrix} x^{i+1} \\ x^{i} \end{pmatrix} = \begin{pmatrix} P^{\dagger}R & P^{\dagger}S \\ I & O \end{pmatrix} \begin{pmatrix} x^{i} \\ x^{i-1} \end{pmatrix} + \begin{pmatrix} P^{\dagger}b \\ O \end{pmatrix},$$

i.e.,

$$y^{i+1} = Wy^{i} + d, \tag{5}$$

where $y^{i+1} = \begin{pmatrix} x^{i+1} \\ x^{i} \end{pmatrix}$, $y^{i} = \begin{pmatrix} x^{i} \\ x^{i-1} \end{pmatrix}$, $W = \begin{pmatrix} P^{\dagger}R & P^{\dagger}S \\ I & O \end{pmatrix}$ and $d = \begin{pmatrix} P^{\dagger}b \\ O \end{pmatrix}$. Here I and O stand for identity and null matrices, respectively with appropriate order. The iteration scheme (5) is *convergent* if $\rho(W) < 1$, and then $A = P - R - S$ is called as a *convergent double splitting*.

Let us recall the definition of a double proper nonnegative splitting introduced by Mishra [23] for a real rectangular matrix.

Definition 4.1. ([23], *Definition* 4.1) A double splitting $A = P - R - S$ of $A \in \mathbb{R}^{m\times n}$ is called double proper nonnegative splitting if $R(P) = R(A)$, $N(P) = N(A)$, $P^{\dagger}R \geq 0$ and $P^{\dagger}S \geq 0$.

Setting $P = U$ and $R + S = V$ in the above Definition, we get a proper nonnegative splitting. We now reproduce a convergence result which relates convergence of single and double splitting.

Theorem 4.2. ([23], Theorem 4.3) *Let $A = P - R - S$ be a double proper nonnegative splitting of $A \in \mathbb{R}^{m\times n}$. Then $\rho(W) < 1$ if and only if $\rho(U^{\dagger}V) < 1$, where $U = P$ and $V = R + S$.*

Another convergence result for double proper nonnegative splitting is recalled from Mishra [23].

Theorem 4.3. ([23], Theorem 4.5) *Let $A^{\dagger}P \geq 0$ and $A = P - R - S$ be a double proper nonnegative splitting of $A \in \mathbb{R}^{m\times n}$, then $\rho(W) < 1$.*

We next proceed to present comparison results for double proper nonnegative splittings. Let $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two double proper nonnegative splittings of A. We then have $W_1 = \begin{pmatrix} P_1^{\dagger}R_1 & P_1^{\dagger}S_1 \\ I & O \end{pmatrix} \geq 0$

and $W_2 = \begin{pmatrix} P_2^\dagger R_2 & P_2^\dagger S_2 \\ I & O \end{pmatrix} \geq 0$. Comparison theorems are very useful for analyzing the rate of convergence of respective methods induced from different splittings. On this background, we obtain a result as follows which is a generalization of [15], Theorem 5 to rectangular matrices.

Theorem 4.4. *Let $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two double proper nonnegative splittings of a non-negative matrix $A \in \mathbb{R}^{m\times n}$. If $P_1^\dagger \geq P_2^\dagger$ and $P_1^\dagger R_1 \geq P_2^\dagger R_2$, then $\rho(W_1) \leq \rho(W_2) < 1$ for $0 < \rho(W_2) < 1$.*

Proof. We have now $W_1 \geq 0$ and $W_2 \geq 0$. So, applying Theorem 2.2 to W_2, there exists a non-negative vector $x = [x_1, x_2]^T$, $x \neq 0$ such that $W_2 x = \rho(W_2)x$, i.e.,

$$\begin{aligned} P_2^\dagger R_2 x_1 + P_2^\dagger S_2 x_2 &= \rho(W_2)x_1 \\ x_1 &= \rho(W_2)x_2. \end{aligned}$$

Then

$$\begin{aligned} W_1 x - \rho(W_2)x &= \begin{pmatrix} P_1^\dagger R_1 x_1 + P_1^\dagger S_1 x_2 - \rho(W_2)x_1 \\ x_1 - \rho(W_2)x_2 \end{pmatrix} \\ &= \begin{pmatrix} (P_1^\dagger R_1 - P_2^\dagger R_2)x_1 + \frac{1}{\rho(W_2)}(P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ x_1 - \rho(W_2)x_2 \end{pmatrix} \end{aligned}$$

The conditions $P_1^\dagger R_1 \geq P_2^\dagger R_2$ and $0 < \rho(W_2) < 1$ again imply

$$\begin{aligned} W_1 x - \rho(W_2)x &\leq \frac{1}{\rho(W_2)} \begin{pmatrix} (P_1^\dagger R_1 - P_2^\dagger R_2)x_1 + (P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\rho(W_2)}[P_1^\dagger (R_1 + S_1) - P_2^\dagger (R_2 + S_2)]x_1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\rho(W_2)}[P_1^\dagger (P_1 - A) - P_2^\dagger (P_2 - A)]x_1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\rho(W_2)}[P_2^\dagger - P_1^\dagger]Ax_1 \\ 0 \end{pmatrix} \leq 0. \end{aligned}$$

We have used the fact $P_1^\dagger P_1 = P_2^\dagger P_2$ since $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ are two double proper splittings. Thus, by Lemma 2.5, we have $\rho(W_1) \leq \rho(W_2) < 1$ for $0 < \rho(W_2) < 1$. □

Theorem 4.4 is true if we replace the condition $P_1^\dagger R_1 \geq P_2^\dagger R_2$ by $R_1 \geq R_2$. But it is more general than the theorem given below.

Theorem 4.5. *Let $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two double proper nonnegative splittings of a non-negative matrix $A \in \mathbb{R}^{m\times n}$. If $P_1^\dagger \geq P_2^\dagger$ and $R_1 \geq R_2$, then $\rho(W_1) \leq \rho(W_2) < 1$ for $0 < \rho(W_2) < 1$.*

Proof. Since $P_1^{\dagger} \geq P_2^{\dagger}$ and $R_1 \geq R_2$, it is clear that $P_1^{\dagger}R_1 \geq P_2^{\dagger}R_2$, then by Theorem 4.4, we have $\rho(W_1) \leq \rho(W_2) < 1$. □

The next result compares convergence rate of two different linear systems, and is an extension of [16], Theorem 6 to rectangular case. The idea of the proof is similar to proof of [16], Theorem 6 and Theorem 4.4, and hence we omit it.

Theorem 4.6. *Let $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be two double proper nonnegative splittings of $A \in \mathbb{R}^{m\times n}$. If $P_1^{\dagger}A_1 \geq P_2^{\dagger}A_2$ and $P_1^{\dagger}R_1 \geq P_2^{\dagger}R_2$, then $\rho(W_1) \leq \rho(W_2) < 1$ for $0 < \rho(W_2) < 1$.*

Other than these generalizations, we have a few new comparison results for semi-monotone matrices, and are discussed below.

Theorem 4.7. *Let $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two double proper nonnegative splittings of a non-negative and semi-monotone matrix $A \in \mathbb{R}^{m\times n}$. If $P_1 \leq P_2$, and $P_i \geq 0$, $R_i \geq 0$ and $S_i \geq 0$ for $i = 1, 2$, then $\rho(W_1) \leq \rho(W_2) < 1$.*

Proof. We have $A^{\dagger}P_i \geq 0$. Theorem 4.3 yields $\rho(W_i) < 1$ for $i = 1, 2$. Now $P_1 \leq P_2$ implies $A + R_1 + S_1 \leq A + R_2 + S_2$ which again gives $R_1 + S_1 \leq R_2 + S_2$, i.e., $R_2 + S_2 \geq R_1 + S_1 \geq 0$. Setting $U = P$ and $V = R + S$, we have $\rho(P_1^{\dagger}(R_1 + S_1)) = \rho(U_1^{\dagger}V_1) \leq \rho(P_2^{\dagger}(R_2 + S_2)) = \rho(U_2^{\dagger}V_2) < 1$ by Theorem 3.3. Lemma 2.7 then yields $\rho(W_1) \leq \rho(W_2) < 1$. □

The above theorem is also true if we replace the condition $P_1 \leq P_2$ by $S_1 \leq S_2$ and $R_1 \leq R_2$ which is shown next.

Theorem 4.8. *Let $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two double proper nonnegative splittings of a semi-monotone matrix $A \in \mathbb{R}^{m\times n}$. If $0 \leq R_1 \leq R_2$ and $0 \leq S_1 \leq S_2$, then $\rho(W_1) \leq \rho(W_2) < 1$.*

Proof. Since $R_1 \leq R_2$, $S_1 \leq S_2$ imply $R_1 + S_1 \leq R_2 + S_2$. Hence by the Theorem 3.3 and Lemma 2.7, we have $\rho(W_1) \leq \rho(W_2) < 1$. □

Another comparison theorem for double proper nonnegative splitting is as follows.

Theorem 4.9. *Let $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two double proper nonnegative splittings of $A \in \mathbb{R}^{m\times n}$. If $P_1^{\dagger}S_1 - P_2^{\dagger}S_2 \leq 0$ and $P_1^{\dagger}S_1 - P_2^{\dagger}S_2 \leq P_2^{\dagger}R_2 - P_1^{\dagger}R_1$, then $\rho(W_1) \leq \rho(W_2) < 1$ for $0 < \rho(W_2) < 1$.*

Proof. Clearly, from the definition of double proper nonnegative splittings, we have $W_1 \geq 0$ and $W_2 \geq 0$. So, applying Theorem 2.2 to W_2, there exists a non-negative vector $x = [x_1, x_2]^T$, $x \neq 0$ such that $W_2x = \rho(W_2)x$, i.e.,

$$\begin{aligned} P_2^{\dagger}R_2x_1 + P_2^{\dagger}S_2x_2 &= \rho(W_2)x_1 \\ x_1 &= \rho(W_2)x_2. \end{aligned}$$

We now have

$$\begin{aligned} W_1x - \rho(W_2)x &= \begin{pmatrix} P_1^\dagger R_1 x_1 + P_1^\dagger S_1 x_2 - \rho(W_2)x_1 \\ x_1 - \rho(W_2)x_2 \end{pmatrix} \\ &= \begin{pmatrix} (P_1^\dagger R_1 - P_2^\dagger R_2)x_1 + \frac{1}{\rho(W_2)}(P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ x_1 - \rho(W_2)x_2 \end{pmatrix} \end{aligned}$$

Using the condition $P_1^\dagger S_1 - P_2^\dagger S_2 \leq 0$ and $0 < \rho(W_2) < 1$, we have $\frac{1}{\rho(W_2)}[P_1^\dagger S_1 - P_2^\dagger S_2]x_1 \leq (P_1^\dagger S_1 - P_2^\dagger S_2)x_1$. Then

$$W_1x - \rho(W_2)x \leq \begin{pmatrix} (P_1^\dagger R_1 - P_2^\dagger R_2)x_1 + (P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ 0 \end{pmatrix}$$

Again, from the condition $P_1^\dagger S_1 - P_2^\dagger S_2 \leq P_2^\dagger R_2 - P_1^\dagger R_1$, we have $P_1^\dagger(R_1 + S_1) - P_2^\dagger(R_2 + S_2) \leq 0$. Then

$$W_1x - \rho(W_2)x \leq \begin{pmatrix} [P_1^\dagger(R_1 + S_1) - P_2^\dagger(R_2 + S_2)]x_1 \\ 0 \end{pmatrix} \leq 0$$

From Lemma 2.5, we have $\rho(W_1) \leq \rho(W_2) < 1$ for $0 < \rho(W_2) < 1$. □

The example given below describes that the converse of above theorem is not true.

Example 4.10. Let $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \end{pmatrix}$. Setting $P_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$, $R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, and $P_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$, $R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $S_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we have $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ two double proper nonnegative splittings with $A^\dagger \geq 0$. Then $\rho(W_1) \leq \rho(W_2) < 1$ for $0 \leq \rho(W_2) < 1$. But $P_1^\dagger S_1 - P_2^\dagger S_2 = \begin{pmatrix} -0.3333 & 0.1167 & 0 \\ 0 & -0.3333 & 0 \\ 0 & 0 & 0 \end{pmatrix} \nleq \begin{pmatrix} -0.1667 & 0 & 0 \\ -0.1667 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = P_2^\dagger R_2 - P_1^\dagger R_1$.

5. Conclusions

Improving convergence rate of iteration scheme (1) is a problem of interest for getting solution faster. In this direction, Climent and Perea [6] proposed multisplitting theory for rectangular matrices while the authors of [9] and [14] studied the same problem for nonsingular matrices by using two stage iterative method. Iterative technique for solving rectangular system also avoids use of the normal system $A^TAx = A^Tb$ where A^TA is frequently ill-conditioned and influenced greatly by roundoff errors (see [10]).

In this note, we have presented a few results for computing least square minimum norm solution of a rectangular system $Ax = b$ in a faster way using theory of proper nonnegative and double proper nonnegative splittings. We also have produced some results which are helpful in choosing an effective preconditioner for solving rectangular linear system in more faster way. While computing Drazin inverse and Drazin inverse solution A^Db to a square singular system $Ax = b$ is still a challenging problem as the solution A^Db lies in the Krylov subspace of (A, b), i.e., $K_s(A, b) = span\{b, Ab, A^2b, \ldots, A^{s-1}b\}$. Using these approach, one can even solve the above stated problem.

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