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Results in Mathematics

On the Structure of Submanifolds in Euclidean Space with Flat Normal Bundle

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Abstract. In this paper we study the structure of an immersed submanifold M^n in a Riemannian manifold with flat normal bundle in two ways. Firstly, we prove that if M^n is compact and satisfies some pointwise pinching condition, and assume further that the ambient space has pure curvature tensor and non-negative isotropic curvature, then the Betti numbers $\beta_p(M) = 0$ for $2 \leq p \leq n-2$. Secondly, suppose that M^n is a complete non-compact submanifold in the Euclidean space with finite total curvature in the sense that its traceless second fundament form has finite L^n-norm, then we show that the spaces of L^2 harmonic p-forms on M^n have finite dimensions for all $2 \leq p \leq n-2$.

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1. Introduction

The geometric structure and topological properties of submanifolds in various ambient space have been studied extensively during past few years. In [2], Cao–Shen–Zhu showed that a complete stable minimal immersed hypersurface M^n in $\mathbb{R}^{n+1}$ with $n \geq 3$ must have only one end. The proof of Cao–Shen–Zhu mainly use a Liouville theorem of Schoen–Yau [11] asserting that a complete stable minimal hypersurface of $\mathbb{R}^{n+1}$ can not admit a non-constant harmonic function with finite Dirichlet integral. This result was generalized by Li–Wang

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[10], they showed that if a complete minimal hypersurface M^n in $\mathbb{R}^{n+1}$ has finite index, then the dimension of the space of L^2 harmonic 1-forms on M is finite, and M has finitely many ends. In [13], Yun proved that for a complete oriented minimal hypersurface M^n in $\mathbb{R}^{n+1}$ with $n \geq 3$, if the L^n-norm of its second fundamental form is less than an explicit constant, then there are no nontrivial L^2 harmonic 1-forms on M, and M has only one end. Fu–Xu [6] obtained that if an oriented complete submanifold M^n $(n \geq 3)$ in $\mathbb{R}^{n+k}$ has finite total mean curvature and finite total curvature, then Li–Wang's conclusion still holds. Recently, Cavalcante, Mirandola and Vitório [4] obtained a generalization of the result of Fu–Xu without any additional hypothesis on the mean curvature.

In general, one is interested in understanding relations between the topology and geometry of a Riemanian manifold M and the spaces of harmonic forms. When M is compact, by Hodge theory, the space of harmonic p-forms on M is isomorphic to its p-th de Rham cohomology group. When M is noncompact, the Hodge theory does not work anymore, and it is natural to consider L^2 harmonic forms, as it is showed that L^2-Hodge theory remains valid in noncompact manifolds as classical Hodge theory work well in the compact case.

We denote the space of all L^2 harmonic p-forms on M^n by $H^p(L^2(M))$. These spaces have a (reduced) L^2-cohomology interpretation. For more results concerning L^2 harmonic p-forms on complete non-compact manifolds, one can consult a very nice survey of Carron [3].

In the case of harmonic p-forms of higher order, there is a little more difficulty in computing the Laplacian of their squared norm because of the Weitzenböck curvature operator, which is the zero order term involving curvature tensor. However, if M is a submanifold with flat normal bundle and the ambient space has pure curvature tensor, we can estimate the Weitzenböck curvature operator in terms of the sectional curvature of the ambient space and the second fundamental form of M by the Gauss equation.

Let M^n be a complete submanifold immersed in a Riemannian manifold N^{n+k}. Fix a point $x \in M$ and a local orthonormal frame $\{e_1, \ldots, e_{n+k}\}$ of N^{n+k} such that $\{e_1, \ldots, e_n\}$ are tangent fields of M^n. For each α, $n+1 \leq \alpha \leq n+k$, define a linear map $A_\alpha : T_xM \to T_xM$ by

$$\langle A_\alpha X, Y\rangle = \langle \overline{\nabla}_X Y, e_\alpha\rangle,$$

where X, Y are tangent fields and $\overline{\nabla}$ is the Riemannian connection of N^{n+k}. Denote by $h_{ij}^\alpha = \langle A_\alpha e_i, e_j\rangle$. The squared norm $|A|^2$ of the second fundamental form and the mean curvature vector H are defined respectively by

$$|A|^2 = \sum_\alpha tr(A_\alpha^2) = \sum_\alpha \sum_{i,j} (h_{ij}^\alpha)^2 \qquad \text{and} \qquad H = \sum_\alpha H^\alpha e_\alpha = \frac{1}{n} \sum_\alpha \sum_i h_{ii}^\alpha e_\alpha.$$

For each α, define a linear map $\phi_\alpha : T_xM \to T_xM$ by

$$\langle \phi_\alpha X, Y\rangle = \langle X, Y\rangle H^\alpha - \langle A_\alpha X, Y\rangle,$$

and a bilinear map $\phi : T_xM \times T_xM \to T_xM^\perp$ by

$$\phi(X, Y) = \sum_\alpha \langle \phi_\alpha X, Y\rangle e_\alpha.$$

It is easy to see that the tensor ϕ is traceless and

$$|\phi|^2 := \sum_\alpha tr(\phi_\alpha^2) = |A|^2 - n|H|^2,$$

which measures how much the immersion deviates from being totally umbilical. We say that M^n has finite total curvature if

$$\int_M |\phi|^n dv < +\infty.$$

If N^{n+k} is a nonpositive curved manifolds, Hoffman and Spruck [7] proved the following L^1 Sobolev inequality

$$\left(\int_M g^{\frac{n}{n-1}} dv\right)^{\frac{n-1}{n}} \leq D(n) \int_M (|\nabla g| + n|H|g)\, dv \quad \forall\, g \in C_0^\infty(M) \tag{1.1}$$

where $D(n) > 0$ is an explicit constant depending only on the dimension n.

Recall that a complete Riemannian manifold N^{n+m} has non-negative $(n-1)$-th Ricci curvature if for any $x \in N$ and any n orthonormal vectors $\{e, e_1, \ldots, e_{n-1}\} \subset T_xN$, the curvature tensor satisfies $\sum_{i=1}^{n-1} \langle \bar{R}(e_i, e)e, e_i\rangle \geq 0$. Let M^n be a complete submanifold in N^{n+k} of non-negative $(n-1)$-th Ricci curvature. Then we have the following inequality due to Leung [8]

$$\mathrm{Ric}_M \geq 2(n-1)H^2 - (n-2)\sqrt{\frac{n-1}{n}}|H|\sqrt{|A|^2 - n|H|^2} - \frac{n-1}{n}|A|^2. \tag{1.2}$$

Our main results in this paper are stated as follows.

Theorem 1.1. *Let M^n be a compact immersed submanifold in N^{n+k} which has non-negative $(n-1)$-th Ricci curvature. Assume that*

$$|A|^2 \leq \frac{n^2|H|^2}{n-1}. \tag{1.3}$$

Then every harmonic 1-form on M is parallel. Furthermore, if the inequality (1.3) is strict at some point, then the Betti numbers $\beta_1(M) = \beta_{n-1}(M) = 0$.

Let us recall that a Riemannian manifold N is said to have nonnegative isotropic curvature if

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq 0$$

for every orthonormal 4-frame $\{e_1, e_2, e_3, e_4\}$ of TN. For harmonic forms of higher order, we have the following result.

Theorem 1.2. *Let M^n be a compact immersed submanifold in N^{n+k} with flat normal bundle. Assume that N^{n+k} has pure curvature tensor and non-negative isotropic curvature, and*

$$|A|^2 \leq \frac{n^2|H|^2}{\max\{p, n-p\}} \tag{1.4}$$

for $2 \leq p \leq n-2$. Then every harmonic p-form on M is parallel. Furthermore, if the inequality (1.4) is strict at some point, then the Betti numbers $\beta_p(M) = 0$ for $2 \leq p \leq n-2$.

Finally, we shall generalize Cavalcante, Mirandola and Vitório's [4] finiteness theorem for L^2 harmonic 1-forms to L^2 harmonic p-forms of higher order.

Theorem 1.3. *Let M^n $(n \geq 3)$ be a complete non-compact immersed submanifold in $\mathbb{R}^{n+k}$ with flat normal bundle and finite total curvature. Then $dim H^p(L^2(M)) < \infty$ for all $2 \leq p \leq n-2$.*

2. An Estimate for the Weitzenböck Curvature Operator

Let M^n be an n-dimensional complete submanifold, and let $\triangle$ be the Hodge Laplace-Beltrami operator of M^n acting on the space of differential p-forms. Denote by R_{ijkl} and $\bar{R}_{ijkl}$ the curvature tensors of M^n and N^{n+k}, respectively. Given two p-forms ω and θ on M, we define a pointwise inner product

$$\langle \omega, \theta \rangle = \sum_{i_1, \ldots, i_p = 1}^{n} \omega(e_{i_1}, \ldots, e_{i_p}) \theta(e_{i_1}, \ldots, e_{i_p}).$$

Observe that we are omitting the normalizing factor $1/p!$. The Weitzenböck formula ([12]) gives

$$\triangle = \nabla^2 - \mathcal{R}_p, \tag{2.1}$$

where ∇^2 is the Bochner Laplacian and $\mathcal{R}_p$ is the Weitzenböck curvature operator, which is an endomorphism depending upon the curvature tensor of M^n. Using an orthonormal basis $\{\theta^1, \ldots, \theta^n\}$ dual to $\{e_1, \ldots, e_n\}$, one may express the curvature term $\mathcal{R}_p$ as

$$\langle \mathcal{R}_p(\omega), \omega \rangle = \left\langle \sum_{j,k=1}^{n} \theta^k \wedge i_{e_j} R(e_k, e_j)\omega, \omega \right\rangle$$

for any p-form ω. In particular, if ω is a 1-form and $\omega^\sharp$ denotes the vector field dual to ω, then

$$\langle \mathcal{R}_1(\omega), \omega \rangle = \mathrm{Ric}(\omega^\sharp, \omega^\sharp).$$

By (2.1), we have

$$\frac{1}{2}\triangle|\omega|^2 = |\nabla\omega|^2 + \left\langle \sum_{j,k=1}^{n} \theta^k \wedge i_{e_j} R(e_k, e_j)\omega, \omega \right\rangle$$
$$= |\nabla\omega|^2 + pF(\omega) \tag{2.2}$$

where

$$F(\omega) = R_{ij}\omega^{ii_2\ldots i_p}\omega^j_{i_2\ldots i_p} - \frac{p-1}{2}R_{ijkl}\omega^{iji_3\ldots i_p}\omega^{kl}_{i_3\ldots i_p}.$$

Here, repeated indices are contracted and summed. The Gauss equation implies that

$$R_{ijkl} = \bar{R}_{ijkl} + h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk},$$

and

$$R_{ij} = \sum_{k=1}^{n}\bar{R}_{ikjk} + nH^\alpha h^\alpha_{ij} - h^\alpha_{ik}h^\alpha_{jk}.$$

Thus,

$$F(\omega) = F_1(\omega) + F_2(\omega)$$

where

$$F_1(\omega) = \sum_{k=1}^{n}\sum_{i,j,i_2,\ldots,i_p}\bar{R}_{ikjk}\omega^{ii_2\ldots i_p}\omega^j_{i_2\ldots i_p} - \frac{p-1}{2}\sum \bar{R}_{ijkl}\omega^{iji_3\ldots i_p}\omega^{kl}_{i_3\ldots i_p} \tag{2.3}$$

and

$$\begin{aligned} F_2(\omega) &= (nH^\alpha h^\alpha_{ij} - h^\alpha_{ik}h^\alpha_{jk})\omega^{ii_2\ldots i_p}\omega^j_{i_2\ldots i_p} \\ &\quad - \frac{p-1}{2}(h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk})\omega^{iji_3\ldots i_p}\omega^{kl}_{i_3\ldots i_p} \\ &= nH^\alpha h^\alpha_{ij}\omega^{ii_2\ldots i_p}\omega^j_{i_2\ldots i_p} - h^\alpha_{ik}h^\alpha_{jk}\omega^{ii_2\ldots i_p}\omega^j_{i_2\ldots i_p} \\ &\quad - (p-1)h^\alpha_{ik}h^\alpha_{jl}\omega^{iji_3\ldots i_p}\omega^{kl}_{i_3\ldots i_p}. \end{aligned} \tag{2.4}$$

To estimate $F_1(\omega)$, we need to put some assumption on the curvature tensor of N^{n+k}. Let us recall that a Riemannian manifold N^{n+k} is said to have pure curvature tensor if for every $x \in N$ there is an orthonormal basis $\{e_1, \ldots, e_{n+k}\}$ of the tangent space T_xN such that $\langle \bar{R}(e_i, e_j)e_k, e_l\rangle = 0$ whenever the set $\{i, j, k, l\}$ contains more than two elements. It is obvious that all 3-manifolds and conformally flat manifolds have pure curvature tensor.

We assume that N^{n+k} has pure curvature tensor and the indices $1 \leq i_1, i_2, \ldots, i_n \leq n$ are distinct with each other in the following discussion. Denote by $K_{ij} = \bar{R}_{ijij} = \langle \bar{R}(e_i, e_j)e_j, e_i\rangle$. Then

$$\begin{aligned}
F_1(\omega) =& \sum_{k=1}^{n} \sum_{i,i_2,\ldots,i_p} \bar{R}_{ikik}\omega^{ii_2\ldots i_p}\omega^{i}_{i_2\ldots i_p} \\
&- \frac{p-1}{2}\sum(\bar{R}_{ijij}\omega^{iji_3\ldots i_p}\omega^{ij}_{i_3\ldots i_p} + \bar{R}_{ijji}\omega^{iji_3\ldots i_p}\omega^{ji}_{i_3\ldots i_p}) \\
=& \sum_{k=1}^{n} K_{ik}\omega^{ii_2\ldots i_p}\omega_{ii_2\ldots i_p} - (p-1)\sum K_{ij}\omega^{iji_3\ldots i_p}\omega_{iji_3\ldots i_p} \\
=& \sum_{k=1}^{n} K_{i_1k}\omega^{i_1i_2\ldots i_p}\omega_{i_1i_2\ldots i_p} \\
&- \sum(K_{i_1i_2} + K_{i_1i_3} + \cdots + K_{i_1i_p})\omega^{i_1i_2i_3\ldots i_p}\omega_{i_1i_2i_3\ldots i_p} \\
=& \sum_{h=p+1}^{n} K_{i_1i_h}\omega^{i_1i_2\ldots i_p}\omega_{i_1i_2\ldots i_p} \\
=& \frac{1}{p}\sum_{t=1}^{p}\sum_{h=p+1}^{n} K_{i_ti_h}\omega^{i_1i_2\cdots i_p}\omega_{i_1i_2\ldots i_p} \\
\geq& \frac{1}{p}\left(\inf_{i_1,\ldots,i_n}\sum_{t=1}^{p}\sum_{h=p+1}^{n} K_{i_ti_h}\right)|\omega|^2.
\end{aligned} \tag{2.5}$$

On the other hand, to estimate the three terms at the right hand side of (2.4) in the case $2 \leq p \leq n-2$, we assume that M has flat normal bundle. Work at a point $x \in M$, we can choose an orthonormal frame $\{e_i\}_{i=1}^{n}$ diagonalizing (h^{α}_{ij}) for all α. Hence

$$\begin{aligned}
F_2(\omega) &= nH^{\alpha}h^{\alpha}_{ii}\omega^{ii_2\ldots i_p}\omega^{i}_{i_2\ldots i_p} - h^{\alpha}_{ii}h^{\alpha}_{ii}\omega^{ii_2\ldots i_p}\omega^{i}_{i_2\ldots i_p} - (p-1)h^{\alpha}_{ii}h^{\alpha}_{jj}\omega^{iji_3\ldots i_p}\omega^{ij}_{i_3\ldots i_p} \\
&= nH^{\alpha}h^{\alpha}_{ii}\omega^{ii_2\ldots i_p}\omega^{i}_{i_2\ldots i_p} - h^{\alpha}_{ii}h^{\alpha}_{ii}\omega^{ii_2\ldots i_p}\omega^{i}_{i_2\ldots i_p} + h^{\alpha}_{jj}h^{\alpha}_{jj}\omega^{iji_3\ldots i_p}\omega^{ij}_{i_3\ldots i_p} \\
&\quad - (h^{\alpha}_{ii} + h^{\alpha}_{jj} + h^{\alpha}_{i_3i_3} + \cdots + h^{\alpha}_{i_pi_p})h^{\alpha}_{jj}\omega^{iji_3\ldots i_p}\omega^{ij}_{i_3\ldots i_p} \\
&= \frac{nH^{\alpha}}{p}(h^{\alpha}_{ii} + h^{\alpha}_{i_2i_2} + h^{\alpha}_{i_3i_3} + \cdots + h^{\alpha}_{i_pi_p})\omega^{ii_2\ldots i_p}\omega^{i}_{i_2\ldots i_p} \\
&\quad - \frac{1}{p}(h^{\alpha}_{ii} + h^{\alpha}_{jj} + h^{\alpha}_{i_3i_3} + \cdots + h^{\alpha}_{i_pi_p})^2\omega^{iji_3\ldots i_p}\omega^{ij}_{i_3\ldots i_p} \\
&= \frac{1}{p}[nH^{\alpha}(h^{\alpha}_{i_1i_1} + \cdots + h^{\alpha}_{i_pi_p}) - (h^{\alpha}_{i_1i_1} + \cdots + h^{\alpha}_{i_pi_p})^2]\omega^{i_1i_2\ldots i_p}\omega^{i_1i_2\ldots i_p} \\
&= \frac{1}{p}(h^{\alpha}_{i_1i_1} + \cdots + h^{\alpha}_{i_pi_p})(h^{\alpha}_{i_{p+1}i_{p+1}} + \cdots + h^{\alpha}_{i_ni_n})|\omega^{i_1i_2\ldots i_p}|^2 \\
&\geq \frac{1}{p}\inf_{i_1,\ldots,i_n}\{(h^{\alpha}_{i_1i_1} + \cdots + h^{\alpha}_{i_pi_p})(h^{\alpha}_{i_{p+1}i_{p+1}} + \cdots + h^{\alpha}_{i_ni_n})\}|\omega|^2
\end{aligned} \tag{2.6}$$

for $2 \leq p \leq n-2$. To estimate the term at the right side of (2.6), we have the following lemma.

Lemma 2.1. *Let M^n be a complete immersed submanifold with flat normal bundle. Choose an orthonormal frame such that $h^\alpha_{ij} = \delta_{ij} h^\alpha_{ij}$. Then*

$$\begin{aligned}&(h^\alpha_{i_1 i_1} + \cdots + h^\alpha_{i_p i_p})(h^\alpha_{i_{p+1} i_{p+1}} + \cdots + h^\alpha_{i_n i_n}) \\ &\quad \geq 2p(n-p)|H|^2 - \frac{p(n-p)}{n}|A|^2 - |2p-n|\sqrt{\frac{p(n-p)}{n}}|H|\sqrt{|A|^2 - n|H|^2}.\end{aligned}$$

Proof. Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|A_\alpha|^2 &= \sum_{s=1}^p (h^\alpha_{i_s i_s})^2 + \sum_{t=p+1}^n (h^\alpha_{i_t i_t})^2 \\ &\geq \frac{1}{p}\left(\sum_{s=1}^p h^\alpha_{i_s i_s}\right)^2 + \frac{1}{n-p}\left(\sum_{t=p+1}^n h^\alpha_{i_t i_t}\right)^2 \\ &= \frac{n}{p(n-p)}\left(\sum_{s=1}^p h^\alpha_{i_s i_s}\right)^2 - \frac{2n}{n-p} H^\alpha \sum_{s=1}^p h^\alpha_{i_s i_s} + \frac{n^2}{n-p}|H^\alpha|^2.\end{aligned}$$

Thus,

$$\left(\sum_{s=1}^p h^\alpha_{i_s i_s}\right)^2 - 2pH^\alpha \sum_{s=1}^p h^\alpha_{i_s i_s} + np|H^\alpha|^2 - \frac{p(n-p)}{n}|A_\alpha|^2 \leq 0,$$

which implies that

$$\begin{aligned}pH^\alpha - \sqrt{\frac{p(n-p)}{n}}\sqrt{|A_\alpha|^2 - n|H^\alpha|^2} &\leq \sum_{s=1}^p h^\alpha_{i_s i_s} \\ &\leq pH^\alpha + \sqrt{\frac{p(n-p)}{n}}\sqrt{|A_\alpha|^2 - n|H^\alpha|^2},\end{aligned} \tag{2.7}$$

and also

$$\left(\sum_{s=1}^p h^\alpha_{i_s i_s}\right)^2 - nH^\alpha \sum_{s=1}^p h^\alpha_{i_s i_s} \leq \frac{p(n-p)}{n}|A_\alpha|^2 - np|H^\alpha|^2 + (2p-n)H^\alpha \sum_{s=1}^p h^\alpha_{i_s i_s}. \tag{2.8}$$

Substituting (2.7) into (2.8) yields

$$\begin{aligned}\left(\sum_{s=1}^p h^\alpha_{i_s i_s}\right)^2 - nH^\alpha \sum_{s=1}^p h^\alpha_{i_s i_s} &\leq \frac{p(n-p)}{n}|A_\alpha|^2 - np|H^\alpha|^2 + p(2p-n)|H^\alpha|^2 \\ &\quad + |2p-n|\sqrt{\frac{p(n-p)}{n}}|H^\alpha|\sqrt{|A_\alpha|^2 - n|H^\alpha|^2}.\end{aligned}$$

Hence using the Cauchy–Schwarz inequality again yields

$$
\begin{aligned}
&(h^\alpha_{i_1i_1}+\cdots+h^\alpha_{i_pi_p})(h^\alpha_{i_{p+1}i_{p+1}}+\cdots+h^\alpha_{i_ni_n})\\
&\quad=\sum_\alpha\left[nH^\alpha\sum_{s=1}^p h^\alpha_{i_si_s}-\left(\sum_{s=1}^p h^\alpha_{i_si_s}\right)^2\right]\\
&\quad\geq 2p(n-p)\sum_\alpha|H^\alpha|^2-\frac{p(n-p)}{n}\sum_\alpha|A_\alpha|^2\\
&\qquad-|2p-n|\sqrt{\frac{p(n-p)}{n}}\sum_\alpha|H^\alpha|\sqrt{|A_\alpha|^2-n|H^\alpha|^2}\\
&\quad\geq 2p(n-p)|H|^2-\frac{p(n-p)}{n}|A|^2-|2p-n|\sqrt{\frac{p(n-p)}{n}}|H|\sqrt{|A|^2-n|H|^2}.
\end{aligned}
$$

□

We first assume that $2p\leq n$. From Lemma 2.1, by the Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
&(h^\alpha_{i_1i_1}+\cdots+h^\alpha_{i_pi_p})(h^\alpha_{i_{p+1}i_{p+1}}+\cdots+h^\alpha_{i_ni_n})\\
&\quad\geq 2p(n-p)|H|^2-\frac{p(n-p)}{n}|A|^2-\frac{(n-2p)^2}{4s}|H|^2-\frac{sp(n-p)}{n}(|A|^2-n|H|^2)\\
&\quad=\left[(2+s)p(n-p)-\frac{(n-2p)^2}{4s}\right]|H|^2-\frac{(1+s)p(n-p)}{n}|A|^2\\
&\quad=(1+s)\left[\frac{(2+s)p(n-p)-\frac{(n-2p)^2}{4s}}{1+s}|H|^2-\frac{p(n-p)}{n}|A|^2\right].
\end{aligned}
\tag{2.9}
$$

for all $s>0$. Denote by

$$f(s)=\frac{(2+s)p(n-p)-\frac{(n-2p)^2}{4s}}{1+s}.$$

A straightforward computation shows that

$$\max_{s>0}f(s)=f\left(\frac{n-2p}{2p}\right)=np.$$

Hence from (2.9), we get

$$
\begin{aligned}
&(h^\alpha_{i_1i_1}+\cdots+h^\alpha_{i_pi_p})(h^\alpha_{i_{p+1}i_{p+1}}+\cdots+h^\alpha_{i_ni_n})\\
&\quad\geq \left(1+\frac{n-2p}{2p}\right)\left[np|H|^2-\frac{p(n-p)}{n}|A|^2\right]\\
&\quad=\frac{1}{2}\left[n^2|H|^2-(n-p)|A|^2\right].
\end{aligned}
\tag{2.10}
$$

If $2p \geq n$, then $2(n-p) \leq n$. Thus replacing p by $n-p$ in (2.9) and (2.10), we have

$$(h^\alpha_{i_1i_1} + \cdots + h^\alpha_{i_pi_p})(h^\alpha_{i_{p+1}i_{p+1}} + \cdots + h^\alpha_{i_ni_n}) \geq \frac{1}{2}(n^2|H|^2 - p|A|^2). \tag{2.11}$$

Combining (2.10) and (2.11), we have

$$(h^\alpha_{i_1i_1} + \cdots + h^\alpha_{i_pi_p})(h^\alpha_{i_{p+1}i_{p+1}} + \cdots + h^\alpha_{i_ni_n}) \geq \frac{1}{2}\left(n^2|H|^2 - \max\{p, n-p\}|A|^2\right) \tag{2.12}$$

for all $2 \leq p \leq n-2$. Combining (2.2), (2.5), (2.6) and (2.12), we conclude that

$$\begin{aligned}\frac{1}{2}\triangle|\omega|^2 \geq |\nabla\omega|^2 &+ \left(\inf_{i_1,\cdots,i_n}\sum_{t=1}^{p}\sum_{h=p+1}^{n} K_{i_ti_h}\right)|\omega|^2\\ &+ \frac{1}{2}\left(n^2|H|^2 - \max\{p, n-p\}|A|^2\right)|\omega|^2.\end{aligned} \tag{2.13}$$

3. Proof of the Main Results

Theorem 3.1. *Let M^n be a compact immersed submanifold in N^{n+k} which has non-negative $(n-1)$-th Ricci curvature. Assume that*

$$|A|^2 \leq \frac{n^2H^2}{n-1}. \tag{3.1}$$

Then every harmonic 1-form on M is parallel. Furthermore, if the inequality (3.1) is strict at some point, then the Betti numbers $\beta_1(M) = \beta_{n-1}(M) = 0$.

Proof. Given $\omega \in H^1(L^2(M))$, it follows from (1.2) and (3.1) that

$$\begin{aligned}F(\omega) &= \mathrm{Ric}(\omega^\sharp, \omega^\sharp)\\ &\geq \left[2(n-1)H^2 - (n-2)\sqrt{\frac{n-1}{n}}|H|\sqrt{|A|^2 - nH^2} - \frac{n-1}{n}|A|^2\right]|\omega|^2\\ &\geq \left[2(n-1) - (n-2)\sqrt{\frac{n-1}{n}}\sqrt{\frac{n^2}{n-1} - n} - \frac{n-1}{n}\frac{n^2}{n-1}\right]H^2|\omega|^2\\ &= 0.\end{aligned}$$

Combining with (2.2), we conclude that

$$\frac{1}{2}\triangle|\omega|^2 = |\nabla\omega|^2 + F(\omega) \geq 0.$$

By the compactness of M and the maximum principle, $|\omega| =$ constant. Hence $|\nabla\omega| = 0 = F(\omega)$. The first equality means that ω is parallel. If (3.1) is strict at some point $x_0 \in M$, then the Ricci curvature of M^n is positive at x_0. Hence $F(\omega) = \mathrm{Ric}(\omega^\sharp, \omega^\sharp) = 0$ implies that $\omega^\sharp(x_0) = 0$, which is equivalent to

$\omega(x_0) = 0$. Since ω is parallel, $\omega = 0$ on M. Hence $\beta_1(M) = 0$. By duality, $\beta_{n-1}(M) = \beta_1(M) = 0$. ☐

Theorem 3.2. *Let M^n be a compact immersed submanifold in N^{n+k} with flat normal bundle. Assume that N^{n+k} has pure curvature tensor and non-negative isotropic curvature, and*

$$|A|^2 \leq \frac{n^2 H^2}{\max\{p, n-p\}} \tag{3.2}$$

for $2 \leq p \leq n-2$. Then every harmonic p-form on M is parallel. Furthermore, if the inequality (3.2) is strict at some point, then the Betti numbers $\beta_p(M) = 0$ for $2 \leq p \leq n-2$.

Proof. Let $\omega \in H^p(L^2(M))$ for $2 \leq p \leq n-2$. Since N^{n+k} has non-negative isotropic curvature, by Lemma 2.2 of [5], we have

$$\inf_{i_1,\ldots,i_n} \sum_{t=1}^{p} \sum_{h=p+1}^{n} K_{i_t i_h} \geq 0, \tag{3.3}$$

which, combining with (2.13) and the hypothesis, implies that

$$\frac{1}{2}\triangle|\omega|^2 \geq |\nabla\omega|^2 + \frac{1}{2}\left(n^2|H|^2 - \max\{p, n-p\}|A|^2\right)|\omega|^2 \geq 0. \tag{3.4}$$

The remaining argument is similar to the proof of Theorem 3.1. ☐

Remark 3.1. It is well known that non-negative sectional curvature implies non-negative isotropic curvature.

To obtain finiteness theorems for L^2 harmonic p-forms under L_n-norm curvature of M^n, we need the following L^2 Sobolev inequality. Putting $g = f^{\frac{2(n-1)}{n-2}}$ with $f \in C_0^1(M)$ in (1.1), using the Hölder inequality yields

$$\begin{aligned}\left(\int_M |f|^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} &\leq \frac{2(n-1)D(n)}{n-2}\int_M f^{\frac{n}{n-2}}|\nabla f| dv + nD(n)\int_M |H| f^{\frac{2(n-1)}{n-2}} dv \\ &\leq \frac{2(n-1)D(n)}{n-2}\left(\int_M |f|^{\frac{2n}{n-2}} dv\right)^{\frac{1}{2}}\left(\int_M |\nabla f|^2 dv\right)^{\frac{1}{2}} \\ &\quad + nD(n)\left(\int_M |f|^{\frac{2n}{n-2}} dv\right)^{\frac{1}{2}}\left(\int_M |H|^2 f^2 dv\right)^{\frac{1}{2}},\end{aligned}$$

which implies the following L^2 Sobolev inequality

$$\left(\int_M |f|^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} \leq c(n)\int_M (|\nabla f|^2 + |H|^2 f^2) dv \quad \forall f \in C_0^1(M) \tag{3.5}$$

for some $c(n) > 0$ depending only on the dimension n.

Theorem 3.3. *Let M^n $(n \geq 3)$ be a complete non-compact immersed submanifold of $\mathbb{R}^{n+k}$ with flat normal bundle and finite total curvature. Then $dim H^p(L^2(M)) < \infty$ for all $2 \leq p \leq n-2$.*

Proof. Let $\omega \in H^p(L^2(M))$ for $2 \le p \le n-2$. Since ω satisfies the refined Kato's inequality ([1])

$$|\nabla\omega|^2 \ge (1+K_p)|\nabla|\omega||^2,$$

where

$$K_p = \begin{cases} \frac{1}{n-p} & \text{if } 2 \le p \le n/2, \\ \frac{1}{p} & \text{if } n/2 \le p \le n-2, \end{cases}$$

we get from (2.13) that

$$\frac{1}{2}\triangle|\omega|^2 \ge (1+K_p)|\nabla|\omega||^2 + \frac{1}{2}\left(n^2|H|^2 - \max\{p, n-p\}|A|^2\right)|\omega|^2. \tag{3.6}$$

On the other hand, we have

$$\frac{1}{2}\triangle|\omega|^2 = |\omega|\triangle|\omega| + |\nabla|\omega||^2,$$

which, combining with (3.6), implies that

$$\begin{aligned} |\omega|\triangle|\omega| &\ge K_p|\nabla|\omega||^2 + \frac{1}{2}\left(n^2|H|^2 - \max\{p, n-p\}|A|^2\right)|\omega|^2 \\ &= K_p|\nabla|\omega||^2 - \frac{n}{2}(|A|^2 - n|H|^2)|\omega|^2 + \frac{1}{2}(n - \max\{p, n-p\})|A|^2|\omega|^2 \\ &= K_p|\nabla|\omega||^2 - \frac{n}{2}|\phi|^2|\omega|^2 + \frac{1}{2}\min\{p, n-p\}|A|^2|\omega|^2. \end{aligned} \tag{3.7}$$

Fix a point $x_0 \in M$ and denote by $\rho(x)$ the geodesic distance on M from x_0 to x. Let us choose $\eta \in C_0^\infty(M)$ satisfying

$$\eta(x) = \begin{cases} 0 & \text{on } B_{x_0}(r_0) \cup (M \backslash B_{x_0}(2r)), \\ \rho(x, x_0) - r_0 & \text{on } B_{x_0}(r_0+1) \backslash B_{x_0}(r_0), \\ 1 & \text{on } B_{x_0}(r) \backslash B_{x_0}(r_0+1), \\ \frac{2r - \rho(x,x_0)}{r} & \text{on } B_{x_0}(2r) \backslash B_{x_0}(r) \end{cases} \tag{3.8}$$

where $r > r_0 + 1$ and r_0 will be determined later. Multiplying (3.7) by η^2 and integrating by parts over M, and using the Cauchy–Schwarz inequality, we get

$$\begin{aligned} 0 \le& \int_M (\eta^2|\omega|\triangle|\omega| - K_p\eta^2|\nabla|\omega||^2)dv + \frac{n}{2}\int_M |\phi|^2\eta^2|\omega|^2 dv \\ &-\frac{1}{2}\min\{p, n-p\}\int_M |A|^2\eta^2|\omega|^2 dv \\ =& -2\int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|\rangle dv - (1+K_p)\int_M \eta^2|\nabla|\omega||^2 dv \\ &+\frac{n}{2}\int_M |\phi|^2\eta^2|\omega|^2 dv - \frac{1}{2}\min\{p, n-p\}\int_M |A|^2\eta^2|\omega|^2 dv \end{aligned} \tag{3.9}$$

$$\leq (b-1-K_p)\int_M \eta^2|\nabla|\omega||^2dv + \frac{1}{b}\int_M |\omega|^2|\nabla\eta|^2dv$$
$$+\frac{n}{2}\int_M |\phi|^2\eta^2|\omega|^2dv - \frac{1}{2}\min\{p,n-p\}\int_M |A|^2\eta^2|\omega|^2dv \tag{3.10}$$

for all $b>0$. On the other hand, it follows from (3.5) and the Hölder inequality that

$$\begin{aligned}\int_M |\phi|^2\eta^2|\omega|^2dv &\leq \left(\int_{\mathrm{supp}(\eta)}|\phi|^n dv\right)^{\frac{2}{n}}\left(\int_M |\eta|\omega||^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}}\\ &\leq c(n)\left(\int_{\mathrm{supp}(\eta)}|\phi|^n dv\right)^{\frac{2}{n}}\int_M \left(|\nabla(\eta|\omega|)|^2+|H|^2\eta^2|\omega|^2\right)dv\\ &= E(\eta)\int_M(\eta^2|\nabla|\omega||^2+|\omega|^2|\nabla\eta|^2+|H|^2\eta^2|\omega|^2)dv\\ &\quad + 2E(\eta)\int_M \eta|\omega|\langle\nabla\eta,\nabla|\omega|\rangle dv \end{aligned}\tag{3.11}$$

where $E(\eta)=c(n)(\int_{\mathrm{supp}(\eta)}|\phi|^n dv)^{\frac{2}{n}}$. Substituting (3.11) into (3.10) and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}0 &\leq \left(b-1-K_p+\frac{nE(\eta)}{2}\right)\int_M \eta^2|\nabla|\omega||^2dv + \left(\frac{1}{b}+\frac{nE(\eta)}{2}\right)\int_M |\omega|^2|\nabla\eta|^2dv\\ &\quad + nE(\eta)\int_M \eta|\omega|\langle\nabla\eta,\nabla|\omega|\rangle dv + \frac{nE(\eta)}{2}\int_M |H|^2\eta^2|\omega|^2dv\\ &\quad - \frac{1}{2}\min\{p,n-p\}\int_M |A|^2\eta^2|\omega|^2dv\\ &\leq \left(b-1-K_p+\frac{nE(\eta)}{2}+\frac{nE(\eta)\gamma}{2}\right)\int_M \eta^2|\nabla|\omega||^2dv\\ &\quad + \left(\frac{1}{b}+\frac{nE(\eta)}{2}+\frac{nE(\eta)}{2\gamma}\right)\int_M |\omega|^2|\nabla\eta|^2dv + \frac{nE(\eta)}{2}\int_M |H|^2\eta^2|\omega|^2dv\\ &\quad - \frac{1}{2}\min\{p,n-p\}\int_M |A|^2\eta^2|\omega|^2dv\end{aligned}$$

for any $\gamma>0$. Thus,

$$\begin{aligned}C\int_M \eta^2|\nabla|\omega||^2dv &\leq D\int_M |\omega|^2|\nabla\eta|^2dv + \frac{nE(\eta)}{2}\int_M |H|^2\eta^2|\omega|^2dv\\ &\quad - \frac{1}{2}\min\{p,n-p\}\int_M |A|^2\eta^2|\omega|^2dv\end{aligned}\tag{3.12}$$

where

$$C = 1 + K_p - b - \frac{(1+\gamma)nE(\eta)}{2},$$
$$D = \frac{(1+\gamma)nE(\eta)}{2\gamma} + \frac{1}{b}.$$

Since M has finite total curvature, we can fix r_1 large enough such that

$$\left(\int_{M\backslash B_{x_0}(r_1)} |\phi|^n dv\right)^{\frac{2}{n}} < \frac{K_p}{nc(n)}.$$

Take $r_0 \geq r_1$, thus $\operatorname{supp}(\eta) \subset M\backslash B_{x_0}(r_1)$, and $E(\eta) < \frac{K_p}{n}$. Let $\gamma = b = 1$, then $C = K_p - nE(\eta) > 0$ and $D = nE(\eta) + 1 > 0$. It follows from (3.5) and the Cauchy-Schwarz inequality that

$$\begin{aligned}\left(\int_M |\eta|\omega||^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} &\leq c(n)\int_M (|\nabla(\eta|\omega|)|^2 + |H|^2\eta^2|\omega|^2)dv \\ &\leq c(n)(1+s)\int_M \eta^2|\nabla|\omega||^2 dv + c(n)\left(1+\frac{1}{s}\right) \\ &\quad\times \int_M |\omega|^2|\nabla\eta|^2 dv + c(n)\int_M |H|^2\eta^2|\omega|^2 dv \end{aligned}\tag{3.13}$$

for all $s > 0$. Substituting (3.12) into (3.13) yields

$$\begin{aligned}\left(\int_M |\eta|\omega||^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} &\leq \left[c(n)(1+s)C^{-1}D + c(n)\left(1+\frac{1}{s}\right)\right]\int_M |\omega|^2|\nabla\eta|^2 dv \\ &\quad+ \left[\frac{n}{2}E(\eta)c(n)(1+s)C^{-1} + c(n)\right]\int_M |H|^2\eta^2|\omega|^2 dv \\ &\quad- \frac{1}{2}\min\{p, n-p\}c(n)(1+s)C^{-1}\int_M |A|^2\eta^2|\omega|^2 dv. \end{aligned}\tag{3.14}$$

By a direct computation, we have

$$\begin{aligned}&\left[\frac{n}{2}E(\eta)c(n)(1+s)C^{-1} + c(n)\right]|H|^2 - \frac{1}{2}\min\{p, n-p\}c(n)(1+s)C^{-1}|A|^2 \\ &\leq c(n)\left[\frac{1}{2}E(\eta)(1+s)C^{-1} + \frac{1}{n} - \frac{1}{2}\min\{p, n-p\}(1+s)C^{-1}\right]|A|^2 \\ &\leq c(n)\left[\frac{1}{2}(E(\eta)-1)(1+s)C^{-1} + \frac{1}{n}\right]|A|^2 \\ &\leq c(n)\left[\frac{1}{2}(\frac{K_p}{n} - 1)K_p^{-1} + \frac{1}{n}\right]|A|^2 \\ &\leq 0.\end{aligned}$$

Hence (3.14) reduces to

$$\left(\int_M |\eta|\omega||^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}} \leq C_1\int_M |\omega|^2|\nabla\eta|^2 dv \tag{3.15}$$

for some constant $C_1 = C_1(n) > 0$.

It follow from (3.8) and (3.15) that

$$\left(\int_{B_{x_0}(r)\setminus B_{x_0}(r_0+1)} |\omega|^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}} \leq C_1\int_{B_{x_0}(r_0+1)\setminus B_{x_0}(r_0)} |\omega|^2 dv + \frac{C_1}{r^2}\int_{B_{x_0}(2r)\setminus B_{x_0}(r)} |\omega|^2 dv.$$

Since $|\omega| \in L^2(M)$, taking $r \to \infty$, we have

$$\left(\int_{M\setminus B_{x_0}(r_0+1)} |\omega|^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}} \leq C_1\int_{B_{x_0}(r_0+1)\setminus B_{x_0}(r_0)} |\omega|^2 dv. \tag{3.16}$$

It follows from the Hölder inequality that

$$\int_{B_{x_0}(r_0+2)\setminus B_{x_0}(r_0+1)} |\omega|^2 dv \leq \mathrm{vol}(B_{x_0}(r_0+2))^{\frac{2}{n}}\left(\int_{B_{x_0}(r_0+2)\setminus B_{x_0}(r_0+1)} |\omega|^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}}.$$

Combining with (3.16), we conclude that there exists a constant $C_2 > 0$ depending on $\mathrm{vol}(B_{x_0}(r_0+2))$ such that

$$\int_{B_{x_0}(r_0+2)} |\omega|^2 dv \leq C_2\int_{B_{x_0}(r_0+1)} |\omega|^2 dv. \tag{3.17}$$

From (3.7), we get

$$|\omega|\triangle|\omega| \geq K_p|\nabla|\omega||^2 - F|\omega|^2 \tag{3.18}$$

where $F : M \to [0,\infty)$ is the function given by

$$F = \frac{n}{2}|\phi|^2 - \frac{1}{2}\min\{p, n-p\}|A|^2.$$

Fix $x \in M$ and take $\varphi \in C_0^1(B_x(1))$. Multiplying (3.18) by $\varphi^2|\omega|^{q-2}$, $q \geq 2$, and integrating by parts, we obtain

$$\begin{aligned}
&K_p\int_{B_x(1)} \varphi^2|\omega|^{q-2}|\nabla|\omega||^2 dv - \int_{B_x(1)} F\varphi^2|\omega|^q dv\\
&\quad\leq -(q-1)\int_{B_x(1)} \varphi^2|\omega|^{q-2}|\nabla|\omega||^2 dv - 2\int_{B_x(1)} \langle|\omega|^{\frac{q}{2}}\nabla\varphi, |\omega|^{\frac{q}{2}-1}\varphi|\nabla|\omega||\rangle dv\\
&\quad\leq (1-q+K_p)\int_{B_x(1)} \varphi^2|\omega|^{q-2}|\nabla|\omega||^2 dv + \frac{1}{K_p}\int_{B_x(1)} |\omega|^q|\nabla\varphi|^2 dv
\end{aligned}$$

which implies that

$$(q-1)\int_{B_x(1)} \varphi^2|\omega|^{q-2}|\nabla|\omega||^2 dv \le \int_{B_x(1)} [F\varphi^2 + \frac{1}{K_p}|\nabla\varphi|^2]|\omega|^q dv. \quad (3.19)$$

On the other hand, using the Cauchy-Schwarz inequality, we have

$$\int_{B_x(1)} |\nabla(\varphi|\omega|^{\frac{q}{2}})|^2 dv \le \left(1 + \frac{q}{2\epsilon}\right)\int_{B_x(1)} |\omega|^q|\nabla\varphi|^2 dv + \frac{q}{2}\left(\frac{q}{2}+\epsilon\right)\int_{B_x(1)} \varphi^2|\omega|^{q-2}|\nabla|\omega||^2 dv$$

for any $\epsilon > 0$. Combining with (3.19) and taking $\epsilon = \frac{1}{2}$, we conclude that

$$\int_{B_x(1)} |\nabla(\varphi|\omega|^{\frac{q}{2}})|^2 dv \le \int_{B_x(1)} \left\{\frac{q(1+q)}{4(q-1)}F\varphi^2 + \left[1+q+\frac{q(1+q)}{4(q-1)K_p}\right]|\nabla\varphi|^2\right\}|\omega|^q dv. \quad (3.20)$$

Applying (3.5) to $\varphi|\omega|^{\frac{q}{2}}$ and using (3.20), we have

$$\left(\int_{B_x(1)} (\varphi|\omega|^{\frac{q}{2}})^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} \le c(n)\int_{B_x(1)} |\nabla(\varphi|\omega|^{\frac{q}{2}})|^2 dv + c(n)\int_{B_x(1)} |H|^2(\varphi|\omega|^{\frac{q}{2}})^2 dv \le \int_{B_x(1)} (\mathcal{A}\varphi^2 + \mathcal{B}|\nabla\varphi|^2)|\omega|^q dv$$

where

$$\mathcal{A} = \frac{q(q+1)c(n)}{4(q-1)}F + c(n)|H|^2 \le qc(n)(F+|H|^2)$$

and

$$\mathcal{B} = c(n)\left[q+1+\frac{q(q+1)}{4(q-1)K_p}\right] \le qnc(n).$$

Thus,

$$\left(\int_{B_x(1)} (\varphi|\omega|^{\frac{q}{2}})^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} \le qC_3\int_{B_x(1)} (\varphi^2 + |\nabla\varphi|^2)|\omega|^q dv \quad (3.21)$$

for a constant $C_3 > 0$ depending only on n, $\mathrm{vol}(B_x(1))$, $\sup_{B_x(1)}F$ and $\sup_{B_x(1)}|H|^2$.

Given an integer $k \ge 0$, we set $q_k = \frac{2n^k}{(n-2)^k}$ and $r_k = \frac{1}{2} + \frac{1}{2^{k+1}}$. Take a function $\varphi_k \in C_0^\infty(B_x(r_k))$ satisfying $\varphi_k \ge 0$, $\varphi_k = 1$ on $B_x(r_{k+1})$ and $|\nabla\varphi| \le 2^{k+3}$. Replacing q and φ in (3.21) by q_k and φ_k respectively, we get

$$\left(\int_{B_x(r_{k+1})}|\omega|^{q_{k+1}}dv\right)^{\frac{1}{q_{k+1}}} \le (q_k C_3 4^{k+4})^{\frac{1}{q_k}}\left(\int_{B_x(r_k)}|\omega|^{q_k}dv\right)^{\frac{1}{q_k}}. \quad (3.22)$$

Applying the Moser iteration to (3.22), we conclude that

$$|\omega|^2(x) \le \|\omega\|^2_{L^\infty(B_x(\frac{1}{2}))} \le C_4\int_{B_x(1)}|\omega|^2dv \quad (3.23)$$

for a constant $C_4 > 0$ depending only on n, $\mathrm{vol}(B_x(1))$, $\sup_{B_x(1)}F$ and $\sup_{B_x(1)}|H|^2$. Take $x \in B_{x_0}(r_0+1)$ such that

$$|\omega|^2(x) = \sup_{B_{x_0}(r_0+1)}|\omega|^2,$$

then (3.23) implies that

$$\sup_{B_{x_0}(r_0+1)}|\omega|^2 \le C_4\int_{B_{x_0}(r_0+2)}|\omega|^2dv.$$

Combining with (3.17), we have

$$\sup_{B_{x_0}(r_0+1)}|\omega|^2 \le C_5\int_{B_{x_0}(r_0+1)}|\omega|^2dv \quad (3.24)$$

where $C_5 > 0$ depends only on n, $\mathrm{vol}(B_{x_0}(r_0+2))$, $\sup_{B_{x_0}(r_0+2)}F$ and $\sup_{B_{x_0}(r_0+2)}|H|^2$.

Finally, let $\mathcal{V}$ be any finite dimensional subspace of $H^p(L^2(M))$. According to Lemma 11 of [9], there exists an $\omega \in \mathcal{V}$ such that

$$\dim\mathcal{V}\int_{B_{x_0}(r_0+1)}|\omega|^2dv \le \mathrm{vol}(B_{x_0}(r_0+1))(\min\{n,\dim\mathcal{V}\})\sup_{B_{x_0}(r_0+1)}|\omega|^2.$$

Combining with (3.24) we conclude that

$$\dim\mathcal{V} \le C_6$$

where $C_6 > 0$ depends only on n, $\mathrm{vol}(B_{x_0}(r_0+2))$, $\sup_{B_{x_0}(r_0+2)}F$ and $\sup_{B_{x_0}(r_0+2)}|H|^2$. This implies that $H^p(L^2(M))$ has finite dimension, which complete the proof of Theorem 3.3. □

Remark 3.2. Cavalcante, Mirandola and Vitório's [4] proved a finiteness theorem for L^2 harmonic 1-forms on complete submanifolds with finite total curvature in Euclidean space.

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