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Results in Mathematics

New Korovkin Type Theorem for Non-Tensor Meyer–König and Zeller Operators

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Abstract. In this paper, we introduce a certain class of linear positive operators via a generating function, which includes the non-tensor MKZ operators and their non-trivial extension. In investigating the approximation properties, we prove a new Korovkin type approximation theorem by using appropriate test functions. We compute the rate of convergence of these operators by means of the modulus of continuity and the elements of modified Lipschitz class functions. Furthermore, we give functional partial differential equations for this class. Using the corresponding equations, we calculate the first few moments of the non-tensor MKZ operators and investigate their approximation properties. Finally, we state the multivariate versions of the results and obtain the convergence properties of the multivariate Meyer–König and Zeller operators.

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1. Introduction

It was Korovkin who introduced the idea of approximation to a function by means of positive linear operators [18]. Different variants of the Korovkin's theorems can be found in [7]. Especially in the last two decades, many mathematicians studied and improved this theory by defining positive linear operators on various function spaces (see [12–14,21]). In the present paper, we are concerned with the celebrated Meyer–König and Zeller operators which were defined in [20] by

$$M_n(f;x)=\sum_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right)\binom{n+k}{k}x^k(1-x)^{n+1},\ x\in[0,1)\,.$$

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Different variants of these operators and their approximation properties have been an area of intensive research during the last five decades (see [2,3,11,15, 23]). The slightly modified form of these operators were given by Cheney and Sharma [9], where they replaced the nodes $\frac{k}{n+k+1}$ by $\frac{k}{n+k}$ and introduced the modified operators, which they called Bernstein power series. In 2005, Altın et al. [6] considered a generating function extension of the Bernstein power series and proved a Korovkin type approximation theorem by using the test functions $\widetilde{f}_i(t) = (\frac{t}{1-t})^i$ $(i = 0, 1, 2)$. It should be noticed that, if one uses the usual Korovkin's theorem (where the test functions are $f_i(t) = t^i$ $(i = 0, 1, 2)$) in investigating the approximation properties of the operators $M_n(f; x)$, it is not easy to calculate the second moment. Alkemade [5] solved this problem, but the result was not sufficiently useful. For the estimation of the higher order moments, we refer [1,8,16]. Another generalization of Bernstein power series was given in [10], where the author used the test functions $\widetilde{f}_i(t)$ to prove a Voronovskaja type asymptotic formula. The uniform approximation properties of the Bernstein power series were obtained very recently in [17].

Taşdelen and Erençin [22] introduced the bivariate tensor type generalization of the Bernstein power series by means of the generating functions. They proved a Korovkin type approximation theorem by introducing the test functions $f_0(s,t) = 1, f_1(s,t) = \frac{s}{1-s},\ f_2(s,t) = \frac{t}{1-t}$ and $f_3(s,t) = (\frac{s}{1-s})^2 + (\frac{t}{1-t})^2$. Recently, the multivariable version of the above mentioned Korovkin theorem involving the matrix summability methods was given in [4].

Let

$$S_A := \{\mathbf{x} = (x,y) : 0 \le x \le A < 1,\ 0 \le y \le A - x\}.$$

In the present paper, we introduce the following non-tensor two variable operators:

$$L_n(f;x,y) = \frac{1}{\Omega_n(u,v;x,y)} \sum_{k,l=0}^{\infty} f\left(\frac{a_{k,l,n}}{a_{k,l,n}+c_{k,l,n}+b_n}, \frac{c_{k,l,n}}{a_{k,l,n}+c_{k,l,n}+b_n}\right) \times P_{k,l}^n(u,v)x^k y^l \tag{1}$$

where

$$\frac{a_{k,l,n}}{a_{k,l,n}+c_{k,l,n}+b_n}, \frac{c_{k,l,n}}{a_{k,l,n}+c_{k,l,n}+b_n} \in S_A, (0 < A < 1)$$

$P_{k,l}^n(u,v) \ge 0$ for all $S_A \subset \mathbb{R}^2$ and $\{\Omega_n(s,t;x,y)\}_{n\in\mathbb{N}}$ is the generating function for the double indexed function sequence $\{P_{k,l}^n(u,v)\}_{k,l\in\mathbb{N}_0}$ given in the form

$$\Omega_n(u,v;x,y) = \sum_{k,l=0}^{\infty} P_{k,l}^n(u,v)x^k y^l. \tag{2}$$

$$(x,y) \in S_A$$

It is clear that $L_n(f;x,y)$ is linear and positive and therefore monotone. Throughout the paper, we assume that the following conditions hold:

(i) $\Omega_n(u,v;x,y) = (1-x-y)\Omega_{n+1}(u,v;x,y)$,
(ii) $a_{k+1,l,n}P^n_{k+1,l}(u,v) = b_{n+1}P^{n+1}_{k,l}(u,v)$, $c_{k,l+1,n}P^n_{k,l+1}(u,v) = b_{n+1}P^{n+1}_{k,l}(u,v)$,
(iii) $a_{k+1,l,n} = a_{k,l,n+1} + d_n$, $a_{0,l,n} = 0$, $|d_n| \leq K_1 < \infty$ $(\forall n \in \mathbb{N})$,
(iv) $c_{k,l+1,n} = c_{k,l,n+1} + e_n$, $c_{k,0,n} = 0$, $|e_n| \leq K_2 < \infty$ $(\forall n \in \mathbb{N})$
(v) $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 1$ and $b_n \neq 0$ $(\forall n \in \mathbb{N})$.

Note that, choosing $\Omega_n(u,v;x,y) = \frac{1}{(1-x-y)^{n+1}}$, we see from (2) that $P^n_{k,l}(u,v) = \frac{(n+k+l)!}{n!k!l!}$. Further, taking $a_{k,l,n} = k$, $c_{k,l,n} = l$ and $b_n = n$ in (1), we obtain the non-tensor bivariate Meyer–König and Zeller operators (see [19])

$$\mathbb{M}_n(f;x,y) = (1-x-y)^{n+1}\sum_{k,l=0}^{\infty}\frac{(n+k+l)!}{n!k!l!}x^ky^lf\left(\frac{k}{k+l+n},\frac{l}{k+l+n}\right).$$

Letting $u,v \geq 0$ to be fixed, and taking $\Omega_n(u,v;x,y) = \frac{1}{(1-x-y)^{u+v+n+1}}$ then $P^n_{k,l}(u,v)$ will be

$$P^n_{k,l}(u,v) = \binom{n+u+v+k+l}{k,l} = \frac{\Gamma(n+u+v+k+l+1)}{\Gamma(n+u+v+1)k!l!}.$$

Choosing, $a_{k,l,n} = k$, $c_{k,l,n} = l$ and $b_n = n+u+v$, we define the generalized form of non-tensor bivariate Meyer–König and Zeller operators:

$$\mathbb{A}^{(u,v)}_n(f;x,y) := (1-x-y)^{n+u+v+1}\sum_{k,l=0}^{\infty}\binom{n+u+v+k+l}{k,l}x^ky^lf \times\left(\frac{k}{k+l+n+u+v},\frac{l}{k+l+n+u+v}\right). \tag{3}$$

Clearly, $\mathbb{M}_n(f;x,y) = \mathbb{A}^{(0,0)}_n(f;x,y)$.

2. Korovkin Type Theorem

Throughout the paper, we consider the following test functions

$$\varphi_0(s,t) = 1,\quad \varphi_1(s,t) = \frac{s}{1-s-t},\quad \varphi_2(s,t) = \frac{t}{1-s-t},$$
$$\varphi_3(s,t) = \left(\frac{s}{1-s-t}\right)^2 + \left(\frac{t}{1-s-t}\right)^2.$$

Clearly, for

$$s = \frac{a_{k_1,k_2,n}}{a_{k_1,k_2,n}+c_{k_1,k_2,n}+b_n},\quad t = \frac{c_{k_1,k_2,n}}{a_{k_1,k_2,n}+c_{k_1,k_2,n}+b_n}$$

we have

$$\varphi_1\left(\frac{a_{k_1,k_2,n}}{a_{k_1,k_2,n}+c_{k_1,k_2,n}+b_n},\frac{c_{k_1,k_2,n}}{a_{k_1,k_2,n}+c_{k_1,k_2,n}+b_n}\right)=\frac{a_{k_1,k_2,n}}{b_n}, \quad (4)$$

$$\varphi_2\left(\frac{a_{k_1,k_2,n}}{a_{k_1,k_2,n}+c_{k_1,k_2,n}+b_n},\frac{c_{k_1,k_2,n}}{a_{k_1,k_2,n}+c_{k_1,k_2,n}+b_n}\right)=\frac{c_{k_1,k_2,n}}{b_n}, \quad (5)$$

$$\varphi_3\left(\frac{a_{k_1,k_2,n}}{a_{k_1,k_2,n}+c_{k_1,k_2,n}+b_n},\frac{c_{k_1,k_2,n}}{a_{k_1,k_2,n}+c_{k_1,k_2,n}+b_n}\right)$$
$$=\left(\frac{a_{k_1,k_2,n}}{b_n}\right)^2+\left(\frac{c_{k_1,k_2,n}}{b_n}\right)^2. \quad (6)$$

In this paper, we consider the following function space

$$\mathcal{H}_\omega(S_A):=\{f\in C(S_A):|f(s,t)-f(x,y)|\le\omega\left(f,|(\varphi_1(s,t),\varphi_2(s,t))\right.$$
$$\left.-(\varphi_1(x,y),\varphi_2(x,y))|\right)\} \quad (7)$$

where $C(S_A)$ denotes the space of continuous functions defined on S_A,

$$|(\varphi_1(s,t),\varphi_2(s,t))-(\varphi_1(x,y),\varphi_2(x,y))|$$
$$=\sqrt{(\varphi_1(s,t)-\varphi_1(x,y))^2+(\varphi_2(s,t)-\varphi_2(x,y))^2}$$

and

$$\omega(f,\delta)=\sup\left\{|f(s,t)-f(x,y)|:(s,t),(x,y)\in S_A,\sqrt{(s-x)^2+(t-y)^2}\le\delta\right\}$$

is the modulus of continuity of f satisfying the following properties:

(a) ω is non-negative and increasing function of δ,
(b) $\omega(f,\delta_1+\delta_2)\le\omega(f,\delta_1)+\omega(f,\delta_2)$,
(c) $\lim_{\delta\to0^+}\omega(f,\delta)=0$.

Clearly, for each $(s,t),(x,y)\in S_A$ and for all $f\in\mathcal{H}_\omega(S_A)$, we have

$$|f(s,t)-f(x,y)|\le\omega(f,\delta)\left(1+\frac{|(\varphi_1(s,t),\varphi_2(s,t))-(\varphi_1(x,y),\varphi_2(x,y))|}{\delta}\right). \quad (8)$$

In the following theorem, we prove one of the main result of the paper.

Theorem 1. *Let $T_n:\mathcal{H}_\omega(S_A)\to C(S_A)$ be a sequence of linear positive operators satisfying*

$$\lim_{n\to\infty}\|T_n(\varphi_i;\cdot,\cdot)-\varphi_i\|_{C(S_A)}=0,\quad (i=0,1,2,3) \quad (9)$$

where $\|\cdot\|_{C(S_A)}$ denotes the usual supremum norm on $C(S_A)$. Then for all $f\in\mathcal{H}_\omega(S_A)$, we have

$$\lim_{n\to\infty}\|T_n(f;\cdot,\cdot)-f\|_{C(S_A)}=0.$$

Proof. Let $f \in \mathcal{H}_\omega(S_A)$ be given. For all $\epsilon > 0$, we have from property (c) that

$$|f(s,t) - f(x,y)| < \epsilon$$

for $(s,t),(x,y) \in S_A$, satisfying

$$\sqrt{\left(\frac{s}{1-s-t} - \frac{x}{1-x-y}\right)^2 + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y}\right)^2} < \delta$$

with some $\delta > 0$. On the other hand, since $f \in C(S_A)$, for $(s,t),(x,y) \in S_A$ with

$$\sqrt{\left(\frac{s}{1-s-t} - \frac{x}{1-x-y}\right)^2 + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y}\right)^2} \geq \delta,$$

we have

$$|f(s,t) - f(x,y)| < \frac{2M}{\delta^2}\left\{\left(\frac{s}{1-s-t} - \frac{x}{1-x-y}\right)^2 + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y}\right)^2\right\},$$

where M is the bound of f. Combining the above inequalities, we get for all $(s,t),(x,y) \in S_A$ and $f \in \mathcal{H}_\omega(S_A)$ that

$$|f(s,t) - f(x,y)| < \epsilon + \frac{2M}{\delta^2}\left\{\left(\frac{s}{1-s-t} - \frac{x}{1-x-y}\right)^2 + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y}\right)^2\right\}. \tag{10}$$

By linearity and positivity of the operators T_n, we can write

$$|T_n(f;x,y) - f(x,y)| \leq T_n\left(|f(s,t) - f(x,y)|;x,y\right) + |f(x,y)|\,|T_n(\varphi_0;x,y) - \varphi_0(x,y)|.$$

Using the inequality (10), we get

$$\begin{aligned}|T_n(f;x,y) - f(x,y)| < \epsilon + \left(\epsilon + M + \frac{2M}{\delta^2}B(A)\right)|T_n(\varphi_0;x,y) - \varphi_0(x,y)| \\ + \frac{4M}{\delta^2}B(A)\left\{|T_n(\varphi_1;x,y) - \varphi_1(x,y)| \right. \\ \left. + |T_n(\varphi_2;x,y) - \varphi_2(x,y)|\right\} \\ + \frac{2M}{\delta^2}|T_n(\varphi_3;x,y) - \varphi_3(x,y)|\end{aligned}$$

where

$$B(A) = \max_{(x,y)\in S_A}\left\{\frac{x^2+y^2}{(1-x-y)^2}, \frac{x}{1-x-y}, \frac{y}{1-x-y}\right\}.$$

Taking into account (9), the proof is completed. □

3. Approximation of $L_n(f;x,y)$ in $\mathcal{H}_\omega(S_A)$

In this section, we prove the convergence of the operators $L_n(f;x,y)$ in the space $\mathcal{H}_\omega(S_A)$. It is obvious from (1) and (2) that $L_n(1;x,y)=1$. For the other moments, we need the following Lemma.

Lemma 2. *Assume that the conditions* (i)–(iv) *are satisfied. Then, for the operators L_n defined by* (1) *and* (2), *we have*

(a) $L_n(\varphi_1;x,y)=\frac{b_{n+1}}{b_n}\varphi_1(x,y)$,
(b) $L_n(\varphi_2;x,y)=\frac{b_{n+1}}{b_n}\varphi_2(x,y)$,
(c) $L_n(\varphi_3;x,y)=\frac{b_{n+1}b_{n+2}}{b_n^2}\varphi_3(x,y)+\frac{b_{n+1}d_n}{b_n^2}\varphi_1(x,y)+\frac{b_{n+1}e_n}{b_n^2}\varphi_2(x,y)$.

Proof. Using (4), we get

$$L_n(\varphi_1;x,y)=\frac{1}{\Omega_n(u,v;x,y)}\sum_{k,l=0}^{\infty}\frac{a_{k,l,n}}{b_n}P_{k,l}^n(u,v)x^ky^l.$$

Considering the conditions (i) and (ii), we have

$$\begin{aligned}L_n(\varphi_1;x,y)&=\frac{1}{(1-x-y)\Omega_{n+1}(u,v;x,y)}\sum_{k=1,l=0}^{\infty}\frac{a_{k,l,n}}{b_n}P_{k,l}^n(u,v)x^ky^l\\&=\frac{b_{n+1}x}{b_n(1-x-y)\Omega_{n+1}(u,v;x,y)}\sum_{k,l=0}^{\infty}P_{k,l}^{n+1}(u,v)x^ky^l.\end{aligned}$$

By (1), we get (a) at once. In a similar manner, using (5), then considering the conditions (i) and the second part of (ii), we get (b).

Finally, by (6),

$$L_n(\varphi_3;x,y)=\frac{1}{\Omega_n(u,v;x,y)}\sum_{k,l=0}^{\infty}\frac{(a_{k,l,n})^2+(c_{k,l,n})^2}{b_n^2}P_{k,l}^n(u,v)x^ky^l.$$

By the conditions (i)–(iv), we obtain

$$\begin{aligned}L_n(\varphi_3;x,y)&=\frac{x}{b_n^2\Omega_n(u,v;x,y)}\sum_{k=1,l=0}^{\infty}(a_{k,l,n})^2P_{k,l}^n(u,v)x^{k-1}y^l\\&\quad+\frac{y}{b_n^2\Omega_n(u,v;x,y)}\sum_{k=0,l=1}^{\infty}(c_{k,l,n})^2P_{k,l}^n(u,v)x^ky^{l-1}\\&=\frac{x}{b_n^2\Omega_n(u,v;x,y)}\sum_{k,l=0}^{\infty}(a_{k+1,l,n})^2P_{k+1,l}^n(u,v)x^ky^l\end{aligned}$$

$$
\begin{aligned}
&+\frac{y}{b_n^2\Omega_n(u,v;x,y)}\sum_{k,l=0}^{\infty}\left(c_{k,l+1,n}\right)^2 P_{k,l+1}^{n}(u,v)x^k y^l\\
&=\frac{b_{n+1}x}{b_n^2\Omega_n(u,v;x,y)}\sum_{k,l=0}^{\infty}a_{k+1,l,n}P_{k,l}^{n+1}(u,v)x^k y^l\\
&\quad+\frac{b_{n+1}y}{b_n^2\Omega_n(u,v;x,y)}\sum_{k,l=0}^{\infty}c_{k,l+1,n}P_{k,l}^{n+1}(u,v)x^k y^l\\
&=\frac{b_{n+1}x}{b_n^2\Omega_n(u,v;x,y)}\sum_{k=1,l=0}^{\infty}a_{k,l,n+1}P_{k,l}^{n+1}(u,v)x^k y^l\\
&\quad+\frac{b_{n+1}y}{b_n^2\Omega_n(u,v;x,y)}\sum_{k=0,l=1}^{\infty}c_{k,l,n+1}P_{k,l}^{n+1}(u,v)x^k y^l\\
&\quad+\frac{xd_nb_{n+1}}{b_n^2\Omega_n(u,v;x,y)}\sum_{k,l=0}^{\infty}P_{k,l}^{n+1}(u,v)x^k y^l\\
&\quad+\frac{ye_nb_{n+1}}{b_n^2\Omega_n(u,v;x,y)}\sum_{k,l=0}^{\infty}P_{k,l}^{n+1}(u,v)x^k y^l\\
&=\frac{b_{n+1}x^2}{b_n^2\Omega_n(u,v;x,y)}\sum_{k,l=0}^{\infty}a_{k+1,l,n+1}P_{k+1,l}^{n+1}(u,v)x^k y^l\\
&\quad+\frac{b_{n+1}y^2}{b_n^2\Omega_n(u,v;x,y)}\sum_{k,l=0}^{\infty}c_{k,l+1,n+1}P_{k,l+1}^{n+1}(u,v)x^k y^l\\
&\quad+\frac{b_{n+1}d_n}{b_n^2}\varphi_1(x,y)+\frac{b_{n+1}e_n}{b_n^2}\varphi_2(x,y)\\
&=\frac{x^2b_{n+1}b_{n+2}}{b_n^2\Omega_n(u,v;x,y)}\sum_{k,l=0}^{\infty}P_{k,l}^{n+2}(u,v)x^k y^l\\
&\quad+\frac{y^2b_{n+1}b_{n+2}}{b_n^2\Omega_n(u,v;x,y)}\sum_{k,l=0}^{\infty}P_{k,l}^{n+2}(u,v)x^k y^l\\
&\quad+\frac{b_{n+1}d_n}{b_n^2}\varphi_1(x,y)+\frac{b_{n+1}e_n}{b_n^2}\varphi_2(x,y)\\
&=\frac{b_{n+1}b_{n+2}}{b_n^2}\varphi_3(x,y)+\frac{b_{n+1}d_n}{b_n^2}\varphi_1(x,y)+\frac{b_{n+1}e_n}{b_n^2}\varphi_2(x,y).
\end{aligned}
$$

Therefore we get (c). □

By Theorem 1 and Lemma 2, we have the following approximation theorem.

Theorem 3. *Let $L_n : \mathcal{H}_\omega(S_A) \to C(S_A)$ be a sequence of linear positive operators defined by* (1) *and* (2)*. Assume that the conditions* (i)–(v) *are satisfied. Then for all $f \in \mathcal{H}_\omega(S_A)$, we have*

$$\lim_{n\to\infty} \left\| L_n (f;\cdot,\cdot) - f \right\|_{C(S_A)} = 0.$$

Proof. According to Theorem 1, we should prove that

$$\lim_{n\to\infty} \left\| L_n (\varphi_i;\cdot,\cdot) - \varphi_i \right\|_{C(S_A)} = 0, \quad (i = 0, 1, 2, 3).$$

By (1) and (2), $L_n(\varphi_0; x, y) = \varphi_0(x, y)$.

Now let $B(A) = \sup_{(x,y)\in S_A} \{\varphi_1(x, y), \varphi_2(x, y), \varphi_3(x, y)\}$. From Lemma 2 (a) and (b), we have

$$\left\| L_n (\varphi_1;\cdot,\cdot) - \varphi_1 \right\|_{C(S_A)} \le \left| \frac{b_{n+1}}{b_n} - 1 \right| B(A)$$

and

$$\left\| L_n (\varphi_2;\cdot,\cdot) - \varphi_2 \right\|_{C(S_A)} \le \left| \frac{b_{n+1}}{b_n} - 1 \right| B(A).$$

Therefore, from (v) we obtain

$$\lim_{n\to\infty} \left\| L_n (\varphi_i;\cdot,\cdot) - \varphi_i \right\|_{C(S_A)} = 0, \ (i = 1, 2).$$

Finally, by Lemma 2 (c) and (iii)–(iv), we get

$$\begin{aligned} L_n (\varphi_3; x, y) - \varphi_3(x, y) &= \left[\frac{b_{n+1} b_{n+2}}{b_n^2} - 1 \right] \varphi_3(x, y) \\ &\quad + \frac{b_{n+1} d_n}{b_n^2} \varphi_1(x, y) + \frac{b_{n+1} e_n}{b_n^2} \varphi_2(x, y) \end{aligned}$$

and hence

$$\begin{aligned} \left| L_n (\varphi_3; x, y) - \varphi_3(x, y) \right| &\le \left| \frac{b_{n+1} b_{n+2}}{b_n^2} - 1 \right| \varphi_3(x, y) \\ &\quad + \frac{b_{n+1} d_n}{b_n^2} \varphi_1(x, y) + \frac{b_{n+1} e_n}{b_n^2} \varphi_2(x, y) \\ &\le \left| \frac{b_{n+1} b_{n+2}}{b_n^2} - 1 \right| \varphi_3(x, y) \\ &\quad + \frac{b_{n+1} K_1}{b_n^2} \varphi_1(x, y) + \frac{b_{n+1} K_2}{b_n^2} \varphi_2(x, y). \end{aligned}$$

Taking supremum over S_A and letting $B(A) = \sup\{\varphi_1(x, y), \varphi_2(x, y), \varphi_3(x, y)\}$, we get

$$\left| L_n (\varphi_3; x, y) - \varphi_3(x, y) \right| \le B(A) \left[\left| \frac{b_{n+1} b_{n+2}}{b_n^2} - 1 \right| + (K_1 + K_2) \frac{b_{n+1}}{b_n^2} \right].$$

Passing to limit as $n \to \infty$ and using (v), the proof is completed. $\square$

4. The Order of Approximation

In this section, we compute the order of approximation of $L_n(f; x, y)$ to $f(x, y)$ in terms of the modulus of continuity and the modified Lipschitz class functionals. We start with the following lemma.

Lemma 4. *Let L_n be a sequence of linear positive operators defined by* (1) *and* (2)*. Then the following estimate*

$$L_n\left(\sqrt{(\varphi_1(s,t)-\varphi_1(x,y))^2+(\varphi_2(s,t)-\varphi_2(x,y))^2}; x, y\right)$$
$$\leq \sqrt{B(A)\left[\left|\frac{b_{n+1}b_{n+2}}{b_n^2}-1\right|+4\left|\frac{b_{n+1}}{b_n}-1\right|+(K_1+K_2)\frac{b_{n+1}}{b_n^2}\right]},$$

holds true, where

$$B(A) = \sup_{(x,y)\in S_A}\{\varphi_1(x,y), \varphi_2(x,y), \varphi_3(x,y)\}$$

and K_1, K_2 are the positive constants given in (iii) *and* (iv)*.*

Proof. Using Cauchy–Schwarz inequality and noting that $L_n(1; x, y) = 1$, we get

$$L_n\left(\sqrt{(\varphi_1(s,t)-\varphi_1(x,y))^2+(\varphi_2(s,t)-\varphi_2(x,y))^2}; x, y\right)$$
$$\leq \sqrt{L_n\left((\varphi_1(s,t)-\varphi_1(x,y))^2+(\varphi_2(s,t)-\varphi_2(x,y))^2; x, y\right)}.$$

On the other hand by Lemma 2, we can write

$$L_n\left((\varphi_1(s,t)-\varphi_1(x,y))^2+(\varphi_2(s,t)-\varphi_2(x,y))^2; x, y\right)$$
$$\leq |L_n(\varphi_3; x,y)-\varphi_3(x,y)| + \frac{2y}{1-x-y}|L_n(\varphi_2; x,y)-\varphi_2(x,y)|$$
$$+\frac{2x}{1-x-y}|L_n(\varphi_1; x,y)-\varphi_1(x,y)|$$
$$\leq B(A)\left[\left|\frac{b_{n+1}b_{n+2}}{b_n^2}-1\right|+4\left|\frac{b_{n+1}}{b_n}-1\right|+(K_1+K_2)\frac{b_{n+1}}{b_n^2}\right].$$

Whence the result. □

Theorem 5. *Let L_n be a sequence of linear positive operators defined by* (1) *and* (2)*. Then for all $f \in \mathcal{H}_\omega(S_A)$, we have*

$$\|L_n(f;\cdot,\cdot)-f\|_{C(S_A)}$$
$$\leq 2\omega\left(f, \sqrt{B(A)\left[\left|\frac{b_{n+1}b_{n+2}}{b_n^2}-1\right|+4\left|\frac{b_{n+1}}{b_n}-1\right|+(K_1+K_2)\frac{b_{n+1}}{b_n^2}\right]}\right),$$

where $B(A), K_1$ and K_2 are the same as in Lemma 4.

Proof. Because of linearity and monotonicity of the operators L_n, we get by (7) and (8) that

$$\begin{aligned} |L_n(f;x,y) - f(x,y)| &\leq L_n\left(|f(s,t) - f(x,y)|;x,y\right) \\ &\leq L_n\left(\omega\left(f, \sqrt{(\varphi_1(s,t) - \varphi_1(x,y))^2 + (\varphi_2(s,t) - \varphi_2(x,y))^2}\right);x,y\right) \\ &\leq \omega(f,\delta_n)\left[1+\frac{L_n\left(\sqrt{(\varphi_1(s,t)-\varphi_1(x,y))^2+(\varphi_2(s,t)-\varphi_2(x,y))^2};x,y\right)}{\delta_n}\right]. \end{aligned}$$

Now using Lemma 4 and choosing

$$\delta_n = \sqrt{B(A)\left[\left|\frac{b_{n+1}b_{n+2}}{b_n^2} - 1\right| + 4\left|\frac{b_{n+1}}{b_n} - 1\right| + (K_1 + K_2)\frac{b_{n+1}}{b_n^2}\right]},$$

we get the result after taking supremum over S_A on both sides of the inequality. □

Now, we are aimed to compute the order of convergence of the operators in terms of the modified Lipschitz class functionals. We introduce the modified Lipschitz class functions by

$$\begin{aligned} Lip_M^*(\alpha) := \Bigg\{ & f \in C(S_A) : |f(s,t) - f(x,y)| \\ & \leq M\left[\left(\frac{s}{1-s-t} - \frac{x}{1-x-y}\right)^2 \right. \\ & \left. + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y}\right)^2\right]^{\alpha/2};\ (t,s),(x,y) \in S_A \Bigg\}, \end{aligned}$$

where M is positive constant depending on f and $\alpha \in (0,1]$.

Theorem 6. *Let L_n be a sequence of linear positive operators defined by (1) and (2). Then for all $f \in Lip_M^*(\alpha)$, we have*

$$\begin{aligned} \|L_n(f;\cdot,\cdot) - f\|_{C(S_A)} \leq M\Bigg[B(A)\Bigg[\left|\frac{b_{n+1}b_{n+2}}{b_n^2} - 1\right| \\ +4\left|\frac{b_{n+1}}{b_n} - 1\right| + (K_1 + K_2)\frac{b_{n+1}}{b_n^2}\Bigg]\Bigg]^{\alpha/2}, \end{aligned}$$

where $B(A), K_1$ and K_2 be the same as in Lemma 4.

Proof. Using linearity and monotonicity properties of the operators L_n and considering that $f \in Lip_M^*(\alpha)$, we get

$$\begin{aligned} |L_n(f;x,y) - f(x,y)| &\leq L_n\left(|f(s,t) - f(x,y)|\,;x,y\right) \\ &\leq M L_n\left(\left[\left(\frac{s}{1-s-t} - \frac{x}{1-x-y}\right)^2 \right.\right. \\ &\qquad \left.\left. + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y}\right)^2\right]^{\alpha/2} ;x,y\right). \end{aligned}$$

Applying Hölder inequality with $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$ and taking into account that $L_n(1;x,y) = 1$, we obtain

$$\begin{aligned} &|L_n(f;x,y) - f(x,y)| \\ &\leq M\left\{L_n\left(\sqrt{(\varphi_1(s,t) - \varphi_1(x,y))^2 + (\varphi_2(s,t) - \varphi_2(x,y))^2};x,y\right)\right\}^{\alpha}. \end{aligned}$$

Taking supremum over S_A on both sides of the above inequality and considering Lemma 4, the proof is completed. □

5. Functional Partial Differential Equations and Their Consequences

In this section, we obtain a functional partial differential equation satisfied by the particular case of the operators $L_n(f;x,y)$. We consider the operators

$$L_n^*(f;x,y) = \frac{1}{\Omega_n(u,v;x,y)} \sum_{k,l=0}^{\infty} f\left(\frac{k}{k+l+b_n}, \frac{l}{k+l+b_n}\right) P_{k,l}^n(u,v) x^k y^l, \tag{11}$$

where $\Omega_n(u,v;x,y)$ is given by (2).

Theorem 7. *Let $(x,y) \in S_A$, $f \in C(S_A)$, $L_n^*(f;x,y)$ be defined by (11) and (2). Assume that*

$$\frac{\partial}{\partial x}\left(\Omega_n(u,v;x,y)\right) = A_n(x,y)\Omega_n(u,v;x,y), \tag{12}$$

$$\frac{\partial}{\partial y}\left(\Omega_n(u,v;x,y)\right) = B_n(x,y)\Omega_n(u,v;x,y). \tag{13}$$

Then $L_n^*(f;x,y)$ satisfy the following functional partial differential equations

$$x\frac{\partial}{\partial x}L_n^*(f;x,y) = -xA_n(x,y)L_n^*(f;x,y) + b_n L_n^*(\varphi_1 f;x,y), \tag{14}$$

$$y\frac{\partial}{\partial y}L_n^*(f;x,y) = -yB_n(x,y)L_n^*(f;x,y) + b_n L_n^*(\varphi_2 f;x,y) \tag{15}$$

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)L_n^*(f;x,y)=\left(-xA_n(x,y)-yB_n(x,y)\right)L_n^*(f;x,y)$$
$$+b_nL_n^*(hf;x,y), \tag{16}$$

where $h(s,t)=\frac{s}{1-s-t}+\frac{t}{1-s-t}$.

Proof. Since $f\in C(S_A)$, it is clear by (2) that the series in (11) converges uniformly for all $(x,y)\in S_A$. Therefore, term by term differentiation is permissible in S_A. Differentiating both sides of (11) with respect to x and considering (12), we get

$$\frac{\partial}{\partial x}L_n^*(f;x,y)=-A_n(x,y)L_n^*(f;x,y)$$
$$+\frac{1}{\Omega_n(u,v;x,y)}\sum_{k=1,l=0}^{\infty}kf\left(\frac{k}{k+l+b_n},\frac{l}{k+l+b_n}\right)P_{k,l}^n(u,v)x^{k-1}y^l.$$

Multiplying both sides by x, we have

$$x\frac{\partial}{\partial x}L_n^*(f;x,y)=-xA_n(x,y)L_n^*(f;x,y)$$
$$+\frac{b_n}{\Omega_n(u,v;x,y)}\sum_{k,l=0}^{\infty}\frac{k}{b_n}f\left(\frac{k}{k+l+b_n},\frac{l}{k+l+b_n}\right)P_{k,l}^n(u,v)x^ky^l$$
$$=-xA_n(x,y)L_n^*(f;x,y)+b_nL_n^*(\varphi_1f;x,y),$$

which gives (14). Now, differentiating both sides of (11) with respect to y and taking into account (13), we get

$$\frac{\partial}{\partial y}L_n^*(f;x,y)=-B_n(x,y)L_n^*(f;x,y)$$
$$+\frac{1}{\Omega_n(u,v;x,y)}\sum_{k=0,l=1}^{\infty}l\ f\left(\frac{k}{k+l+b_n},\frac{l}{k+l+b_n}\right)P_{k,l}^n(u,v)x^ky^{l-1}.$$

Multiplying both sides by y, we obtain

$$y\frac{\partial}{\partial y}L_n^*(f;x,y)=-xB_n(x,y)L_n^*(f;x,y)$$
$$+\frac{b_n}{\Omega_n(u,v;x,y)}\sum_{k,l=0}^{\infty}\frac{l}{b_n}f\left(\frac{k}{k+l+b_n},\frac{l}{k+l+b_n}\right)P_{k,l}^n(u,v)x^ky^l$$
$$=-yB_n(x,y)L_n^*(f;x,y)+b_nL_n^*(\varphi_2f;x,y).$$

This proves (15). Adding (14) and (15), then taking into account that L_n^* is linear, we get (16) □

Obviously, letting $\Omega_n(u,v;x,y)=\frac{1}{(1-x-y)^{n+1}}$ and $b_n=n$ in (11), we get the non-tensor bivariate Meyer–König and Zeller operators

$$\mathbb{M}_n(f;x,y)=(1-x-y)^{n+1}\sum_{k,l=0}^{\infty}\frac{(n+k+l)!}{n!k!l!}x^k y^l f\left(\frac{k}{k+l+n},\frac{l}{k+l+n}\right).$$

Hence we have the following Corollary at once.

Corollary 8. *Let $(x,y)\in S_A,\ f\in C(S_A)$. Then $\mathbb{M}_n(f;x,y)$ satisfy the following functional partial differential equations*

$$x\frac{\partial}{\partial x}\mathbb{M}_n(f;x,y)=\frac{-x(n+1)}{1-x-y}\mathbb{M}_n(f;x,y)+n\mathbb{M}_n(\varphi_1 f;x,y), \quad (17)$$

$$y\frac{\partial}{\partial y}\mathbb{M}_n(f;x,y)=\frac{-y(n+1)}{1-x-y}\mathbb{M}_n(f;x,y)+n\mathbb{M}_n(\varphi_2 f;x,y) \quad (18)$$

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\mathbb{M}_n(f;x,y)=\frac{-(n+1)}{1-x-y}(x+y)\,\mathbb{M}_n(f;x,y)+n\mathbb{M}_n(hf;x,y),$$

where $h(s,t)=\frac{s}{1-s-t}+\frac{t}{1-s-t}$.

Although, one can write the first few moments of the operators $\mathbb{M}_n(f;x,y)$ directly from Lemma 2, we saw in the proof of this lemma that the computations are rigorous. The above corollary is very useful in computing the moments easily. For instance taking $f=\varphi_0=1$ in (17), we get

$$\mathbb{M}_n(\varphi_1;x,y)=\frac{n+1}{n}\varphi_1(x,y).$$

Under the same choice in (18), we obtain

$$\mathbb{M}_n(\varphi_2;x,y)=\frac{n+1}{n}\varphi_2(x,y).$$

Now set $h_1(x,y)=(\frac{x}{1-x-y})^2$ and $h_2(x,y)=(\frac{y}{1-x-y})^2$. Choosing $f=\varphi_1$, in (17) and $f=\varphi_2$ in (18), we have

$$\mathbb{M}_n(h_1;x,y)=\frac{(n+1)(n+2)}{n^2}h_1(x,y)+\frac{(n+1)}{n^2}\varphi_1(x,y)$$

and

$$\mathbb{M}_n(h_2;x,y)=\frac{(n+1)(n+2)}{n^2}h_2(x,y)+\frac{(n+1)}{n^2}\varphi_2(x,y),$$

respectively. Adding both sides of the above equalities, we get

$$\mathbb{M}_n(\varphi_3;x,y)=\frac{(n+1)(n+2)}{n^2}\varphi_3(x,y)+\frac{(n+1)}{n^2}\left(\varphi_1(x,y)+\varphi_2(x,y)\right).$$

We should note that, the formulae for the moments $\mathbb{M}_n(\varphi_i;x,y),\ i=1,2,3,$ agrees with the corresponding formulae from Lemma 2. As a consequence of Theorem 1, we can state the following corollary.

Corollary 9. *For all $f \in \mathcal{H}_\omega(S_A)$, we have*

$$\lim_{n\to\infty} \|\mathbb{M}_n(f;\cdot,\cdot) - f\|_{C(S_A)} = 0.$$

Following the similar procedures as in the proofs of Theorems 5 and 6, we can state the following corollaries, which gives the order of approximation by the non-tensor bivariate Meyer–König and Zeller operators in terms of modulus of continuity and Lipschitz class functions, respectively.

Corollary 10. *For all $f \in \mathcal{H}_\omega(S_A)$, we have*

$$\|\mathbb{M}_n(f;\cdot,\cdot) - f\|_{C(S_A)} \le 2\omega\left(f, \sqrt{B(A)\left[\frac{9n+4}{n^2}\right]}\right),$$

where $B(A) = \sup_{(x,y)\in S_A}\{\varphi_1(x,y), \varphi_2(x,y), \varphi_3(x,y)\}$.

Corollary 11. *Let $\alpha \in (0,1]$. For all $f \in Lip_M^*(\alpha)$, we have*

$$\|\mathbb{M}_n(f;\cdot,\cdot) - f\|_{C(S_A)} \le M\left[B(A)\left[\frac{9n+4}{n^2}\right]\right]^{\alpha/2},$$

where $B(A) = \sup_{(x,y)\in S_A}\{\varphi_1(x,y), \varphi_2(x,y), \varphi_3(x,y)\}$.

6. Multivariate Korovkin Type Theorem

In this section we consider the domain

$$\mathbb{S}_A := \left\{\mathbf{x} = (x_1,\ldots,x_m) \in \mathbb{R}^m : x_i \ge 0\ (i=1,\ldots,m),\ \sum_{i=1}^m x_i \le A < 1\right\},$$

and introduce the space of functions

$$\begin{aligned}\mathcal{H}_\omega(\mathbb{S}_A) : &= \{f \in C(\mathbb{S}_A) : |f(\mathbf{s}) - f(\mathbf{x})| \\ &\le \ \omega\left(f, |(\varphi_1(\mathbf{s}),\ldots,\varphi_m(\mathbf{s})) - (\varphi_1(\mathbf{x}),\ldots,\varphi_m(\mathbf{x}))|\right)\}\end{aligned}$$

where $\mathbf{x} = (x_1,\ldots,x_m),\ \mathbf{s} = (s_1,\ldots,s_m), \varphi_i(\mathbf{s}) = \frac{s_i}{1-|\mathbf{s}|}\ (i=1,\ldots,m),\ |\mathbf{s}| := \sum_{i=1}^m s_i$ and

$$|(\varphi_1(\mathbf{s}),\ldots,\varphi_m(\mathbf{s})) - (\varphi_1(\mathbf{x}),\ldots,\varphi_m(\mathbf{x}))| = \sqrt{\sum_{i=1}^m (\varphi_i(\mathbf{s}) - \varphi_i(\mathbf{x}))^2}.$$

In a similar manner as in Theorem 1, we can give the $m-$dimensional version of our Korovkin type theorem as follows:

Theorem 12. *Let $T_n : \mathcal{H}_\omega(\mathbb{S}_A) \to C(\mathbb{S}_A)$ be a sequence of linear positive operators satisfying*

$$\lim_{n\to\infty} \|T_n(\varphi_i;\cdot,\cdot) - \varphi_i\|_{C(S_A)} = 0, \quad i = 0,1,2,\ldots,m+1$$

where

$$\begin{aligned}\varphi_0(\mathbf{s}) &= 1,\\ \varphi_i(\mathbf{s}) &= \frac{s_i}{1-|\mathbf{s}|}\ (i=1,\ldots,m),\\ \varphi_{m+1}(\mathbf{s}) &= \sum_{i=1}^{m}\left(\frac{s_i}{1-|\mathbf{s}|}\right)^2,\end{aligned}$$

and $\|\cdot\|_{C(\mathbb{S}_A)}$ *denotes the usual supremum norm on* $C(\mathbb{S}_A)$*. Then for all* $f \in \mathcal{H}_\omega(\mathbb{S}_A)$*, we have*

$$\lim_{n\to\infty}\|T_n(f;\cdot,\cdot)-f\|_{C(\mathbb{S}_A)}=0.$$

Now, lets consider the multivariate Meyer–König and Zeller operators (see [19])

$$\mathbb{M}_n(f;\mathbf{x}) = \frac{1}{(1-|\mathbf{x}|)^{n+1}}\sum_{\mathbf{k}=\mathbf{0}}^{\infty}\binom{n+|\mathbf{k}|}{\mathbf{k}}\mathbf{x}^{\mathbf{k}},\quad (n\in\mathbb{N},\mathbf{x}\in\mathbb{S}_A)$$

where $\mathbf{x}=(x_1,\ldots,x_m)$, $\mathbf{k}=(k_1,\ldots,k_m)$, $\mathbf{0}=(0,\ldots,0^{m-th})$, $\mathbf{x}^{\mathbf{k}}=x_1^{k_1}\cdots x_m^{k_m}$, $|\mathbf{x}|=x_1+\cdots+x_m$ and

$$\binom{n+|\mathbf{k}|}{\mathbf{k}} = \frac{(n+k_1+\cdots+k_m)!}{n!k_1!\cdots k_m!}.$$

Theorem 13. *Let* $\mathbf{x}\in\mathbb{S}_A$, $f\in C(\mathbb{S}_A)$*. Then* $\mathbb{M}_n(f;\mathbf{x})$ *satisfy the following functional partial differential equations*

$$x_i\frac{\partial}{\partial x_i}\mathbb{M}_n(f;\mathbf{x}) = -\frac{(n+1)x_i}{1-|\mathbf{x}|}\mathbb{M}_n(f;\mathbf{x}) + n\mathbb{M}_n(\varphi_i f;\mathbf{x}),\ (i=1,\ldots,m),$$

$$\left(\sum_{i=1}^{m}x_i\frac{\partial}{\partial x_i}\right)\mathbb{M}_n(f;\mathbf{x}) = -(n+1)\frac{|\mathbf{x}|}{1-|\mathbf{x}|}\mathbb{M}_n(f;\mathbf{x}) + n\mathbb{M}_n(hf;\mathbf{x}),$$

where $h(\mathbf{s})=\frac{|\mathbf{s}|}{1-|\mathbf{s}|}$.

Using the above theorem, the moments are computed in the following lemma.

Lemma 14. *For the multivariate Meyer–König and Zeller operators, we have*

$$\begin{aligned}\mathbb{M}_n(\varphi_0;\mathbf{x}) &= 1,\\ \mathbb{M}_n(\varphi_i,\mathbf{x}) &= \frac{n+1}{n}\varphi_i(\mathbf{x}),\ (i=1,\ldots,m),\\ \mathbb{M}_n(\varphi_{m+1},\mathbf{x}) &= \frac{(n+1)(n+2)}{n^2}\varphi_{m+1}(\mathbf{x}) + \frac{(n+1)}{n^2}\sum_{i=1}^{m}\varphi_i(\mathbf{x}).\end{aligned}$$

Therefore, as a consequence of Theorem 12 and Lemma 14, we can state the following approximation theorem.

Theorem 15. *For all $f \in \mathcal{H}_\omega(\mathbb{S}_A)$, we have*

$$\lim_{n\to\infty} \|\mathbb{M}_n(f;\cdot,\cdot) - f\|_{C(\mathbb{S}_A)} = 0.$$

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