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Results in Mathematics

Asymptotic Formulas and Inequalities for the Gamma Function in Terms of the Tri-Gamma Function

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Abstract. In the paper, the authors establish some asymptotic formulas and double inequalities for the factorial $n!$ and the gamma function Γ in terms of the tri-gamma function ψ'.

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1. Introduction

We recall that the classical Euler's gamma function may be defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}\, t \tag{1}$$

for $\Re(z) > 0$, that the logarithmic derivative of $\Gamma(x)$ is called the psi or digamma function and denoted by

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}\, x} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{2}$$

for $x > 0$, that the derivatives $\psi'(x)$ and $\psi''(x)$ for $x > 0$ are respectively called the tri-gamma and tetra-gamma functions, and that the derivatives $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ and $x > 0$ are called the polygamma functions.

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We also recall from [3, Chapter XIII] and [17, Chapter IV] that a function $f(x)$ is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies $0 \le (-1)^n f^{(n)}(x) < \infty$ for $x \in I$ and all integers $n \ge 0$. The class of completely monotonic functions may be characterized by the celebrated Bernstein-Widder Theorem [17, p. 160, Theorem 12a] which reads that a necessary and sufficient condition that $f(x)$ should be completely monotonic in $0 \le x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} \,\mathrm{d}\,\alpha(t), \tag{3}$$

where $\alpha(t)$ is bounded and non-decreasing and the integral converges for $0 \le x < \infty$.

In [16, Theorem 2.1], it was proved that the function

$$F_\alpha(x) = \ln\Gamma(x+1) - x\ln x + x - \frac{1}{2}\ln x - \frac{1}{2}\ln(2\pi) - \frac{1}{12}\psi'(x+\alpha) \tag{4}$$

is completely monotonic on $(0,\infty)$ if and only if $\alpha \ge \frac{1}{2}$ and that the function $-F_\alpha(x)$ is completely monotonic on $(0,\infty)$ if and only if $\alpha = 0$. Consequently, the double inequality

$$\frac{x^x}{e^x}\sqrt{2\pi x}\,\exp\left(\frac{1}{12}\psi'\left(x+\frac{1}{2}\right)\right) < \Gamma(x+1) < \frac{x^x}{e^x}\sqrt{2\pi x}\,\exp\left(\frac{1}{12}\psi'(x)\right) \tag{5}$$

was derived in [16, Corollary 2.1]. These results were also established in the preprint [10] independently from a different origin and by a different motivation. For some more information on bounding the gamma function Γ, please refer to the newly published papers [4–8, 14], the survey articles [11–13], and plenty of references collected therein.

The goal of this paper is to discover best asymptotic formulas and double inequalities for the factorial $n! = \Gamma(n+1)$ and the gamma function $\Gamma(x)$ in terms of the tri-gamma function $\psi'\left(x+\frac{1}{2}\right)$. These results have something to do with the function $F_\alpha(x)$ and the double inequality (5).

2. An Asymptotic Formula and a Double Inequality for $n!$

In this section, we establish a best asymptotic formula and a double inequality for the factorial $n! = \Gamma(n+1)$ in terms of the tri-gamma function $\psi'\left(x+\frac{1}{2}\right)$.

Theorem 1. *As $n \to \infty$, the asymptotic formula*

$$n! \sim \frac{n^n}{e^n}\sqrt{2\pi n}\,\exp\left(\frac{1}{12}\psi'\left(n+\frac{1}{2}\right)\right) \tag{6}$$

is the most accurate one among all approximations of the form

$$n! \sim \frac{n^n}{e^n}\sqrt{2\pi n}\,\exp\left(\frac{1}{12}\psi'(n+a)\right), \tag{7}$$

where $a \in \mathbb{R}$.

Proof. For $n \ge 1$, define a sequence w_n by

$$n! = \Gamma(n+1) = \sqrt{2\pi}\, n^{n+1/2} e^{-n} \exp\left(\frac{1}{12}\psi'(n+a)\right) \exp w_n.$$

Taking into account

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + (-1)^k \frac{k!}{z^{k+1}} \tag{8}$$

for $k = 1$, see [1, p. 260, 6.4.6], yields

$$w_{n+1} - w_n = 1 + \ln(n+1) - \left(n + \frac{3}{2}\right) \ln(n+1) + \left(n + \frac{1}{2}\right) \ln n + \frac{1}{12(n+a)^2}$$

and

$$w_{n+1} - w_n = \left(-\frac{1}{6}a + \frac{1}{12}\right)\frac{1}{n^3} + \left(\frac{1}{4}a^2 - \frac{3}{40}\right)\frac{1}{n^4} + O\left(\frac{1}{n^5}\right).$$

Hence, we have

$$\lim_{n\to\infty} \left\{n^3 \left[w_{n+1} - w_n\right]\right\} = \frac{1}{12} - \frac{1}{6}a.$$

Lemma 1.1 in [4,15] states that if the sequence $\{\omega_n : n \in \mathbb{N}\}$ converges to 0 and

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = \ell \in \mathbb{R} \tag{9}$$

for $k > 1$, then

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{\ell}{k-1}. \tag{10}$$

Consequently, the sequence w_n converges fastest only if $a = \frac{1}{2}$. □

Theorem 2. *For every integer $n \ge 1$, we have*

$$\exp\left(\frac{1}{240n^3} - \frac{11}{6720n^5}\right) < \frac{e^n n!}{n^n\sqrt{2\pi n}\exp\left(\frac{1}{12}\psi'\left(n+\frac{1}{2}\right)\right)} < \exp\frac{1}{240n^3}. \tag{11}$$

Proof. The double inequality (11) may be rewritten as

$$f(n) = \ln\Gamma(n+1) - \left(n+\frac{1}{2}\right)\ln n + n - \frac{1}{2}\ln(2\pi) - \frac{1}{12}\psi'\left(n+\frac{1}{2}\right) - \frac{1}{240n^3} \le 0 \tag{12}$$

and

$$g(n) = \ln\Gamma(n+1) - \left(n+\frac{1}{2}\right)\ln n + n - \frac{1}{2}\ln(2\pi) - \frac{1}{12}\psi'\left(n+\frac{1}{2}\right) - \frac{1}{240n^3} + \frac{11}{6720n^5} \ge 0. \tag{13}$$

Employing the recurrence formula (8) applied to $k = 1$ and straightforwardly computing reveal that $f(n+1) - f(n) = u(n)$ and $g(n+1) - g(n) = v(n)$, where

$$u(x) = 1 + \ln(x+1) - \left(x + \frac{3}{2}\right)\ln(x+1) + \left(x + \frac{1}{2}\right)\ln x + \frac{1}{12\left(x+\frac{1}{2}\right)^2} - \frac{1}{240(x+1)^3} + \frac{1}{240x^3}$$

and

$$v(x) = 1 + \ln(x+1) - \left(x + \frac{3}{2}\right)\ln(x+1) + \left(x + \frac{1}{2}\right)\ln x + \frac{1}{12\left(x+\frac{1}{2}\right)^2} - \frac{1}{240(x+1)^3} + \frac{1}{240x^3} + \frac{11}{6720(x+1)^5} - \frac{11}{6720x^5}.$$

It is not difficult to verify that

$$u''(x) = \frac{13x + 74x^2 + 232x^3 + 391x^4 + 330x^5 + 110x^6 + 1}{20x^5(x+1)^5(2x+1)^4} > 0$$

and

$$v''(x) = -\frac{Q(x)}{1120x^7(x+1)^7(2x+1)^4} < 0,$$

where

$$Q(x) = 825x + 5499x^2 + 21325x^3 + 52589x^4 + 83867x^5 + 83881x^6 + 47936x^7 + 11984x^8 + 55.$$

This shows that $u(x)$ is strictly convex and $v(x)$ is strictly concave on $(0, \infty)$. Further considering $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} v(x) = 0$, we obtain that $u(x) > 0$ and $v(x) < 0$ on $(0, \infty)$. Consequently, the sequence $f(n)$ is strictly increasing and $g(n)$ is strictly decreasing while they both converge to 0. As a result, we conclude that $f(n) < 0$ and $g(n) > 0$ for every integer $n \geq 1$. The proof of Theorem 2 is complete. □

3. An Asymptotic Series and a Double Inequality for Γ

We now discover an asymptotic series and a double inequality for the gamma function $\Gamma(x)$ in terms of the tri-gamma function $\psi'\left(x + \frac{1}{2}\right)$.

Theorem 3. *As $x \to \infty$, we have*

$$\Gamma(x+1) \sim \sqrt{2\pi}\, x^{x+1/2} \exp\left(\frac{1}{12}\psi'\left(x + \frac{1}{2}\right) - x + \frac{1}{240}\frac{1}{x^3} - \frac{11}{6720}\frac{1}{x^5} + \frac{107}{80640}\frac{1}{x^7} - \frac{2911}{1520640}\frac{1}{x^9} + \cdots\right). \quad (14)$$

Proof. Motivated by the inequality (11), we now consider a new function $h(x)$ defined by

$$\Gamma(x+1) = \sqrt{2\pi}\, x^{x+1/2} e^{-x} \exp\left(\frac{1}{12}\psi'\left(x+\frac{1}{2}\right)\right) \exp h(x),$$

that is,

$$h(x) = \left[\ln\Gamma(x+1) - \left(x+\frac{1}{2}\right)\ln x + x - \ln\sqrt{2\pi}\right] - \frac{1}{12}\psi'\left(x+\frac{1}{2}\right).$$

Using the formulas

$$\ln\Gamma(x+1) - \left(x+\frac{1}{2}\right)\ln x + x - \ln\sqrt{2\pi} = \sum_{m=1}^{\infty}\frac{B_{2m}}{2m(2m-1)x^{2m-1}}$$

and

$$\psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \sum_{m=1}^{\infty}\frac{B_{2m}}{x^{2m+1}} = \sum_{m=1}^{\infty}\frac{B_{m-1}}{x^m},$$

see [1, p. 257, 6.1.40] and [1, p. 260, 6.4.11], figures out

$$h(x) = \sum_{m=1}^{\infty}\frac{B_{2m}}{2m(2m-1)x^{2m-1}} - \sum_{m=1}^{\infty}\frac{B_{m-1}}{12\left(x+\frac{1}{2}\right)^m}, \tag{15}$$

where B_k for $k \geq 0$ denote the Bernoulli numbers which may be generated by

$$\frac{z}{e^z-1} = \sum_{k=0}^{\infty} B_k\frac{z^k}{k!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi.$$

Making use of

$$\begin{aligned}\sum_{k=1}^{m}\frac{a_k}{\left(x+\frac{1}{2}\right)^k} &= \sum_{k=1}^{m} a_k\left(1+\frac{1}{2x}\right)^{-k}\frac{1}{x^k}\\ &= \sum_{k=1}^{m} a_k\left[\sum_{i=0}^{\infty}\binom{-k}{i}\frac{1}{2^i x^i}\right]\frac{1}{x^k} = \sum_{k=1}^{m}\sum_{i=0}^{\infty}\frac{a_k}{2^i}\binom{-k}{i}\frac{1}{x^{k+i}}\end{aligned}$$

in (15), where a_k is any sequence and

$$\binom{-k}{i} = \frac{1}{i!}\prod_{\ell=0}^{i-1}(-k-\ell),$$

we obtain that

$$h(x) = \frac{1}{240}\frac{1}{x^3} - \frac{11}{6720}\frac{1}{x^5} + \frac{107}{80640}\frac{1}{x^7} - \frac{2911}{1520640}\frac{1}{x^9} + O\left(\frac{1}{x^{11}}\right).$$

The proof of Theorem 3 is complete. □

By truncation of the series (14), under- and upper- approximations can be obtained. The method for proving this fact is illustrated in the next theorem. For sake of simplicity, we choose to prove (11).

Theorem 4. *Inequality (11) holds true, for every real number $n \geq 1$.*

Proof. Let $f(x)$ and $g(x)$ for $x \in [1, \infty)$ be defined by (12) and (13) respectively. Making use of inequalities

$$\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} < \ln\Gamma(x+1) - \left(x + \frac{1}{2}\right)\ln x + x - \frac{1}{2}\ln(2\pi) < \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5}$$

and

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7},$$

which may be deduced from [2, Theorem 2 and Corollary 1], finds that $f(x) < a(x)$ and $g(x) > b(x)$, where

$$\begin{aligned}
a(x) &= \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{12}\left[\frac{1}{x+\frac{1}{2}} + \frac{1}{2\left(x+\frac{1}{2}\right)^2} \right. \\
&\quad \left. + \frac{1}{6\left(x+\frac{1}{2}\right)^3} - \frac{1}{30\left(x+\frac{1}{2}\right)^5}\right] - \frac{1}{240x^3} \\
&= -\frac{A(x-1)}{5040x^5(2x+1)^5} \\
&< 0, \\
b(x) &= \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} - \frac{1}{12}\left[\frac{1}{x+\frac{1}{2}} + \frac{1}{2\left(x+\frac{1}{2}\right)^2} \right. \\
&\quad \left. + \frac{1}{6\left(x+\frac{1}{2}\right)^3} - \frac{1}{30\left(x+\frac{1}{2}\right)^5} + \frac{1}{42\left(x+\frac{1}{2}\right)^7}\right] - \frac{1}{240x^3} + \frac{11}{6720x^5} \\
&= \frac{B(x-1)}{20160x^7(2x+1)^7} \\
&> 0, \\
A(x) &= 3760x + 6565x^2 + 5310x^3 + 1980x^4 + 264x^5 + 785, \\
B(x) &= 93268x + 263179x^2 + 382830x^3 \\
&\quad + 315336x^4 + 147504x^5 + 35952x^6 + 3424x^7 + 12547.
\end{aligned}$$

The proof of Theorem 4 is thus complete. □

Remark 1. This paper is a slightly revised version of the preprint [9].

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