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Results in Mathematics

Rigidity Theorems for Relative Tchebychev Hypersurfaces

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Abstract. In this paper, we calculate the Laplacian of the norm of the cubic Simon form for a hypersurface with a relative normalization. The method used here is developed in the rigidity theory of minimal submanifolds. As consequences, we obtain some local and global rigidity theorems about relative Tchebychev hypersurfaces.

Mathematics Subject Classification. 53A15.

Keywords. Relative Tchebychev hypersurface, cubic form, cubic Simon form, Tchebychev vector field, rigidity theorem, quadric.

Introduction

Relative differential geometry is a general theory of the affine geometry of hypersurfaces in an affine space. It contains equiaffine and centroaffine geometry as special cases. For details of relative differential geometry, one refers to [18, 28].

Affine spheres are the most simple but very important objects in equiaffine differential geometry. Many authors contribute their efforts to this subject. Some properties and more references about affine spheres can be found in [18, 25, 28]. In a series of papers [3, 8–13, 20–22], symmetric affine spheres were intensively studied by F. Dillen, Z. Hu, C. Li, H. Li, X. Li, U. Simon, L. Vrancken, etc.

Wang [30] introduced the concepts of a Tchebychev operator and of Tchebychev hypersurfaces in centroaffine geometry. The conformal classification of Tchebychev hypersurfaces is given in [23]. For some results about Tchebychev hypersurfaces we refer to [14, 17, 19, 23, 24, 29].

The authors of the paper [17] introduced the concept of a relative Tchebychev hypersurface (in relative geometry) which includes the concepts of an

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affine sphere in equiaffine geometry and of a Tchebychev hypersurface in centroaffine geometry as special cases. They also stated some local and global Ricci-pinching theorems for relative Tchebychev hypersurfaces.

In the past two decades, the studies of affine hypersurfaces and of Lagrangian submanifolds interacted each other. Recently, [20] gives a direct corresponding between symmetric equiaffine spheres and symmetric minimal Lagrangian submanifolds in certain complex space forms. Inspired by the work on rigidity theorems of minimal submanifolds and Lagrangian submanifolds, such as [5,6,16], we are going to study rigidity problems about cubic forms for relative Tchebychev hypersurfaces.

This paper is organized as follows. In Sect. 1, we review some basic facts in relative geometry. In Sect. 2, first we review some basic facts about cubic Simon forms and relative Tchebychev operators. Then we give a detailed calculation of the Laplacian of the norm of the cubic Simon form. In Sect. 3, we obtain some local and global rigidity results about the cubic form for relative Tchebychev hypersurfaces.

1. Preliminaries

In this section, we would like to review the fundamental equations of an affine hypersurface with a relative normalization. For details of the contents, one refers to [18,28].

1.1. Relative Normalization of a Nondegenerate Hypersurface

Let A be an affine space which is modelled on a real vector space V of dimension $n+1$. Let V^* be the dual space of V, and $\langle\ ,\ \rangle : V \times V^* \to \mathbb{R}$ the canonical pairing. A has a smooth manifold structure. Moreover $T_pA \cong V$ and $T_p^*A \cong V^*$ for each point $p \in M$. Let $\bar{\nabla}$ be a flat affine connection on TA, and $\bar{\nabla}^*$ the dual connection on T^*A.

Let $x : M \to A$ be an immersed connected oriented smooth manifold M of dimension n. Then for each point $p \in M$, $dx(T_pM)$ is an n dimensional subspace of $T_{x(p)}A$, and defines a one dimensional subspace $C_pM = \{v_p^* \in V^* | \ker v_p^* = dx(T_pM)\} \subset T_p^*A$. The trivial line bundle $CM = \bigcup_p C_pM$ is called the conormal line bundle of x.

Let Y be a nowhere vanishing section of CM. If $\mathrm{rank}(dY, Y) = n+1$, then x is called a nondegenerate hypersurface. The nondegeneracy is independent of the choice of the conormal field Y. In this paper, we will only discuss nondegenerate hypersurfaces. Let $y : M \to V$ be a vector field with $\mathrm{rank}(y, dx) = n+1$. If $\langle Y, dy\rangle = 0$, then y is called a relative normal field. Furthermore, if $\langle Y, y\rangle = 1$, then the pair $\{Y, y\}$ is called a relative normalization of x. From now on , all normalizations considered are relative normalizations. A nondegenerate hypersurface x with a given relative normalization $\{Y, y\}$ will be denoted by a triple $\{x, Y, y\}$.

We state the structure equations of such a triple $\{x, Y, y\}$ as follows:

$$\begin{aligned}\bar{\nabla}_v y &= dy(v) = -dx(S(v)),\\ \bar{\nabla}_v dx(w) &= dx(\nabla_v w) + h(v,w)y,\\ \bar{\nabla}^*_v dY(w) &= dY(\nabla^*_v w) - \hat{S}(v,w)Y,\end{aligned}$$

where S is the affine shape operator, h is a nondegenerate symmetric $(0,2)$-tensor, ∇ and ∇^* are torsion free affine connections on TM respectively, and $\hat{S}$ is a symmetric (0,2)-tensor. h is definite if and only if $x(M)$ is locally strongly convex; in this case, choosing an appropriate orientation of the normalization, h is positive definite and thus a Riemannian metric (In the case that h is nondegenerate one can consider h as a semi-Riemannian metric).

These geometric quantities satisfy the relations

$$\begin{aligned}\hat{S}(v,w) &= h(S(v),w) = h(v,S(w)),\\ dh(v_1,v_2) &= h(\nabla v_1, v_2) + h(v_1, \nabla^* v_2).\end{aligned} \tag{1.1}$$

∇ and ∇^* are torsion free affine connections on the tangent bundle TM of the manifold M. If the triple $\{\nabla, h, \nabla^*\}$ satisfies (1.1), then ∇ and ∇^* are called conjugate connections.

1.2. Consequences of Conjugate Connections

For a triple of conjugate connections $\{\nabla, h, \nabla^*\}$, one can define $C = \frac{1}{2}(\nabla - \nabla^*) \in \Omega^1(M, \mathrm{End}(TM))$. By (1.1), the $(0,3)$-tensor $\hat{C} := h \circ C$ is totally symmetric and called the cubic form of $\{\nabla, h, \nabla^*\}$. For $\{\nabla, h, \nabla^*\}$, the Tchebychev form $\hat{T}$ is defined as the normalized trace of C,

$$\hat{T} = \frac{1}{n}\mathrm{tr}C.$$

The Tchebychev field T is the dual of $\hat{T}$ with respect to h.

For conjugate connections $\{\nabla, h, \nabla^*\}$, the curvature tensors R, R^* and R^h of ∇, ∇^* and ∇^h, respectively, are related as follows

$$R = R^h + [\nabla^h, C] + C \wedge C, \tag{1.2}$$

$$R^* = R^h - [\nabla^h, C] + C \wedge C, \tag{1.3}$$

where ∇^h is the Levi-Civita connection of the semi-Riemannian metric h. The Ricci tensors have the relations

$$\mathrm{Ric} = \mathrm{Ric}^h + \mathrm{div}^h C - n\nabla^h \hat{T} + n\langle \hat{T}, C\rangle - \alpha, \tag{1.4}$$

$$\mathrm{Ric}^* = \mathrm{Ric}^h - \mathrm{div}^h C + n\nabla^h \hat{T} + n\langle \hat{T}, C\rangle - \alpha, \tag{1.5}$$

where α denotes the following symmetric (0,2)-tensor

$$\alpha(U,V) := \mathrm{tr}\{w \to C(C(w)U)V\}. \tag{1.6}$$

For the conjugate connections $\{\nabla, h, \nabla^*\}$, induced by an affine hypersurface $x(M)$ with a relative normalization $\{Y, y\}$, one can prove that ∇ and ∇^*

are Ricci symmetric, and that the Tchebychev form is closed $d\hat{T}=0$; see e.g. [28].

1.3. Integrability Conditions

We have the following equations from the integrability of an affine hypersurface $x(M)$ with a relative normalization $\{Y,y\}$,

$$(\nabla_U S)(V)=(\nabla_V S)(U), \tag{1.7}$$
$$R(U,V)W=-(h(U,W)S(V)-h(V,W)S(U)), \tag{1.8}$$

and

$$(\nabla^*_U \hat{S})(V,W)=(\nabla^*_V \hat{S})(U,W), \tag{1.9}$$
$$R^*(U,V)W=-(\hat{S}(U,W)V-\hat{S}(V,W)U). \tag{1.10}$$

Using (1.2) and (1.3), it is clear that (1.7)–(1.10) imply the following complete integrability conditions in terms of ∇^h,

$$\begin{aligned}R^h(U,V)W=&-\frac{1}{2}\left[h(U,W)S(V)-h(V,W)S(U)+\hat{S}(U,W)V-\hat{S}(V,W)U\right]\\&+\left[C(C(U)W)V-C(C(V)W)U\right],\end{aligned} \tag{1.11}$$

$$\begin{aligned}&(\nabla^h_U C)(V)W-(\nabla^h_V C)(U)W\\&\quad=-\frac{1}{2}\left[h(U,W)S(V)-h(V,W)S(U)-\hat{S}(U,W)V+\hat{S}(V,W)U\right],\end{aligned} \tag{1.12}$$

and

$$(\nabla^h_U S)V-(\nabla^h_V S)U=C(SU)V-C(SV)U. \tag{1.13}$$

We list here two formulas about the Ricci curvature and scalar curvature as they will be used soon after,

$$\mathrm{Ric}^h=\alpha-n\langle\hat{T},C\rangle+\frac{1}{2}((\mathrm{tr}S)h+(n-2)\hat{S}), \tag{1.14}$$

and

$$n(n-1)\kappa=:\mathrm{tr}\mathrm{Ric}^h=(n-1)\mathrm{tr}S+\|C\|^2-n^2\|T\|^2; \tag{1.15}$$

κ is called the normalized scalar curvature.

Note that (1.14) and (1.15) follow directly from the integrability condition (1.11).

2. Some Properties of Cubic Forms

2.1. Cubic Simon Forms, Quadrics and Relative Tchebychev Hypersurfaces

Let $x:M\to A$ be a nondegenerate n-dimensional hypersurface immersed into an $(n+1)$-dimensional affine space A. For any relative normalization $\{Y,y\}$ of

$x(M)$, the cubic Simon form $\tilde{C}$ as a (1,2)-tensor field is defined as the traceless part of the cubic form C:

$$\tilde{C}(U,V) := C(U,V) - \frac{n}{n+2}\{h(U,V)T + h(U,T)V + h(V,T)U\}. \tag{2.1}$$

It is proved that $\tilde{C}$ is independent of the choice of the relative normalization. Moreover, the equation $\tilde{C} = 0$ characterizes quadrics (cf. [28]). The following (1,1)-tensor fields

$$L := \frac{1}{2}S - \frac{n}{n+2}\nabla^h T \quad \text{and} \quad \tilde{L} := L - \frac{1}{n}\mathrm{tr}L\ \mathrm{id}$$

are defined in [17]. Following [17], a hypersurface $x(M)$ is called a relative Tchebychev hypersurface with respect to a relative normalization $\{Y, y\}$, if $L = \lambda\ \mathrm{id}$, or equivalently $\tilde{L} = 0$. One should note that the concept of a relative Tchebychev hypersurface depends on the choice of the normalization.

Lemma 1 [17]. *Let $x(M)$ be a nondegenerate hypersurface with a relative normalization $\{Y, y\}$. Then the following equation holds*

$$\begin{aligned}&(\nabla^h_U\tilde{C})(V,W) - (\nabla^h_V\tilde{C})(U,W)\\ &\quad = -\left[h(U,W)\tilde{L}(V) - h(V,W)\tilde{L}(U) - h(\tilde{L}(U),W)V + h(\tilde{L}(V),W)U\right].\end{aligned} \tag{2.2}$$

It implies that $\mathrm{div}^h\tilde{C} = -nh \circ \tilde{L}$. *As consequences, the following conditions are equivalent*

(i) *$x(M)$ is a relative Tchebychev hypersurface;*
(ii) *$(\nabla^h_U\tilde{C})(V,W) = (\nabla^h_V\tilde{C})(U,W)$ for any vector fields $U, V, W \in \Gamma(TM)$;*
(iii) *$\mathrm{div}^h\tilde{C} = 0$.*

2.2. Estimates of $\Delta^h\|\tilde{C}\|^2$

Let $x : M \to A$ be a nondegenerate immersed hypersurface of $\dim M = n$. Let $\{Y, y\}$ be a relative normalization of $x(M)$. A local frame field $e_1, \ldots, e_n, e_{n+1}$ is called an adapted relative affine frame field if $e_{n+1} = y$ and $e_1, \ldots, e_n$ are local orthonormal tangent vector fields with respect to h. Their dual frame fields are $\theta^1, \ldots, \theta^n, \theta^{n+1} = Y$. In this paper we adopt the index range

$$1 \le i, j, k, l, p, q, s, t \le n.$$

Then from (1.11) and (2.1), we have

$$R_i{}^j{}_{kl} = -\frac{1}{2}\left(\delta_{ki}S^j_l - \delta_{li}S^j_k + S_{ki}\delta^j_l - S_{li}\delta^j_k\right) + \left(C^t_{ki}C^j_{tl} - C^t_{li}C^j_{tk}\right), \tag{2.3}$$

and

$$\begin{aligned}\tilde{C}^k_{ij} &= C^k_{ij} - \frac{n}{n+2}\left(T^k\delta_{ij} + T^i\delta_{jk} + T^j\delta_{ki}\right)\\ &= C^k_{ij} - B^k_{ij},\end{aligned} \tag{2.4}$$

where $R_{i\ kl}^{\ j} = h\left(R^h(e_k, e_l)e_i, e_j\right), C_{ij}^k = h(C(e_i)e_j, e_k), T^k = h(T, e_k)$ and

$$B_{ij}^k := \frac{n}{n+2}\left(T^k\delta_{ij} + T^i\delta_{jk} + T^j\delta_{ki}\right).$$

Theorem 1. *Let $x(M)$ be an affine hypersurface with a given relative normalization $\{Y, y\}$. Then the Laplacian of $\|\tilde{C}\|^2$ with respect to the metric h has the following expression:*

$$\begin{aligned}\frac{1}{2}\Delta^h\|\tilde{C}\|^2 = \|\nabla^h\tilde{C}\|^2 &- (n+2)\sum_{ijk}\tilde{C}_{ij}^k\tilde{L}_{i,j}^k + \frac{n+2}{2}\sum_{jp}\tilde{\alpha}_{jp}S_{jp} + \frac{1}{2}\mathrm{tr}S\|\tilde{C}\|^2\\ &+ \sum_{ij}N\left(\tilde{C}^i\tilde{C}^j - \tilde{C}^j\tilde{C}^i\right) + \|\tilde{\alpha}\|^2\\ &- n\sum_{jkst}\tilde{C}_{ij}^k\tilde{C}_{jt}^k\tilde{C}_{ti}^sT^s - \frac{n^2}{n+2}\sum_{ijksp}\tilde{C}_{ij}^k\tilde{C}_{pj}^kT^iT^p - \frac{n^2}{n+2}\|\tilde{C}\|^2\|T\|^2,\end{aligned} \tag{2.5}$$

where the notation $N(A) := \mathrm{tr}A^{\mathrm{t}}A = \sum_{ij}(a_{ij})^2$ is used for a matrix $A = (a_{ij})$ as usual (cf. [5,6,16]), $\tilde{C}^i$ denotes the matrix $(\tilde{C}_{kt}^i)$ and $\tilde{\alpha}_{ij} := \sum_{kt}\tilde{C}_{kt}^i\tilde{C}_{kt}^j$.

Proof. By definition, one has

$$\frac{1}{2}\Delta^h\|\tilde{C}\|^2 = \|\nabla^h\tilde{C}\|^2 + \sum_{ijk}\tilde{C}_{ij}^k\Delta^h\tilde{C}_{ij}^k,$$

where $\Delta^h\tilde{C}_{ij}^k := \sum_t\tilde{C}_{ij,tt}^k$; using the local calculus, the comma "," indicates covariant differentiation in terms of ∇^h.

We reformulate (2.2) under the adapted frame field as

$$\tilde{C}_{ij,t}^k - \tilde{C}_{it,j}^k = -\left(\delta_{ti}\tilde{L}_j^k - \delta_{ji}\tilde{L}_t^k - \tilde{L}_{ti}\delta_j^k + \tilde{L}_{ji}\delta_t^k\right). \tag{2.6}$$

The Ricci identity reads

$$\tilde{C}_{ij,ts}^k - \tilde{C}_{ij,st}^k = \tilde{C}_{pj}^kR_{i\ ts}^{\ p} + \tilde{C}_{ip}^kR_{j\ ts}^{\ p} - \tilde{C}_{ij}^pR_{p\ ts}^{\ k}. \tag{2.7}$$

By (2.6) and (2.7), we have

$$\begin{aligned}\sum_t\tilde{C}_{ij,tt}^k &= \sum_t\tilde{C}_{it,jt}^k - \sum_t\left(\delta_{ti}\tilde{L}_{j,t}^k - \delta_{ji}\tilde{L}_{t,t}^k - \tilde{L}_{ti,t}\delta_j^k + \tilde{L}_{ji,t}\delta_t^k\right)\\ &= \sum_t\tilde{C}_{it,tj}^k + \sum_t\left(\tilde{C}_{pt}^kR_{i\ jt}^{\ p} + \tilde{C}_{ip}^kR_{t\ jt}^{\ p} - \tilde{C}_{it}^pR_{p\ jt}^{\ k}\right)\\ &\quad - \sum_t\left(\delta_{ti}\tilde{L}_{j,t}^k - \delta_{ji}\tilde{L}_{t,t}^k - \tilde{L}_{ti,t}\delta_j^k + \tilde{L}_{ji,t}\delta_t^k\right)\\ &= \sum_t\tilde{C}_{tt,ij}^k - \sum_t\left(\delta_{tt}\tilde{L}_{i,j}^k - \delta_{it}\tilde{L}_{t,j}^k - \tilde{L}_{tt,j}\delta_i^k + \tilde{L}_{it,j}\delta_t^k\right)\end{aligned}$$

$$+\sum_{t}\left(\tilde{C}^k_{pt}R_i{}^p{}_{jt}+\tilde{C}^k_{ip}R_t{}^p{}_{jt}-\tilde{C}^p_{it}R_p{}^k{}_{jt}\right)$$
$$-\sum_{t}\left(\delta_{ti}\tilde{L}^k_{j,t}-\delta_{ji}\tilde{L}^k_{t,t}-\tilde{L}_{ti,t}\delta^k_j+\tilde{L}_{ji,t}\delta^k_t\right). \tag{2.8}$$

Since $\tilde{C}$ is traceless, by (2.3) and (2.8), we get

$$\begin{aligned}
\sum_{ijk}\tilde{C}^k_{ij}\Delta^h\tilde{C}^k_{ij} &= \sum_{ijkt}\tilde{C}^k_{ij}\tilde{C}^k_{ij,tt}\\
&= -(n+2)\sum_{ijk}\tilde{C}^k_{ij}\tilde{L}^k_{i,j}+\sum_{ijktp}\tilde{C}^k_{ij}\left(\tilde{C}^k_{pt}R_i{}^p{}_{jt}+\tilde{C}^k_{ip}R_t{}^p{}_{jt}-\tilde{C}^p_{it}R_p{}^k{}_{jt}\right)\\
&= -(n+2)\sum_{ijk}\tilde{C}^k_{ij}\tilde{L}^k_{i,j}\\
&\quad+\sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^k_{pt}\left[-\frac{1}{2}\left(\delta_{ji}S^p_t-\delta_{ti}S^p_j+S_{ji}\delta^p_t-S_{ti}\delta^p_j\right)+\left(C^s_{ji}C^p_{st}-C^s_{ti}C^p_{sj}\right)\right]\\
&\quad+\sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^k_{ip}\left[-\frac{1}{2}\left(\delta_{jt}S^p_t-\delta_{tt}S^p_j+S_{jt}\delta^p_t-S_{tt}\delta^p_j\right)+\left(C^s_{jt}C^p_{st}-C^s_{tt}C^p_{sj}\right)\right]\\
&\quad-\sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^p_{it}\left[-\frac{1}{2}\left(\delta_{jp}S^k_t-\delta_{tp}S^k_j+S_{jp}\delta^k_t-S_{tp}\delta^k_j\right)+\left(C^s_{jp}C^k_{st}-C^s_{tp}C^k_{sj}\right)\right]\\
&= -(n+2)\sum_{ijk}\tilde{C}^k_{ij}\tilde{L}^k_{i,j}+\frac{n+2}{2}\sum_{ijkp}\tilde{C}^k_{ij}\tilde{C}^k_{pi}S_{jp}+\frac{1}{2}\mathrm{tr}S\|\tilde{C}\|^2\\
&\quad+\underbrace{\sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^k_{pt}\left(C^s_{ji}C^p_{st}-C^s_{ti}C^p_{sj}\right)}_{\mathrm{I}}+\underbrace{\sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^k_{ip}\left(C^s_{jt}C^p_{st}-C^s_{tt}C^p_{sj}\right)}_{\mathrm{II}}\\
&\quad-\underbrace{\sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^p_{it}\left(C^s_{jp}C^k_{st}-C^s_{tp}C^k_{sj}\right)}_{\mathrm{III}}.
\end{aligned}$$

The three terms I, II, III have the following expressions,

$$\begin{aligned}
\mathrm{I} &= \sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^k_{pt}\left(\tilde{C}^s_{ji}\tilde{C}^p_{st}-\tilde{C}^s_{ti}\tilde{C}^p_{sj}\right)\\
&\quad+\sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^k_{pt}\left(\tilde{C}^s_{ji}B^p_{st}+B^s_{ji}\tilde{C}^p_{st}-\tilde{C}^s_{ti}B^p_{sj}-B^s_{ti}\tilde{C}^p_{sj}\right)\\
&\quad+\sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^k_{pt}\left(B^s_{ji}B^p_{st}-B^s_{ti}B^p_{sj}\right)\\
&= \sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^k_{pt}\left(\tilde{C}^s_{ji}\tilde{C}^p_{st}-\tilde{C}^s_{ti}\tilde{C}^p_{sj}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2n}{n+2}\sum_{ijkst}\tilde{C}^k_{ij}\tilde{C}^k_{jt}\tilde{C}^s_{ti}T^s - \frac{2n^2}{(n+2)^2}\sum_{ijksp}\tilde{C}^k_{ij}\tilde{C}^k_{pj}T^iT^p\\
&- \frac{n^2}{(n+2)^2}\|\tilde{C}\|^2\|T\|^2,\\
\mathrm{II} = &\sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^k_{ip}\left(\tilde{C}^s_{jt}\tilde{C}^p_{st} - \tilde{C}^s_{tt}\tilde{C}^p_{sj}\right)\\
&+ \sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^k_{ip}\left(\tilde{C}^s_{jt}B^p_{st} + B^s_{jt}\tilde{C}^p_{st} - \tilde{C}^s_{tt}B^p_{sj} - B^s_{tt}\tilde{C}^p_{sj}\right)\\
&+ \sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^k_{ip}\left(B^s_{jt}B^p_{st} - B^s_{tt}B^p_{sj}\right)\\
= &\sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^k_{ip}\tilde{C}^s_{jt}\tilde{C}^p_{st} - \frac{n(n-2)}{n+2}\sum_{ijksp}\tilde{C}^k_{ij}\tilde{C}^k_{ip}\tilde{C}^p_{sj}T^s\\
&- \frac{n^2(n-2)}{(n+2)^2}\sum_{ijksp}\tilde{C}^k_{ij}\tilde{C}^k_{ip}T^jT^p\\
&- \frac{n^3}{(n+2)^2}\|\tilde{C}\|^2\|T\|^2,\\
\mathrm{III} = &\sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^p_{it}\left(\tilde{C}^s_{jp}\tilde{C}^k_{st} - \tilde{C}^s_{tp}\tilde{C}^k_{sj}\right)\\
&+ \sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^p_{it}\left(\tilde{C}^s_{jp}B^k_{st} + B^s_{jp}\tilde{C}^k_{st} - \tilde{C}^s_{tp}B^k_{sj} - B^s_{tp}\tilde{C}^k_{sj}\right)\\
&+ \sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^p_{it}\left(B^s_{jp}B^k_{st} - B^s_{tp}B^k_{sj}\right)\\
= &\sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^p_{it}\left(\tilde{C}^s_{jp}\tilde{C}^k_{st} - \tilde{C}^s_{tp}\tilde{C}^k_{sj}\right)\\
&+ \frac{2n}{n+2}\sum_{ijkst}\tilde{C}^k_{ij}\tilde{C}^j_{it}\tilde{C}^k_{st}T^s + \frac{2n^2}{(n+2)^2}\sum_{ijkt}\tilde{C}^k_{ij}\tilde{C}^j_{it}T^kT^t\\
&+ \frac{n^2}{(n+2)^2}\|\tilde{C}\|^2\|T\|^2.
\end{aligned}
$$

Combining the above formulae about I, II and III, we have

$$
\begin{aligned}
\frac{1}{2}\Delta^h\|\tilde{C}\|^2 &= \|\nabla^h\tilde{C}\|^2 + \sum_{ijk}\tilde{C}^k_{ij}\Delta^h\tilde{C}^k_{ij}\\
&= \|\nabla^h\tilde{C}\|^2 - (n+2)\sum_{ijk}\tilde{C}^k_{ij}\tilde{L}^k_{i,j} + \frac{n+2}{2}\sum_{ijkp}\tilde{C}^k_{ij}\tilde{C}^k_{pi}S_{jp} + \frac{1}{2}\mathrm{tr}S\|\tilde{C}\|^2\\
&\quad + \sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^k_{pt}\left(\tilde{C}^s_{ji}\tilde{C}^p_{st} - \tilde{C}^s_{ti}\tilde{C}^p_{sj}\right) + \sum_{ijkstp}\tilde{C}^k_{ij}\tilde{C}^k_{ip}\tilde{C}^s_{jt}\tilde{C}^p_{st}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{ijkstp} \tilde{C}^k_{ij}\tilde{C}^p_{it}\left(\tilde{C}^s_{jp}\tilde{C}^k_{st} - \tilde{C}^s_{tp}\tilde{C}^k_{sj}\right)\\
&- n\sum_{jkst}\tilde{C}^k_{ij}\tilde{C}^k_{jt}\tilde{C}^s_{ti}T^s - \frac{n^2}{n+2}\sum_{ijksp}\tilde{C}^k_{ij}\tilde{C}^k_{pj}T^iT^p - \frac{n^2}{n+2}\|\tilde{C}\|^2\|T\|^2\\
= {}& \|\nabla^h\tilde{C}\|^2 - (n+2)\sum_{ijk}\tilde{C}^k_{ij}\tilde{L}^k_{i,j} + \frac{n+2}{2}\sum_{jp}\tilde{\alpha}_{jp}S_{jp} + \frac{1}{2}\mathrm{tr}S\|\tilde{C}\|^2\\
&- \frac{n^2}{n+2}\|\tilde{C}\|^2\|T\|^2\\
&+ \sum_{ij} N\left(\tilde{C}^i\tilde{C}^j - \tilde{C}^j\tilde{C}^i\right) + \|\tilde{\alpha}\|^2 - n\sum_{jkst}\tilde{C}^k_{ij}\tilde{C}^k_{jt}\tilde{C}^s_{ti}T^s\\
&- \frac{n^2}{n+2}\sum_{ijksp}\tilde{C}^k_{ij}\tilde{C}^k_{pj}T^iT^p.
\end{aligned}
$$

□

From Theorem 1, we have some estimates of $\Delta^h\|\tilde{C}\|^2$. For locally strongly convex hypersurfaces, the following theorem holds.

Theorem 2. *Let $x(M)$ be a locally strongly convex hypersurface with a relative normalization $\{Y, y\}$. Then*

$$
\begin{aligned}
\frac{1}{2}\Delta^h\|\tilde{C}\|^2 \geq {}& -(n+2)\sum_{ijk}\tilde{C}^k_{ij}\tilde{L}^k_{i,j}\\
&+ \|\tilde{C}\|^2\left[(n+1)\lambda_{\min}(S) - \frac{n^2(n+10)}{4(n+2)}\|T\|^2 + \frac{1}{n}\|\tilde{C}\|^2\right],
\end{aligned}
$$

where $\lambda_{\min}(S)$ denotes the minimal eigenvalue of S. Moreover, if $x(M)$ is a locally strongly convex relative Tchebychev hypersurface, then $\tilde{L} = 0$ and

$$
\frac{1}{2}\Delta^h\|\tilde{C}\|^2 \geq \|\tilde{C}\|^2\left[(n+1)\lambda_{\min}(S) - \frac{n^2(n+10)}{4(n+2)}\|T\|^2 + \frac{1}{n}\|\tilde{C}\|^2\right]. \tag{2.9}
$$

Proof. On the set where T is not zero, we can choose an adapted local frame such that $e_1 = T/\|T\|$. Then

$$
\begin{aligned}
\frac{1}{2}\Delta^h\|\tilde{C}\|^2 = {}& \|\nabla^h\tilde{C}\|^2 - (n+2)\sum_{ijk}\tilde{C}^k_{ij}\tilde{L}^k_{i,j} + \frac{n+2}{2}\sum_{jp}\tilde{\alpha}_{jp}S_{jp} + \frac{1}{2}\mathrm{tr}S\|\tilde{C}\|^2\\
&- \frac{n^2}{n+2}\|\tilde{C}\|^2\|T\|^2 + \sum_{ij} N\left(\tilde{C}^i\tilde{C}^j - \tilde{C}^j\tilde{C}^i\right)\\
&+ \sum_{ij}\tilde{\alpha}^2_{ij} - n\sum_{ij}\tilde{\alpha}_{ij}\tilde{C}^1_{ij}\|T\| - \frac{n^2}{n+2}\sum_{ij}(\tilde{C}^1_{ij})^2\|T\|^2\\
= {}& \|\nabla^h\tilde{C}\|^2 - (n+2)\sum_{ijk}\tilde{C}^k_{ij}\tilde{L}^k_{i,j} + \frac{n+2}{2}\sum_{jp}\tilde{\alpha}_{jp}S_{jp} + \frac{1}{2}\mathrm{tr}S\|\tilde{C}\|^2
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{n^2}{n+2}\|\tilde{C}\|^2\|T\|^2 + \sum_{ij} N\left(\tilde{C}^i\tilde{C}^j - \tilde{C}^j\tilde{C}^i\right) \\
&+ \sum_{ij}\left(\tilde{\alpha}_{ij} - \frac{n}{2}\tilde{C}^1_{ij}\|T\|\right)^2 - \frac{n^2(n+6)}{4(n+2)}\sum_{ij}(\tilde{C}^1_{ij})^2\|T\|^2 \\
\geq\ & -(n+2)\sum_{ijk}\tilde{C}^k_{ij}\tilde{L}^k_{i,j} + (n+1)\lambda_{\min}(S)\|\tilde{C}\|^2 \\
&+ \frac{1}{n}\|\tilde{C}\|^4 - \frac{n^2(n+10)}{4(n+2)}\|T\|^2\|\tilde{C}\|^2.
\end{aligned}
$$

We have used the following inequalities

$$
\sum_{ij} N\left(\tilde{C}^i\tilde{C}^j - \tilde{C}^j\tilde{C}^i\right) \geq \frac{1}{n}\|\tilde{C}\|^4, \tag{2.10}
$$

and

$$
\sum_{jp}\tilde{\alpha}_{jp}S_{jp} = \operatorname{tr}(\tilde{\alpha}S) \geq \operatorname{tr}(\tilde{\alpha})\lambda_{\min}(S) = \|\tilde{C}\|^2\lambda_{\min}(S), \tag{2.11}
$$

as $\tilde{\alpha}$ is semi-positive for locally strongly convex hypersurfaces, and S is symmetric.

On the set where $T = 0$, one easily has the required inequalities. □

For nondegenerate relative spheres, i.e., $S = \lambda$ id for some constant λ, we have the following estimate.

Theorem 3. *Let $x(M)$ be a nondegenerate relative sphere with respect to a normalization $\{Y, y\}$, thus $S = \lambda$ id. If $x(M)$ is also a relative Tchebychev hypersurface then*

$$
\frac{1}{2}\Delta^h\|\tilde{C}\|^2 \geq \|\tilde{C}\|^2\left[(n+1)\lambda - \frac{n^2(n+10)}{4(n+2)}\|T\|^2 + \frac{1}{n}\|\tilde{C}\|^2\right]. \tag{2.12}
$$

The proof of Theorem 3 is similar to that of Theorem 2, except (2.11). In this case, (2.11) is an equation

$$
\sum_{jp}\tilde{\alpha}_{jp}S_{jp} = \lambda\|\tilde{C}\|^2.
$$

More specifically, if we consider nondegenerate equiaffine spheres then the following inequalities hold as a corollary of Theorem 3.

Corollary 1. *Let $x(M)$ be a nondegenerate equiaffine sphere, thus $S = \lambda$ id. As $T = 0$ here, M is automatically a relative Tchebychev hypersurface, and then (2.12) has a simpler form*

$$
\frac{1}{2}\Delta^h\|C\|^2 \geq \|C\|^2\left[(n+1)\lambda + \frac{1}{n}\|C\|^2\right]. \tag{2.13}
$$

Remark 1. We compare our results with classical results about the Pick invariant [15, 18, 25–27],

$$\frac{1}{2}\Delta^h\|C\|^2 \geq \|C\|^2\left[(n+1)\lambda + \frac{n+1}{n(n-1)}\|C\|^2\right], \tag{2.14}$$

we observe that our estimates are not optimal for equiaffine spheres; in this case one needs more careful estimates.

In the case $n > 2$, the following expression of $\Delta^h\|\tilde{C}\|^2$ was obtained in the paper [17] for the purpose of Ricci-pinching theory,

$$\begin{aligned}\frac{1}{2}\Delta^h\|\tilde{C}\|^2 &= \|\nabla^h\tilde{C}\|^2 - (n+2)\sum_{ijk}\tilde{C}^k_{ij}\tilde{L}^k_{i,j} + \|W\|^2\\ &\quad + \frac{n+2}{n-2}\sum_{ij}\tilde{\alpha}_{ij}\left(R_{ij} - \frac{2n}{n+2}\kappa\delta_{ij}\right),\end{aligned} \tag{2.15}$$

where W is the Weyl conformal curvature tensor,

$$\begin{aligned}W_{ijkl} := R_{ijkl} &- \frac{1}{n-2}\left(R_{ik}\delta_{jl} + R_{jl}\delta_{ik} - R_{il}\delta_{jk} - R_{jk}\delta_{il}\right)\\ &+ \frac{n\kappa}{n-2}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right).\end{aligned}$$

Comparing the formulae (2.5) and (2.15), one gets a formula about the norm of the Weyl tensor.

Corollary 2. *The norm of the Weyl tensor can be expressed in terms of* $\tilde{C}$ *and* h *only, namely:*

$$\begin{aligned}\|W\|^2 &= \frac{2}{(n-1)(n-2)}\|\tilde{C}\|^4 + \sum_{ij}N\left(\tilde{C}^i\tilde{C}^j - \tilde{C}^j\tilde{C}^i\right) - \frac{4}{n-2}\sum_{ij}\tilde{\alpha}^2_{ij}\\ &\quad + \frac{2n^2}{n+2}\sum_{ijksp}\tilde{C}^k_{ij}\tilde{C}^k_{pj}T^iT^p + \frac{n^2}{n+2}\|\tilde{C}\|^2\|T\|^2.\end{aligned} \tag{2.16}$$

Proof. First one notes that

$$\tilde{\alpha} = \alpha - \frac{4n}{n+2}\langle\hat{T}, C\rangle - \frac{n^2(n-2)}{(n+2)^2}\hat{T}\otimes\hat{T} + \frac{2n^2}{(n+2)^2}\|\hat{T}\|^2 h. \tag{2.17}$$

Using a local adapted frame field, one combines (1.14), (2.4) and (2.17) to get

$$\begin{aligned}&\frac{n+2}{n-2}\sum_{ij}\tilde{\alpha}_{ij}R_{ij}\\ &\quad = \frac{n+2}{2}\sum_{ij}\tilde{\alpha}_{ij}S_{ij} + \frac{1}{2}\left(1 + \frac{4}{n-2}\right)\mathrm{tr}S\|\tilde{C}\|^2 + \left(1 + \frac{4}{n-2}\right)\sum_{ij}\tilde{\alpha}^2_{ij}\end{aligned}$$

$$-n\sum_{jkst}\tilde C_{ij}^k\tilde C_{jt}^k\tilde C_{ti}^sT^s-\frac{3n^2}{n+2}\sum_{ijksp}\tilde C_{ij}^k\tilde C_{pj}^kT^iT^p$$
$$-\frac{n^2}{n+2}\left(2+\frac{2}{n-2}\right)\|\tilde C\|^2\|T\|^2. \tag{2.18}$$

By (2.4), (2.18) and (1.15), we have

$$\begin{aligned}
&\|W\|^2+\frac{n+2}{n-2}\sum_{ij}\tilde\alpha_{ij}\left(R_{ij}-\frac{2n}{n+2}\kappa\delta_{ij}\right)\\
&=\|W\|^2+\frac{n+2}{n-2}\sum_{ij}\tilde\alpha_{ij}R_{ij}-\frac{2n}{n-2}\kappa\|\tilde C\|^2=\|W\|^2+\frac{n+2}{n-2}\sum_{ij}\tilde\alpha_{ij}R_{ij}\\
&\quad-\frac{2\|\tilde C\|^2}{(n-1)(n-2)}\left[(n-1)\mathrm{tr}S+\|\tilde C\|^2-\frac{n-1}{n+2}n^2\|T\|^2\right]\\
&=\frac{n+2}{2}\sum_{ij}\tilde\alpha_{ij}S_{ij}+\frac{1}{2}\mathrm{tr}S\|\tilde C\|^2+\sum_{ij}\tilde\alpha_{ij}^2+\sum_{ij}N\left(\tilde C^i\tilde C^j-\tilde C^j\tilde C^i\right)\\
&\quad-n\sum_{jkst}\tilde C_{ij}^k\tilde C_{jt}^k\tilde C_{ti}^sT^s-\frac{n^2}{n+2}\sum_{ijksp}\tilde C_{ij}^k\tilde C_{pj}^kT^iT^p-\frac{n^2}{n+2}\|\tilde C\|^2\|T\|^2\\
&\quad+\left[\frac{4}{n-2}\sum_{ij}\tilde\alpha_{ij}^2-\frac{2}{(n-1)(n-2)}\|\tilde C\|^4-\sum_{ij}N\left(\tilde C^i\tilde C^j-\tilde C^j\tilde C^i\right)\right.\\
&\quad\left.-\frac{2n^2}{n+2}\sum_{ijksp}\tilde C_{ij}^k\tilde C_{pj}^kT^iT^p-\frac{n^2}{n+2}\|\tilde C\|^2\|T\|^2\right]+\|W\|^2.
\end{aligned}$$

Then the corollary follows form (2.5) and (2.15). □

Remark 2. If one uses the expression (2.15), by (1.14), (1.15), (2.4) and (2.17), there is also a formula similar to (2.5). But the derived estimates will not be better than Theorems 2 and 3.

3. Rigidity Theorems for Relative Tchebychev Hypersurfaces

As applications of Theorems 2 and 3, in this section we are going to prove some rigidity theorems for relative Tchebychev hypersurfaces.

Theorem 4. *Let $x(M)$ be a locally strongly convex relative Tchebychev hypersurface with respect to the relative normalization $\{Y, y\}$. Assume that $x(M)$ satisfies*

(i) $\|\tilde C\|^2=\text{constant}$,

(ii) $\|T\|^2<\dfrac{4(n+2)}{n^2(n+10)}\left((n+1)\lambda_{\min}(S)+\dfrac{1}{n}\|\tilde C\|^2\right)$.

Then $x(M)$ is a quadric.

Proof. The assumptions (i) and (2.9) imply that

$$0 \geq \|\tilde{C}\|^2 \left[(n+1)\lambda_{\min}(S) - \frac{n^2(n+10)}{4(n+2)}\|T\|^2 + \frac{1}{n}\|\tilde{C}\|^2\right] \geq 0.$$

By the above formula and (ii), one gets $\|\tilde{C}\|^2 = 0$ on $x(M)$. So $x(M)$ is a quadric. □

The classification of locally strongly convex equiaffine hypersurfaces with parallel cubic form shows that there exist many non-quadratic hypersurfaces in this class, they are hyperbolic affine spheres (cf. [12,13,20]). For such hypersurfaces, we have an upper bound of the norm of the cubic form.

Corollary 3. *Let $x(M)$ be a locally strongly hypersurface with equiaffine normalization. Let $\nabla^h C = 0$ and $C \neq 0$. It is well known that M is a hyperbolic affine sphere and $S = \lambda$ id. Then*

$$\|C\|^2 \leq n(n+1)|\lambda|.$$

Proof. The apolarity condition for the equiaffine normalization states that $T = 0$, and this implies $\tilde{C} = C$. As already stated $\nabla^h C = 0$ implies that M is a hyperbolic affine sphere (cf. [3,8]); moreover $\nabla^h C = 0$ also gives that $\|C\|^2 =$ constant. Finally we get that $(n+1)\lambda + \frac{1}{n}\|C\|^2 \leq 0$. □

For a hyperovaloid, we have the following gap theorem.

Theorem 5. *Let $x(M)$ be a hyperovaloid and additionally a relative Tchebychev hypersurface with respect to a relative normalization $\{Y, y\}$. Assume that $x(M)$ satisfies*

$$\|T\|^2 < \frac{4(n+2)}{n^2(n+10)}\left((n+1)\lambda_{\min}(S) + \frac{1}{n}\|\tilde{C}\|^2\right).$$

Then $x(M)$ is an ellipsoid.

Proof. Since M is closed, we have

$$0 = \int_M \frac{1}{2}\Delta^h\|\tilde{C}\|^2 \geq \int_M \|\tilde{C}\|^2 \left[(n+1)\lambda_{\min}(S) - \frac{n^2(n+10)}{4(n+2)}\|T\|^2 + \frac{1}{n}\|\tilde{C}\|^2\right].$$

By the assumption, $\|\tilde{C}\|^2$ must be zero everywhere. As $x(M)$ is locally strongly convex this implies $\tilde{C} = 0$, and thus $x(M)$ is an ellipsoid. □

Remark 3. In the paper [24], Liu and Wang proved that relative Tchebychev ovaloids in A^3 must be ellipsoids. So Theorem 5 gives new information only for dimension $n \geq 3$.

When the normalization of the closed manifold M is centroaffine, then $x(M)$ is always a relative sphere and satisfies $S =$ id. In this case, we have the following theorem of Blaschke–Deicke [1,2,4,7,25] as a corollary of Theorem 5.

Corollary 4 (Blaschke–Deicke). *Let $x(M)$ be a hyperovaloid with centroaffine normalization $\{Y, -x\}$. If the centroafine Tchebychev field satisfies $T = 0$ then $C = 0$ and $x(M)$ is an ellipsoid.*

Proof. $T = 0$ implies that $x(M)$ is a centroaffine Tchebychev hypersurface. In this case $\tilde{C} = C$, and the following inequality holds

$$0 = \|T\|^2 < \frac{4(n+2)}{n^2(n+10)} \left((n+1) + \frac{1}{n}\|C\|^2 \right).$$

Then Theorem 5 implies that $C = 0$ and $x(M)$ is an ellipsoid. □

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