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Results in Mathematics

Barker–Larman Problem for Convex Polygons in the Hyperbolic Plane

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Abstract. We solve an analogue of the Barker–Larman problem for convex polygons in the hyperbolic plane.

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1. Introduction

This note is motivated by a conjecture proposed by Barker and Larman in the frames of the geometric tomography theory [1,2].

Conjecture. *Let P, Q, $M \subset \mathbb{R}^n$ be convex bodies such that M belongs to the interiors of P and Q. Assume that whenever $H \subset \mathbb{R}^n$ is a hyperplane supporting M, the $(n-1)$-volumes vol$(P \cap H)$ and vol$(Q \cap H)$ are equal. Then P coincides with Q.*

For the general case this problem remains open, the answer is known to be affirmative for some particular cases only. For instance, the Barker–Larman conjecture is shown to be true if P, Q are convex polygons and M is a strongly convex centrally-symmetric body with analytical boundary in $\mathbb{R}^2$, see [3], or if P, Q are convex polyhedra and M is a ball in $\mathbb{R}^n$, $n \geq 3$, see [4].

It is quite natural to ask whether the Barker–Larman conjecture holds true for convex bodies in *non-Euclidean geometries* like hyperbolic or spherical one. Comparing to the Euclidean case, one has to consider geodesic lines and totally geodesic hyperplanes instead of straight lines and hyperplanes, and the volumes of involved sets have to be measured with respect to corresponding non-Euclidean metric structures.

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The theorem bellow affirms the Barker–Larman conjecture for convex polygons in the hyperbolic plane $\mathbb{H}^2$.

Theorem. *Let P, $Q \subset \mathbb{H}^2$ be convex polygons, which contain a disk Ω of radius $t > 0$ in their interiors. Assume the lengths of segments $P \cap \tau$ and $Q \cap \tau$ are equal for every geodesic $\tau \subset \mathbb{H}^2$ tangent to the circle $\Sigma = \partial\Omega$. Then P coincides with Q.*

Our proof is based on ideas developed by Yaskin in [4] for convex polygons in the Euclidean plane. The principal observation, which follows from the proof, is that the Barker–Larman conjecture in the two-dimensional case does not depend on the particular metric structure of $\mathbb{R}^2$ or $\mathbb{H}^2$. Apparently, one can accomplish the proof for a large class of metric spaces realized in terms of $\mathbb{R}^2$ or domains of $\mathbb{R}^2$ equipped with non-Euclidean distance functions possessing generic analyticity properties. We discuss the case of $\mathbb{H}^2$ just as a simple and illustrative example only.

Notice that a more general case of convex polyhedra in $\mathbb{R}^n$, $n \geq 3$, was treated in [4] in a slightly different manner, so it would be interesting to adapt Yaskin's technics in order to prove a multi-dimensional analogue of the theorem for convex bodies in $\mathbb{H}^n$ or in other non-Euclidean metric spaces with dimension $n \geq 3$.

2. Proof

There are various models for the hyperbolic geometry. We will apply the classical Beltrami–Klein model $\mathbb{H}^2 = (D^2, d_{\mathbb{H}})$, where the hyperbolic plane is viewed as the disc $D^2 \subset \mathbb{R}^2$ of radius R with the distance between points x, $y \in D^2$ given by

$$d_{\mathbb{H}}(x, y) = \frac{R}{2} \ln \left(\frac{bx}{ax} \cdot \frac{ay}{by} \right), \tag{1}$$

here $a, b \in \partial D^2$ are the endpoints of the chord in D^2 passing through x and y, they are arranged so that the direction from a to b coincides with the direction from x to y; besides, $\frac{bx}{ax}$ and $\frac{ya}{ya}$ denote quotients of directed Euclidean segments in D^2. The constant R determines the curvature $-1/R^2$ of the hyperbolic plane.

Geodesics of the hyperbolic plane are represented by chords of D^2, hence convex polygons in the hyperbolic plane are represented by convex polygons in D^2. Besides, if the center of a circle in the hyperbolic plane corresponds to the center O of the disk D^2, then this circle is represented by a circle in D^2, and the hyperbolic and Euclidean radii are related by the formula $r_{\mathbb{E}} = \tanh \frac{r_{\mathbb{H}}}{R}$.

Applying the Beltrami–Klein model, we will treat P and Q as convex polygons inside the disk D^2. By assumption, the interiors of P and Q have a non-empty intersection, which contains a circle Σ of radius $r_{\mathbb{H}} = t$. Without

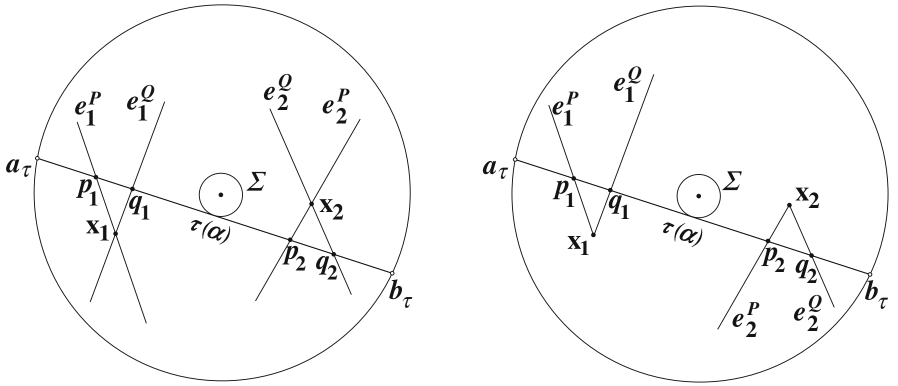

FIGURE 1. Circle Σ and tangent chord $\tau(\alpha)$, segments e_j^P of ∂P, segments e_j^Q of ∂Q, intersection points p_j and q_j

loss of generality, we assume that the center of Σ corresponds to the center O of D^2, this may be always provided by using an isometry of $\mathbb{H}^2$. Hence Σ is represented by a Euclidean circle in D^2 centered at O with Euclidean radius $\varepsilon = \tanh\frac{t}{R}$.

Draw an arbitrary chord τ in D^2 tangent to Σ. Clearly, τ intersects P and Q along some segments p_1p_2 and q_1q_2 respectively. By assumption, the lengths of segments calculated with the help of (1) are equal,

$$d_{\mathbb{H}}(p_1, p_2) = d_{\mathbb{H}}(q_1, q_2). \tag{2}$$

Let us demonstrate, that if (2) holds true for every chord τ tangent to the circle Σ, then P coincides with Q.

Suppose P does not coincide with Q. Because of (2), neither of these polygons can be a subset of the other, so their boundaries ∂P, ∂Q have a non-empty intersection. Moreover, there exists a point $\mathbf{x}_1 \in \partial P \cap \partial Q$, where ∂P meets transversally ∂Q, i.e. a segment of ∂P meets transversally a segment of ∂Q at $\mathbf{x}_1$. Denote these line segments by $e_1^P \subset \partial P$ and $e_1^Q \subset \partial Q$. The point x_1 may be either an intrinsic point of both segments (see Fig. 1, left), or their common endpoint (see Fig. 1, right), the latter happens if x_1 is a vertex for ∂P or for ∂Q.

Parameterize the circle Σ by a natural angle parameter α. The chord in D^2 tangent to Σ at α will be denoted by $\tau(\alpha)$.

There exists a chord $\tau(\alpha_0)$ passing through $\mathbf{x}_1$. Clearly, this chord intersects ∂P and ∂Q at the point $\mathbf{x}_1$ and at some other point $\mathbf{x}_2$, which is also an intersection point for ∂P and ∂Q due to (2). Denote corresponding segments of ∂P and ∂Q, which contain $\mathbf{x}_2$, by e_2^P and e_2^Q respectively.

It is easy to see, that there exist a sufficiently small neighborhood U of α_0 in Σ (or a semi-neighborhood, if $\mathbf{x}_1$ or $\mathbf{x}_2$ is a vertex for P or Q) such that every chord $\tau(\alpha)$, $\alpha \in U$, intersects the segments e_1^P, e_1^Q and e_2^P, e_2^Q, the intersection points are still denoted by p_1, q_1 and p_2, q_2 respectively, see Fig.1.

By assumption, the points p_j, q_j satisfy (2) whenever $\alpha \in U$. In particular, it immediately follows from (2) that e_2^P and e_2^Q are transversal at $\mathbf{x}_2$, since e_1^P and e_1^Q are transversal at $\mathbf{x}_1$.

In order to apply (2) with the help of (1), consider the endpoints of $\tau(\alpha)$ denoted by a_τ and b_τ. Then (2) reads as follows:

$$\frac{R}{2}\ln\left(\frac{b_\tau p_1}{a_\tau p_1}\cdot\frac{a_\tau p_2}{b_\tau p_2}\right) = \frac{R}{2}\ln\left(\frac{b_\tau q_1}{a_\tau q_1}\cdot\frac{a_\tau q_2}{b_\tau q_2}\right), \tag{3}$$

and hence we get

$$\frac{b_\tau p_1}{a_\tau p_1}\cdot\frac{a_\tau p_2}{b_\tau p_2}\cdot\frac{a_\tau q_1}{b_\tau q_1}\cdot\frac{b_\tau q_2}{a_\tau q_2} = 1. \tag{4}$$

By assumption, (4) holds true for all $\alpha \in U$. Notice that both (3) and (4) remains true if one interchanges a_τ and b_τ, so one can ignore how the endpoints a_τ and b_τ in $\tau(\alpha)$ are arranged with respect to p_j and q_j.

Let us rewrite (4) in Cartesian coordinates. Denote by $\hat{e}_j^P$, $\hat{e}_j^Q$, $\hat{\tau}(\alpha)$ the straight lines in $\mathbb{R}^2$ containing the segments e_j^P, e_j^Q, $\tau(\alpha)$ respectively. Besides, denote by $\boldsymbol{\xi}_j$, $\boldsymbol{\eta}_j$, $\boldsymbol{\tau}$ unit direction vectors of $\hat{e}_j^P$, $\hat{e}_j^Q$, $\hat{\tau}(\alpha)$ respectively. The vector $\boldsymbol{\tau}$ being the unit speed vector of Σ with respect to the parametrization α, rearrange the endpoints a_τ, b_τ of $\tau(\alpha)$ so that $\boldsymbol{\tau}$ is directed from a_τ to b_τ.

The straight line $\hat{e}_j^P$ is represented by $\mathbf{r} = \mathbf{x}_j + t_j\boldsymbol{\xi}_j$, where t_j stands for an arc length of $\hat{e}_j^P$. Similarly, the straight line $\hat{e}_j^Q$ is represented by $\mathbf{r} = \mathbf{x}_j + s_j\boldsymbol{\eta}_j$, where s_j denotes an arc length of $\hat{e}_j^Q$. Finally, the straight line $\hat{\tau}(\alpha)$ is represented by $\mathbf{r} = \varepsilon\boldsymbol{\tau}^\perp + \sigma\boldsymbol{\tau}$, where σ stands for the arc length of $\hat{\tau}(\alpha)$ measured from the point $\alpha \in \Sigma$ in the direction of $\boldsymbol{\tau}$, and $\boldsymbol{\tau}^\perp$ denotes a unit vector orthogonal to $\boldsymbol{\tau}$.

For the intersection points p_j and q_j we have:

$$\mathbf{p}_j = \mathbf{x}_j + t_j(p_j)\boldsymbol{\xi}_j = \varepsilon\boldsymbol{\tau}^\perp + \sigma(p_j)\boldsymbol{\tau}, \tag{5}$$

$$\mathbf{q}_j = \mathbf{x}_j + s_j(q_j)\boldsymbol{\eta}_j = \varepsilon\boldsymbol{\tau}^\perp + \sigma(q_j)\boldsymbol{\tau}. \tag{6}$$

From (5)–(6) we get:

$$\sigma(p_j) = \frac{\varepsilon\langle\boldsymbol{\xi}_j,\boldsymbol{\tau}\rangle + \langle\mathbf{x}_j,\boldsymbol{\tau}\rangle\langle\boldsymbol{\xi}_j,\boldsymbol{\tau}^\perp\rangle - \langle\mathbf{x}_j,\boldsymbol{\tau}^\perp\rangle\langle\boldsymbol{\xi}_j,\boldsymbol{\tau}\rangle}{\langle\boldsymbol{\xi}_j,\boldsymbol{\tau}^\perp\rangle} = \frac{\varepsilon\langle\boldsymbol{\xi}_j,\boldsymbol{\tau}\rangle + \left[\boldsymbol{\xi}_j,\mathbf{x}_j\right]}{\langle\boldsymbol{\xi}_j,\boldsymbol{\tau}^\perp\rangle}, \tag{7}$$

$$\sigma(q_j) = \frac{\varepsilon\langle\boldsymbol{\eta}_j,\boldsymbol{\tau}\rangle + \langle\mathbf{x}_j,\boldsymbol{\tau}\rangle\langle\boldsymbol{\eta}_j,\boldsymbol{\tau}^\perp\rangle - \langle\mathbf{x}_j,\boldsymbol{\tau}^\perp\rangle\langle\boldsymbol{\eta}_j,\boldsymbol{\tau}\rangle}{\langle\boldsymbol{\eta}_j,\boldsymbol{\tau}^\perp\rangle} = \frac{\varepsilon\langle\boldsymbol{\eta}_j,\boldsymbol{\tau}\rangle + \left[\boldsymbol{\eta}_j,\mathbf{x}_j\right]}{\langle\boldsymbol{\eta}_j,\boldsymbol{\tau}^\perp\rangle}, \tag{8}$$

where $\langle , \rangle$ and $[,]$ stand for the scalar and vector products respectively; it is assumed that the pair $\boldsymbol{\tau}^{\perp}$, $\boldsymbol{\tau}$ is positively oriented, so $[\boldsymbol{\tau}^{\perp}, \boldsymbol{\tau}] = 1$. Moreover, $\sigma(a_\tau) = -\rho$, $\sigma(b_\tau) = \rho$, where $\rho = \sqrt{R^2 - \varepsilon^2}$.

Consequently, we obtain:

$$a_\tau p_j = \sigma(p_j) + \rho, \quad b_\tau p_j = \sigma(p_j) - \rho, \tag{9}$$

$$a_\tau q_j = \sigma(q_j) + \rho, \quad b_\tau q_j = \sigma(q_j) - \rho. \tag{10}$$

Substituting (9)–(10) and (7)–(8) into (4), we get:

$$\frac{\rho - \frac{\varepsilon\langle \boldsymbol{\xi}_1, \boldsymbol{\tau}\rangle + [\boldsymbol{\xi}_1, \mathbf{x}_1]}{\langle \boldsymbol{\xi}_1, \boldsymbol{\tau}^{\perp}\rangle}}{\rho + \frac{\varepsilon\langle \boldsymbol{\xi}_1, \boldsymbol{\tau}\rangle + [\boldsymbol{\xi}_1, \mathbf{x}_1]}{\langle \boldsymbol{\xi}_1, \boldsymbol{\tau}^{\perp}\rangle}} \cdot \frac{\rho + \frac{\varepsilon\langle \boldsymbol{\xi}_2, \boldsymbol{\tau}\rangle + [\boldsymbol{\xi}_2, \mathbf{x}_2]}{\langle \boldsymbol{\xi}_2, \boldsymbol{\tau}^{\perp}\rangle}}{\rho - \frac{\varepsilon\langle \boldsymbol{\xi}_2, \boldsymbol{\tau}\rangle + [\boldsymbol{\xi}_2, \mathbf{x}_2]}{\langle \boldsymbol{\xi}_2, \boldsymbol{\tau}^{\perp}\rangle}} \cdot \frac{\rho + \frac{\varepsilon\langle \boldsymbol{\eta}_1, \boldsymbol{\tau}\rangle + [\boldsymbol{\eta}_1, \mathbf{x}_1]}{\langle \boldsymbol{\eta}_1, \boldsymbol{\tau}^{\perp}\rangle}}{\rho - \frac{\varepsilon\langle \boldsymbol{\eta}_1, \boldsymbol{\tau}\rangle + [\boldsymbol{\eta}_1, \mathbf{x}_1]}{\langle \boldsymbol{\eta}_1, \boldsymbol{\tau}^{\perp}\rangle}} \cdot \frac{\rho - \frac{\varepsilon\langle \boldsymbol{\eta}_2, \boldsymbol{\tau}\rangle + [\boldsymbol{\eta}_2, \mathbf{x}_2]}{\langle \boldsymbol{\eta}_2, \boldsymbol{\tau}^{\perp}\rangle}}{\rho + \frac{\varepsilon\langle \boldsymbol{\eta}_2, \boldsymbol{\tau}\rangle + [\boldsymbol{\eta}_2, \mathbf{x}_2]}{\langle \boldsymbol{\eta}_2, \boldsymbol{\tau}^{\perp}\rangle}} = 1.$$

Finally, this leads to the following equality:

$$\begin{aligned}
&\left(\rho\langle \boldsymbol{\xi}_1, \boldsymbol{\tau}^{\perp}\rangle - \varepsilon\langle \boldsymbol{\xi}_1, \boldsymbol{\tau}\rangle - [\boldsymbol{\xi}_1, \mathbf{x}_1]\right) \left(\rho\langle \boldsymbol{\xi}_2, \boldsymbol{\tau}^{\perp}\rangle + \varepsilon\langle \boldsymbol{\xi}_2, \boldsymbol{\tau}\rangle + [\boldsymbol{\xi}_2, \mathbf{x}_2]\right)\\
&\quad \cdot \left(\rho\langle \boldsymbol{\eta}_1, \boldsymbol{\tau}^{\perp}\rangle + \varepsilon\langle \boldsymbol{\eta}_1, \boldsymbol{\tau}\rangle + [\boldsymbol{\eta}_1, \mathbf{x}_1]\right) \left(\rho\langle \boldsymbol{\eta}_2, \boldsymbol{\tau}^{\perp}\rangle - \varepsilon\langle \boldsymbol{\eta}_2, \boldsymbol{\tau}\rangle - [\boldsymbol{\eta}_2, \mathbf{x}_2]\right)\\
&= \left(\rho\langle \boldsymbol{\xi}_1, \boldsymbol{\tau}^{\perp}\rangle + \varepsilon\langle \boldsymbol{\xi}_1, \boldsymbol{\tau}\rangle + [\boldsymbol{\xi}_1, \mathbf{x}_1]\right) \left(\rho\langle \boldsymbol{\xi}_2, \boldsymbol{\tau}^{\perp}\rangle - \varepsilon\langle \boldsymbol{\xi}_2, \boldsymbol{\tau}\rangle - [\boldsymbol{\xi}_2, \mathbf{x}_2]\right)\\
&\quad \cdot \left(\rho\langle \boldsymbol{\eta}_1, \boldsymbol{\tau}^{\perp}\rangle - \varepsilon\langle \boldsymbol{\eta}_1, \boldsymbol{\tau}\rangle - [\boldsymbol{\eta}_1, \mathbf{x}_1]\right) \left(\rho\langle \boldsymbol{\eta}_2, \boldsymbol{\tau}^{\perp}\rangle + \varepsilon\langle \boldsymbol{\eta}_2, \boldsymbol{\tau}\rangle + [\boldsymbol{\eta}_2, \mathbf{x}_2]\right).
\end{aligned} \tag{11}$$

This equality holds true for all $\alpha \in U$. Moreover, if one introduces Cartesian coordinates in $\mathbb{R}^2$ with the origin O so that $\boldsymbol{\tau}^{\perp} = (\cos\alpha, \sin\alpha)$, $\boldsymbol{\tau} = (-\sin\alpha, \cos\alpha)$, then (11) will be rewritten as a trigonometric identity with respect to α, its coefficients depend both on coordinates of $\mathbf{x}_j$ and on coordinates of $\boldsymbol{\xi}_j, \boldsymbol{\eta}_j$. *Since this trigonometric identity holds true for all α in an open $U \subset \Sigma$, it holds true for all $\alpha \in \Sigma$.*

Now, consider a pair of straight lines in $\mathbb{R}^2$ passing through $\mathbf{x}_1$ and tangent to Σ. These straight lines don't coincide, since $t > 0$. One of them is just $\hat{\tau}(\alpha_0)$, which passes through $\mathbf{x}_1$ and $\mathbf{x}_2$. The other one is some $\hat{\tau}(\alpha_0^*)$, which passes through $\mathbf{x}_1$ and does not pass through $\mathbf{x}_2$. For this tangent straight line, i.e. for $\alpha = \alpha_0^*$, the equality (11) reduces to the following:

$$\begin{aligned}
&\left(\rho\langle \boldsymbol{\xi}_2, \boldsymbol{\tau}^{\perp}\rangle + \varepsilon\langle \boldsymbol{\xi}_2, \boldsymbol{\tau}\rangle + [\mathbf{x}_2, \boldsymbol{\xi}_2]\right) \cdot \left(\rho\langle \boldsymbol{\eta}_2, \boldsymbol{\tau}^{\perp}\rangle - \varepsilon\langle \boldsymbol{\eta}_2, \boldsymbol{\tau}\rangle - [\mathbf{x}_2, \boldsymbol{\eta}_2]\right)\\
&= \left(\rho\langle \boldsymbol{\xi}_2, \boldsymbol{\tau}^{\perp}\rangle - \varepsilon\langle \boldsymbol{\xi}_2, \boldsymbol{\tau}\rangle - [\mathbf{x}_2, \boldsymbol{\xi}_2]\right) \cdot \left(\rho\langle \boldsymbol{\eta}_2, \boldsymbol{\tau}^{\perp}\rangle + \varepsilon\langle \boldsymbol{\eta}_2, \boldsymbol{\tau}\rangle + [\mathbf{x}_2, \boldsymbol{\eta}_2]\right).
\end{aligned} \tag{12}$$

Simplifying (12), we get:

$$\rho\,[\boldsymbol{\xi}_2, \boldsymbol{\eta}_2]\left(\varepsilon - \langle \mathbf{x}_2, \boldsymbol{\tau}^{\perp}\rangle\right) = 0. \tag{13}$$

On the other hand, ρ does not vanish because $\varepsilon < R$. Next, $[\boldsymbol{\xi}_2, \boldsymbol{\eta}_2]$ does not vanish because $\boldsymbol{\xi}_2$, $\boldsymbol{\eta}_2$ are transversal by assumption. Finally, $\varepsilon - \langle \mathbf{x}_2, \boldsymbol{\tau}^{\perp}\rangle$ does not vanish because $\mathbf{x}_2$ does not belong to $\hat{\tau}(\alpha_0^*)$. Thus, the equality (13) does not hold true. This contradiction shows that our initial assumption, $P \neq Q$, is false, q.e.d.

3. Remarks

Remark 1. Evidently, the proved theorem remains true, if one replaces Ω with an arbitrary convex domain in $D^2 \subset \mathbb{R}^2$ bounded by a regular curve with an analytical support function $h : S^1 \to \mathbb{R}^1$. All that we need to modify is to replace ε and ρ in (11) by h and by $\sqrt{R^2 - h^2}$ respectively. The analyticity of $h(\alpha)$ allows us to maintain the key point of the proof: if (11) holds for $\alpha \in U$, then it holds for $\alpha \in S^1$.

Remark 2. A similar statement can be proved for convex polygons in the spherical geometry, when instead of $\mathbb{R}^2$ and $\mathbb{H}^2$ we consider a half-sphere $\mathbb{S}^2_+$ of radius R.

Namely, *let P, Q be convex polygons in $\mathbb{S}^2_+$, which contain in their interiors a disk Ω of radius $t > 0$ centered at the pole of $\mathbb{S}^2_+$. Assume the lengths of segments $P \cap \tau$ and $Q \cap \tau$ are equal for every geodesic $\tau \subset \mathbb{S}^2_+$ tangent to the circle $\Sigma = \partial\Omega$. Then P coincides with Q.*

Here by segments we mean geodesics (arcs of great circles) in $\mathbb{S}^2_+$, their lengths are measured with respect to the intrinsic distance function in $\mathbb{S}^2_+$.

In order to prove this statement, one can apply to $\mathbb{S}^2_+ \subset \mathbb{R}^3$ the projection map Π from the center of sphere to the plane $\mathbb{R}^2$ tangent to $\mathbb{S}^2_+$ at its pole. Under such one-to-one projection, points of $\mathbb{S}^2_+$ correspond to points of $\mathbb{R}^2$, geodesic curves in $\mathbb{S}^2_+$ correspond to straight lines in $\mathbb{R}^2$, convex polygons in $\mathbb{S}^2_+$ are represented by convex polygons in $\mathbb{R}^2$, the disc Ω in $\mathbb{S}^2_+$ is represented by some disc in $\mathbb{R}^2$. So, we have to prove Theorem not for convex polygons in $\mathbb{S}^2_+$ but for convex polygons in $\mathbb{R}^2$, the only essential difference is that the distance between points x, y in $\mathbb{R}^2$ must be defined as the distance between pre-image points $\Pi^{-1}(x)$, $\Pi^{-1}(y)$ in $\mathbb{S}^2_+$. So, instead of the classical Euclidean distance $d_{\mathbb{E}}$ or the hyperbolic distance $d_{\mathbb{H}}$ given by (1), we have to consider in $\mathbb{R}^2$ the distance function given by

$$d_{\mathbb{S}}(x, y) = R \arccos \frac{\langle \mathbf{x}, \mathbf{y} \rangle + R^2}{\sqrt{|\mathbf{x}|^2 + R^2}\sqrt{|\mathbf{y}|^2 + R^2}}. \tag{14}$$

The method of proof remains the same as in the hyperbolic case.

Remark 3. It would be natural to prove our theorem for the case of convex polygons in $\mathbb{R}^2$ equipped with a distance function different from $d_{\mathbb{E}}$, $d_{\mathbb{H}}$, $d_{\mathbb{S}}$. The question is to what extent the class of distance functions, for which the Barker–Larman conjecture remains true, can be extended.

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