Results. Math. 68 (2015), 203–225

1422-6383/15/010203-23
published online January 1, 2015
DOI 10.1007/s00025-014-0430-2

Results in Mathematics

About Extensions of Generalized Apostol-Type Polynomials

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Abstract. Under a slight modification on the parameters associated to the generalized Apostol-type polynomials and the use of the generating method, we obtain some new results concerning extensions of generalized Apostol-type polynomials. We state some algebraic and differential properties for a new class of extensions of generalized Apostol-type polynomials, as well as, some others identities which connect this polynomial class with the Stirling numbers of second kind, the Jacobi polynomials, the generalized Bernoulli polynomials, the Genocchi polynomials and the Apostol–Euler polynomials, respectively.

Mathematics Subject Classification. 11C08, 11M35, 11B65, 05A10, 33B99.

Keywords. Generalized Apostol-type polynomials, Apostol–Bernoulli polynomials of higher order, Apostol–Euler polynomials of higher order, Apostol–Genocchi polynomials of higher order, stirling numbers of second kind.

1. Introduction

The generalized Apostol-type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ in the variable x, parameters $\lambda,\mu,\nu \in \mathbb{C}$ and order $\alpha \in \mathbb{C}$ [21, 27] are defined by means of the generating function

$$\left(\frac{2^\mu z^\nu}{\lambda e^z+1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)\frac{z^n}{n!}, \tag{1}$$

Partially supported by the Research Grant Program 2009–2014 from Universidad del Atlántico-Colombia.

where $|z| < 2\pi$ when $\lambda = 1$, $|z| < \pi$ when $\lambda = -1$, $|z| < |log(-\lambda)|$ when $\lambda \in \mathbb{C}\backslash\{-1, 1\}$ and $1^\alpha := 1$.

This class of polynomials has been introduced recently in [27] and it provides a unification of another three families of polynomials, namely, Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials of order α, and hence, it also generalizes to the classical classes of Apostol, Bernoulli, Euler and Genocchi polynomials (cf. [1,2,4–6,9,14–28,34–36,39–44] and the references therein).

The seminal idea underlaying both earlier as recent studies about the generalized Apostol-type polynomials has been to make appropriate modifications to the generating functions associated to the classical classes of Apostol, Bernoulli, Euler and Genocchi polynomials, respectively, and get similar algebraic and/or differential properties as these polynomials (see, for instance [5,6,9,15–19,22–28,31,35–37,39–44]).

The interested reader may find recent literature which contains a large number of new and interesting properties involving these polynomials, for instance, four Special Issues entitled 'Proceedings of the International Congress' in Honour of Professor Hari M. Srivastava and published by the following SpringerOpen journals: Boundary Value Problems, Fixed Point Theory and Applications and Journal of Inequalities and Applications contain a broad information about new properties, applications in combinatorics, number theory, numerical analysis and new research trends about this class of polynomials and several of its subclasses (cf. [9,13–15,19,21,22,32]).

An apart mention deserves the remarkable work [18] since it states a unification for generalized Apostol-type polynomials based on the definition of the Apostol–Bernoulli and Apostol–Euler polynomials of order α [34,35] and the use of a generating function linked to the Mittag–Leffler function [7–9,28]

$$E_{1,m+1}(z) := \frac{z^m}{e^z - \sum_{l=0}^{m-1} \frac{z^l}{l!}}, \quad m \in \mathbb{N}. \tag{2}$$

More precisely, the author in [18] introduces three new classes of generalized Apostol–Bernoulli polynomials, Apostol–Euler polynomials and Apostol–Genocchi polynomials with parameters $a, c \in \mathbb{R}^+$ by means of the following generating functions, defined in a suitable neighborhood of $z = 0$.

$$\left(\frac{z^m}{\lambda c^z - \sum_{l=0}^{m-1} \frac{(z \ln a)^l}{l!}}\right)^\alpha c^{xz} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{[m-1,\alpha]}(x; c, a; \lambda)\frac{z^n}{n!}, \tag{3}$$

$$\left(\frac{2^m}{\lambda c^z + \sum_{l=0}^{m-1} \frac{(z \ln a)^l}{l!}}\right)^\alpha c^{xz} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{[m-1,\alpha]}(x; c, a; \lambda)\frac{z^n}{n!}, \tag{4}$$

$$\left(\frac{(2z)^m}{\lambda c^z + \sum_{l=0}^{m-1} \frac{(z \ln a)^l}{l!}}\right)^\alpha c^{xz} = \sum_{n=0}^{\infty} \mathfrak{G}_n^{[m-1,\alpha]}(x; c, a; \lambda)\frac{z^n}{n!}. \tag{5}$$

In the sequel, we call the above polynomials generalized Apostol–Bernoulli polynomials, Apostol–Euler polynomials and Apostol–Genocchi polynomials in the variable x, parameters $\lambda \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level $m \in \mathbb{N}$, respectively. Furthermore, it is clear that the polynomials defined by (3)–(5) belong to some subclass of Apostol-type polynomials (1) when $m = 1$ and $c = \exp(1)$ [27], and, they belong to the class of Appell polynomials, for example, when $\alpha = 1$ and $c = \exp(1)$ [28].

In this paper we focus our attention on a light perturbation of the Mittag-Leffler function (2) by adding of five new parameters in order to define a new unification (which is different from aforementioned ones) of the generalized Apostol-type polynomials. So, we can prove that such new polynomial class preserves some similar algebraic and differential properties as the generalized Apostol-type polynomials and as an immediate consequence we recover many known algebraic and differential properties of such polynomials.

Our methods are analytic and based mainly on the recent developments done in [9,18,21,27,30]. However, the same scheme have been previously exploited by Srivastava, Todorov, Natalini, Bernardini, Bretti, Luo, Tremblay, Gaboury, Fugère, Pintér, Boyadzhiev, He, Wang, Kurt, Simsek, Liu, Ozden, Özarslan, Garg, Choudhary, Choi and their collaborators (see for instance, [8,9,13,15–17,20,22–26,28]).

The paper is organized as follows. Section 2 contains the basic background about Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials in the variable x, parameters $\lambda \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level $m \in \mathbb{N}$, and some other auxiliary results which will be used throughout the paper. In Sect. 3, we define our unified presentation of the generalized Apostol-type polynomials and prove some relevant algebraic and differential properties of them, as well as, their relation with the Stirling numbers of second kind. Finally, the main results of Sect. 4, gathered in Theorems 2, 3, 4, and 5, show the corresponding relations between our unified presentation of the generalized Apostol-type polynomials and Jacobi polynomials, generalized Bernoulli polynomials, Genocchi polynomials and Apostol–Euler polynomials, respectively.

2. Background and Previous Results

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ the sets of natural, nonnegative integer, real, positive real and complex numbers, respectively. For $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}_0$, $(\lambda)_k$ denotes the rising factorial, i.e.,

$$(\lambda)_0 = 1 \quad \text{and} \quad (\lambda)_k = \lambda(\lambda+1)\cdots(\lambda+k-1).$$

Also, as usual, the numbers given by

$$\begin{aligned}
\mathfrak{B}_n^{[m-1,\alpha]}(c, a; \lambda) &:= \mathfrak{B}_n^{[m-1,\alpha]}(0; c, a; \lambda),\\
\mathfrak{E}_n^{[m-1,\alpha]}(c, a; \lambda) &:= \mathfrak{E}_n^{[m-1,\alpha]}(0; c, a; \lambda),\\
\mathfrak{G}_n^{[m-1,\alpha]}(c, a; \lambda) &:= \mathfrak{G}_n^{[m-1,\alpha]}(0; c, a; \lambda),
\end{aligned}$$

denote the corresponding Apostol–Bernoulli numbers, Apostol–Euler numbers and Apostol–Genocchi numbers with parameters $\lambda \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level $m \in \mathbb{N}$.

The following proposition summarizes some elementary properties of the generalized Apostol–Bernoulli polynomials, generalized Apostol–Euler polynomials and generalized Apostol–Genocchi polynomials with parameters λ, a, c, order α and level m, (cf. [18]).

Proposition 1. *For a fixed $m \in \mathbb{N}$, let $\{\mathfrak{B}_n^{[m-1,\alpha]}(x; c, a; \lambda)\}_{n\geq 0}$, $\{\mathfrak{E}_n^{[m-1,\alpha]}(x; c, a; \lambda)\}_{n\geq 0}$ and $\{\mathfrak{G}_n^{[m-1,\alpha]}(x; c, a; \lambda)\}_{n\geq 0}$ be the sequences of generalized Apostol–Bernoulli polynomials, generalized Apostol–Euler polynomials and generalized Apostol–Genocchi polynomials in the variable x, parameters $\lambda \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level m, respectively. Then the following statements hold.*

1. *Special values. For every $n \in \mathbb{N}_0$,*
$$\mathfrak{B}_n^{[m-1,0]}(x; c, a; \lambda) = \mathfrak{E}_n^{[m-1,0]}(x; c, a; \lambda) = \mathfrak{G}_n^{[m-1,0]}(x; c, a; \lambda) = (x \ln c)^n.$$
2. *Summation formulas.*
$$\begin{aligned}
\mathfrak{B}_n^{[m-1,\alpha]}(x; c, a; \lambda) &= \sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_{n-k}^{[m-1,\alpha]}(c, a; \lambda)(x \ln c)^k, \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_{n-k}^{[m-1,\alpha-1]}(c, a; \lambda)\mathfrak{B}_k^{[m-1,1]}(x; c, a; \lambda), \\
\mathfrak{E}_n^{[m-1,\alpha]}(x; c, a; \lambda) &= \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_{n-k}^{[m-1,\alpha]}(c, a; \lambda)(x \ln c)^k, \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_{n-k}^{[m-1,\alpha-1]}(c, a; \lambda)\mathfrak{E}_k^{[m-1,1]}(x; c, a; \lambda), \\
\mathfrak{G}_n^{[m-1,\alpha]}(x; c, a; \lambda) &= \sum_{k=0}^{n} \binom{n}{k} \mathfrak{G}_{n-k}^{[m-1,\alpha]}(c, a; \lambda)(x \ln c)^k, \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathfrak{G}_{n-k}^{[m-1,\alpha-1]}(c, a; \lambda)\mathfrak{G}_k^{[m-1,1]}(x; c, a; \lambda)
\end{aligned}$$
3. [18, Theorem 2, Corollary 1] *Difference equations.*
$$\begin{aligned}
&\lambda\mathfrak{B}_n^{[m-1,\alpha]}(x+1; c, a; \lambda) - \mathfrak{B}_n^{[m-1,\alpha]}(x; c, a; \lambda) \\
&\quad = n\sum_{k=0}^{n-1} \binom{n-1}{k} \mathfrak{B}_{n-k}^{[m-1,\alpha]}(x; c, a; \lambda)\mathfrak{B}_{n-k-1}^{[m-1,-1]}(x; c, a; \lambda), \\
&\lambda\mathfrak{E}_n^{[m-1,\alpha]}(x+1; c, a; \lambda) + \mathfrak{E}_n^{[m-1,\alpha]}(x; c, a; \lambda) \\
&\quad = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_{n-k}^{[m-1,\alpha]}(x; c, a; \lambda)\mathfrak{E}_{n-k}^{[m-1,-1]}(x; c, a; \lambda).
\end{aligned}$$

4. *Differential relations (Appell polynomial sequences). For fixed* α, λ, a, c, $m \in \mathbb{N}$ *and* $n, j \in \mathbb{N}_0$ *with* $0 \leq j \leq n$, *we have*

$$[\mathfrak{B}_n^{[m-1,\alpha]}(x;c,a;\lambda)]^{(j)} = \frac{n!}{(n-j)!}(\ln c)^j\, \mathfrak{B}_{n-j}^{[m-1,\alpha]}(x;c,a;\lambda),$$

$$[\mathfrak{E}_n^{[m-1,\alpha]}(x;c,a;\lambda)]^{(j)} = \frac{n!}{(n-j)!}(\ln c)^j\, \mathfrak{E}_{n-j}^{[m-1,\alpha]}(x;c,a;\lambda),$$

$$[\mathfrak{G}_n^{[m-1,\alpha]}(x;c,a;\lambda)]^{(j)} = \frac{n!}{(n-j)!}(\ln c)^j\, \mathfrak{G}_{n-j}^{[m-1,\alpha]}(x;c,a;\lambda).$$

5. *Integral formulas. For* $m \in \mathbb{N}$ *and fixed* α, λ, a, c, *we have*

$$\begin{aligned}
\int_{x_0}^{x_1} \mathfrak{B}_n^{[m-1,\alpha]}(x;c,a;\lambda)dx &= \frac{\ln c}{n+1}[\mathfrak{B}_{n+1}^{[m-1,\alpha]}(x_1;c,a;\lambda) \\
&\quad -\mathfrak{B}_{n+1}^{[m-1,\alpha]}(x_0;c,a;\lambda)] \\
&= \ln c \sum_{k=0}^{n} \frac{1}{n-k+1}\binom{n}{k}\mathfrak{B}_k^{[m-1,\alpha]}(c,a;\lambda)((x_1 \ln c)^{n-k+1} - (x_0 \ln c)^{n-k+1}), \\
&\times \int_{x_0}^{x_1} \mathfrak{E}_n^{[m-1,\alpha]}(x;c,a;\lambda)dx = \frac{\ln c}{n+1}[\mathfrak{E}_{n+1}^{[m-1,\alpha]}(x_1;c,a;\lambda) \\
&\quad -\mathfrak{E}_{n+1}^{[m-1,\alpha]}(x_0;c,a;\lambda)] \\
&= \ln c \sum_{k=0}^{n} \frac{1}{n-k+1}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(c,a;\lambda)((x_1 \ln c)^{n-k+1} - (x_0 \ln c)^{n-k+1}), \\
&\times \int_{x_0}^{x_1} \mathfrak{G}_n^{[m-1,\alpha]}(x;c,a;\lambda)dx = \frac{\ln c}{n+1}[\mathfrak{G}_{n+1}^{[m-1,\alpha]}(x_1;c,a;\lambda) \\
&\quad -\mathfrak{G}_{n+1}^{[m-1,\alpha]}(x_0;c,a;\lambda)] \\
&= \ln c \sum_{k=0}^{n} \frac{1}{n-k+1}\binom{n}{k}\mathfrak{G}_k^{[m-1,\alpha]}(c,a;\lambda)((x_1 \ln c)^{n-k+1} - (x_0 \ln c)^{n-k+1}),
\end{aligned}$$

6. [18, Theorem 1] *Addition theorem of the argument. For* $\alpha, \beta \in \mathbb{C}$, $c \in \mathbb{R}^+$ *and* $m \in \mathbb{N}$, *we have*

$$\mathfrak{B}_n^{[m-1,\alpha+\beta]}(x+y;c,a;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_{n-k}^{[m-1,\alpha]}(x;c,a;\lambda)\mathfrak{B}_{n-k}^{[m-1,\beta]}(y;c,a;\lambda),$$

$$\mathfrak{E}_n^{[m-1,\alpha+\beta]}(x+y;c,a;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_{n-k}^{[m-1,\alpha]}(x;c,a;\lambda)\mathfrak{E}_{n-k}^{[m-1,\beta]}(y;c,a;\lambda),$$

$$\mathfrak{G}_n^{[m-1,\alpha+\beta]}(x+y;c,a;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_{n-k}^{[m-1,\alpha]}(x;c,a;\lambda)\mathfrak{G}_{n-k}^{[m-1,\beta]}(y;c,a;\lambda).$$

In the table below, we introduce the standard notation for several subclasses of generalized Apostol–Bernoulli polynomials, Apostol–Euler polynomials and Apostol–Genocchi polynomials with parameters $\lambda \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level $m \in \mathbb{N}$, (cf. [1, 2, 5, 8, 9, 13–31, 34–37, 39, 41–44]).

n-th generalized Bernoulli polynomial of level m	$B_n^{[m-1]}(x) := \mathfrak{B}_n^{[m-1,1]}(x; e, e; 1)$
n-th generalized Apostol–Bernoulli polynomial	$B_n^{(\alpha)}(x;\lambda) := \mathfrak{B}_n^{[0,\alpha]}(x; e, a; \lambda)$
n-th generalized Apostol–Euler polynomial	$\mathcal{E}_n^{(\alpha)}(x;\lambda) := \mathfrak{E}_n^{[0,\alpha]}(x; e, a; \lambda)$
n-th generalized Apostol–Genocchi polynomial	$\mathcal{G}_n^{(\alpha)}(x;\lambda) := \mathfrak{G}_n^{[0,\alpha]}(x; e, a; \lambda)$
n-th Apostol–Bernoulli polynomial	$B_n(x;\lambda) := B_n^{(1)}(x;\lambda)$
n-th Apostol–Euler polynomial	$\mathcal{E}_n(x;\lambda) := \mathcal{E}_n^{(1)}(x;\lambda)$
n-th Apostol–Genocchi polynomial	$\mathcal{G}_n(x;\lambda) := \mathcal{G}_n^{(1)}(x;\lambda)$
n-th generalized Bernoulli polynomial	$B_n^{(\alpha)}(x) := B_n^{(\alpha)}(x;1)$
n-th generalized Euler polynomial	$E_n^{(\alpha)}(x) := \mathcal{E}_n^{(\alpha)}(x;1)$
n-th generalized Genocchi polynomial	$G_n^{(\alpha)}(x) := \mathcal{G}_n^{(\alpha)}(x;1)$
n-th Bernoulli polynomial	$B_n(x) := B_n^{(1)}(x)$
n-th Euler polynomial	$E_n(x) := E_n^{(1)}(x)$
n-th Genocchi polynomial	$G_n(x) := G_n^{(1)}(x)$

Finally, we recall the definitions of the Stirling numbers of second kind and the Jacobi polynomials of parameters $\alpha, \beta > -1$. Also, we present some well-known identities satisfied by these polynomials and certain families of polynomials mentioned above, which we need in the next sections.

Definition 1 ([10, p. 207, Theorem B]). For $n \in \mathbb{N}_0$ and $x \in \mathbb{C}$, the Stirling numbers of second kind $S(n,k)$ are defined by means of the following expansion

$$x^n = \sum_{k=0}^{n} \binom{x}{k} k! S(n,k), \tag{6}$$

so that $S(n,0) = \delta_{n,0}$, $S(n,1) = S(n,n) = 1$ and $S(n,n-1) = \binom{n}{2}$.

For $n \in \mathbb{N}_0$ and $\kappa, \beta > -1$, the n-th Jacobi polynomial $P_n^{(\kappa,\beta)}(x)$ may be defined by means of Rodrigues' formula

$$P_n^{(\kappa,\beta)}(x) = (1-x)^{-\kappa}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\{(1-x)^{n+\kappa}(1+x)^{n+\kappa}\},$$
$$x \in \mathbb{C}\backslash\{-1,1\}, \tag{7}$$

and the value at $x = \pm 1$ is given by

$$P_n^{(\kappa,\beta)}(1) = \binom{n+\kappa}{n}, \quad P_n^{(\kappa,\beta)}(-1) = (-1)^n \binom{n+\beta}{n}.$$

There are many equivalent definitions of $P_n^{(\kappa,\beta)}(x)$, for instance, its expression as a ${}_2F_1$ hypergeometric function is very well-known in the literature (see [3,33,38]). The connection between the n-th monomial x^n and the n-th Jacobi polynomial $P_n^{(\kappa,\beta)}(x)$ may be written as follows (see [33, Equation (2), p. 262]).

$$x^n = n! \sum_{k=0}^{n} \binom{n+\kappa}{n-k} (-1)^k \frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{n+1}} P_k^{(\kappa,\beta)}(1-2x). \tag{8}$$

Proposition 2. *For $\lambda \in \mathbb{C}$ and a fixed $m \in \mathbb{N}$. Let $\{B_n^{[m-1]}(x)\}_{n\geq 0}$, $\{G_n(x)\}_{n\geq 0}$ and $\{\mathcal{E}_n(x;\lambda)\}_{n\geq 0}$ be the sequences of generalized Bernoulli polynomials of level m, Genocchi polynomials and Apostol–Euler polynomials, respectively. Then, the following identities are satisfied.*

1. [28, Equation (2.6)].

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(k+m)!} B_{n-k}^{[m-1]}(x), \tag{9}$$

2. [24, Remark 7].

$$x^n = \frac{1}{2(n+1)} \left[\sum_{k=0}^{n+1} \binom{n+1}{k} G_k(x) + G_{n+1}(x) \right], \tag{10}$$

3. [26, Equation (31)].

$$x^n = \frac{1}{2} \left[\lambda \sum_{k=0}^{n} \binom{n}{k} \mathcal{E}_k(x;\lambda) + \mathcal{E}_n(x;\lambda) \right]. \tag{11}$$

3. The Polynomials $\mathcal{Q}_n^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu)$ and Their Properties

In view of the results in Sect. 2 and following the same methodology given in [27,30] we focus our attention on new unified presentations of the generalized Apostol-type polynomials. More precisely, based on (1), (2) and (3)–(5) we consider the following Mittag–Leffler type function

$$E_{1,m+1}^{(c,a;\lambda;\mu;\nu)}(z) := \frac{(2^{\mu} z^{\nu})^m}{\lambda c^z + \sum_{l=0}^{m-1} \frac{(z \ln a)^l}{l!}}, \quad m \in \mathbb{N},\ a, c \in \mathbb{R}^+,\ \lambda, \mu, \nu \in \mathbb{C}. \tag{12}$$

The reason for perturbing the genuine Mittag–Leffler function (2) with the parameters $a, c\lambda, \mu$ and ν is purely technical and allows us to introduce a unified presentation of the generalized Apostol-type polynomials (1) and the generalized Apostol–Bernoulli polynomials, Apostol–Euler polynomials and Apostol–Genocchi polynomials in the variable x, parameters $\lambda \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level $m \in \mathbb{N}$, (3)–(5) (cf. [18,21,27]).

Definition 2. For $m \in \mathbb{N}$, $\alpha, \lambda, \mu, \nu \in \mathbb{C}$ and $a, c \in \mathbb{R}^+$, the generalized Apostol-type polynomials in the variable x, parameters c, a, λ, μ, ν, order α and level m, are defined by means of the following generating function.

$$(E_{1,m+1}^{(c,a;\lambda;\mu;\nu)}(z))^\alpha c^{xz} = \sum_{n=0}^{\infty} \mathcal{Q}_n^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu)\frac{z^n}{n!}, \tag{13}$$

where $|z| < 2\pi$ when $\lambda = 1$, $|z| < \pi$ when $\lambda = -1$, $(|z \ln(\frac{c}{a})| < |\log(-\lambda)|)$ when $\lambda \in \mathbb{C}\backslash\{-1, 1\}$, and $1^\alpha := 1$.

Remark 1. It is worthwhile to recall that in order to ensure that the generating function in (13) is analytic throughout the prescribed open disks in the complex z-plane (centered at the origin $z = 0$), and consequently, to get the convergence of the corresponding Taylor–Maclaurin series expansions we need to impose the above constraint on $|z|$. In a similar way as [27, Remark 1], is tacitly assumed that the exceptional cases $\lambda = \pm 1$ are to be treated separately and the corresponding constraints on $|z|$ should be modified accordingly (see [27] and the references therein).

The numbers given by

$$\mathcal{Q}_n^{[m-1,\alpha]}(c, a; \lambda; \mu; \nu) := \mathcal{Q}_n^{[m-1,\alpha]}(0; c, a; \lambda; \mu; \nu), \tag{14}$$

denote the corresponding unified presentation of the generalized Apostol-type numbers of parameters $\lambda \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level $m \in \mathbb{N}$.

By comparing Definition 2 with (1) and (3)–(5) with $e = \exp(1)$, we have

$$(-1)^\alpha \mathcal{Q}_n^{[m-1,\alpha]}(x; c, a; -\lambda; 0; 1) = \mathfrak{B}_n^{[m-1,\alpha]}(x; c, a; \lambda), \tag{15}$$

$$\mathcal{Q}_n^{[m-1,\alpha]}(x; c, a; \lambda; 1; 0) = \mathfrak{E}_n^{[m-1,\alpha]}(x; c, a; \lambda), \tag{16}$$

$$\mathcal{Q}_n^{[m-1,\alpha]}(x; c, a; \lambda; 1; 1) = \mathfrak{G}_n^{[m-1,\alpha]}(x; c, a; \lambda), \tag{17}$$

$$\mathcal{Q}_n^{[0,\alpha]}(x; e, 1; \lambda; \mu; \nu) = \mathcal{F}_n^{(\alpha)}(x; \lambda; \mu; \nu). \tag{18}$$

Consequently, from the relations (15)–(16) we have that the generating function of $\mathcal{Q}_n^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu)$ in (13) includes, as its special cases, not only the generating function of the generalized Apostol-type polynomials $\mathcal{F}_n^{(\alpha)}(x; \lambda; \mu; \nu)$ in (1) and the generating functions investigated by Ozden et al. (cf. [30, p. 2779, Equation (1.1)] and [27, p. 5726, Equation (175)]), and indeed also the generating functions of $\mathfrak{B}_n^{[m-1,\alpha]}(x; c, a; \lambda)$, $\mathfrak{E}_n^{[m-1,\alpha]}(x; c, a; \lambda)$, $\mathfrak{G}_n^{[m-1,\alpha]}(x; c, a; \lambda)$ and their corresponding special cases (cf. [1,2,5,8,9,13–24,28,31,34–36,39–44]).

Under the appropriate choice on the parameters, level and order, it is possible to provide some illustrative examples (with the help of MAPLE) which show that exist polynomials $\mathcal{Q}_n^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu)$ different from the polynomials given by (1) and (3)–(5).

Example 1. For any $\lambda \in \mathbb{C}\backslash\{-1,1\}$, $m=4$, $c=2$, $a=3$, $\alpha=1$, $\mu=2$ and $\nu=5$, the first generalized Apostol–Bernoulli polynomials and generalized Apostol-type polynomials of level m are:

$$\begin{aligned}
\mathfrak{B}_j^{[3,1]}(x;2,3;\lambda) &= 0, \quad j=0,1,2,3,\\
\mathfrak{B}_4^{[3,1]}(x;2,3;\lambda) &= \frac{24}{\lambda-1},\\
\mathfrak{B}_5^{[3,1]}(x;2,3;\lambda) &= \frac{120[\ln(2)x\lambda-\ln(2)x-\lambda\ln(2)+\ln(3)]}{(\lambda-1)^2}.
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_j^{[3,1]}(x;2,3;\lambda;2;5) &= 0, \quad j=0,1,2,3,4,\\
\mathcal{Q}_5^{[3,1]}(x;2,3;\lambda;2;5) &= 640\,\mathfrak{B}_4^{[3,1]}(x;2,3;\lambda),\\
\mathcal{Q}_6^{[3,1]}(x;2,3;\lambda;2;5) &= 3(2^8)\,\mathfrak{B}_5^{[3,1]}(x;2,3;\lambda).
\end{aligned}$$

Example 2. For any $\lambda \in \mathbb{C}\backslash\{-1,1\}$, $m=3$, $c=e$, $a=3$, $\alpha=\sqrt{2}$, $\mu=\nu=4$, the first generalized Apostol–Euler polynomials and generalized Apostol-type polynomials of level m are:

$$\begin{aligned}
\mathfrak{E}_0^{[2,\sqrt{2}]}(x;e,3;\lambda) &= \left(\frac{8}{\lambda+1}\right)^{\sqrt{2}},\\
\mathfrak{E}_1^{[2,\sqrt{2}]}(x;e,3;\lambda) &= \frac{8^{\sqrt{2}}}{(\lambda+1)^{1+\sqrt{2}}}[x\lambda+x-\sqrt{2}\lambda-\sqrt{2}\ln(3)],\\
\mathfrak{E}_2^{[2,\sqrt{2}]}(x;e,3;\lambda) &= \frac{8^{\sqrt{2}}}{(\lambda+1)^{2+\sqrt{2}}}[(\lambda+1)^2x^2-2\sqrt{2}(\lambda+1)(\lambda+\ln(3))x\\
&\quad -2\sqrt{2}\lambda(1-\ln(3))^2+2(\lambda+\ln(3))^2],\\
\mathfrak{E}_3^{[2,\sqrt{2}]}(x;e,3;\lambda) &= \frac{8^{\sqrt{2}}}{(\lambda+1)^{3+\sqrt{2}}}[(\lambda+1)^3x^3-3\sqrt{2}(\lambda+1)^2(\ln(3)+\lambda)x^2\\
&\quad -3(\lambda+1)(-2\lambda^2-2\lambda\ln(3)\sqrt{2}+\sqrt{2}\lambda\\
&\quad +\sqrt{2}(\ln(3))^2\lambda-4\ln(3)\lambda-2(\ln(3))^2)x+6\lambda^2\\
&\quad +3(\ln(3)^3\lambda\sqrt{2}+6(\ln(3))^3\lambda+\lambda^2\sqrt{2}\\
&\quad -9\sqrt{2}(\ln(3))^2\lambda+3\lambda\ln(3)\sqrt{2}-2\lambda^3\sqrt{2}\\
&\quad +6\ln(3)\lambda-12(\ln(3))^2\lambda-12\lambda^2\ln(3)-9\lambda^2\ln(3)\sqrt{2}\\
&\quad +3(\ln(3))^2\lambda^2\sqrt{2}-(\ln(3))^3\sqrt{2}+6(\ln(3))^2\lambda^2-\sqrt{2}\lambda].
\end{aligned}$$

$$\mathcal{Q}_j^{[2,\sqrt{2}]}(x;e,3;\lambda;4;4) = 2^{\frac{5}{2}}\,\mathfrak{E}_j^{[2,\sqrt{2}]}(x;e,3;\lambda), \quad j=0,1,2,3.$$

Example 3. For any $\lambda \in \mathbb{C}\backslash\{-1,1\}$, $m=2$, $c=3$, $a=e$, $\alpha=\frac{1}{3}$, $\mu=5$ and $\nu=4$, the first generalized Apostol–Genocchi polynomials and generalized

Apostol-type polynomials of level m are:

$$\mathfrak{G}_0^{[1,\frac{1}{3}]}(x;3,e;\lambda) = \left(\frac{4}{\lambda+1}\right)^{\frac{1}{3}},$$

$$\mathfrak{G}_1^{[1,\frac{1}{3}]}(x;3,e;\lambda) = \frac{2^{\frac{2}{3}}}{3(\lambda+1)^{1+\frac{1}{3}}}[3\ln(3)(\lambda+1)x - \ln(3)\lambda - 1],$$

$$\begin{aligned}\mathfrak{G}_2^{[1,\frac{1}{3}]}(x;3,e;\lambda) = \frac{2^{\frac{2}{3}}}{9(\lambda+1)^{2+\frac{1}{3}}}&[9(\ln(3))^2(\lambda+1)^2x^2 - 6\ln(3)\\ &\times(\lambda+1)(\ln(3)\lambda+1)x - 3(\ln(3))^2\lambda\\ &+8\ln(3)\lambda + 4 + (\ln(3))^2\lambda^2],\end{aligned}$$

$$\begin{aligned}\mathfrak{G}_3^{[1,\frac{1}{3}]}(x;3,e;\lambda) = \frac{2^{\frac{2}{3}}}{27(\lambda+1)^{3+\frac{1}{3}}}&[27(\ln(3))^3(\lambda+1)^3x^3 - 27(\ln(3))^2\\ &\times(\lambda+1)^2(\ln(3)\lambda+1)x^2\\ &+9\ln(3)(\lambda+1)(-3(\ln(3))^2\lambda + 8\ln(3)\lambda + 4\\ &+(\ln(3))^2\lambda^2)x - (\ln(3))^3\lambda^3\\ &+18(\ln(3))^3\lambda^2 - 9(\ln(3))^3\lambda - 48(\ln(3))^2\lambda^2\\ &+36(\ln(3))^2\lambda - 84\ln(3)\lambda - 28],\end{aligned}$$

$$\begin{aligned}\mathfrak{G}_4^{[1,\frac{1}{3}]}(x;3,e;\lambda) = \frac{2^{\frac{2}{3}}}{81(\lambda+1)^{4+\frac{1}{3}}}&\{81(\ln(3))^4(\lambda+1)^4x^4 - 108(\ln(3))^3\\ &\times(\lambda+1)^3(\ln(3)\lambda+1)x^3\\ &+54(\ln(3))^2(\lambda+1)^2(-3(\ln(3))^2\lambda + 8\ln(3)\lambda\\ &+4 + (\ln(3))^2\lambda^2)x^2\\ &-12\ln(3)(\lambda+1)[(\ln(3))^3\lambda^3 - 18(\ln(3))^3\lambda^2\\ &+48(\ln(3))^2\lambda^2 + 9(\ln(3))^3\lambda\\ &+84\ln(3)\lambda - 36(\ln(3))^2\lambda + 28]x + 280\\ &-81(\ln(3))^4\lambda^3 + 1176(\ln(3))^2\lambda^2\\ &+1120\ln(3)\lambda - 504(\ln(3))^2\lambda + 144(\ln(3))^3\lambda\\ &-27(\ln(3))^4\lambda + (\ln(3))^4\lambda^4\\ &-720(\ln(3))^3\lambda^2 + 171(\ln(3))^4\lambda^2 + 256(\ln(3))^3\lambda^3\}.\end{aligned}$$

$$\mathcal{Q}_0^{[1,\frac{1}{3}]}(x;3,e;\lambda;5;4) = 0,$$

$$\mathcal{Q}_1^{[1,\frac{1}{3}]}(x;3,e;\lambda;5;4) = 2^{\frac{5}{3}}\,\mathfrak{G}_0^{[1,\frac{1}{3}]}(x;3,e;\lambda),$$

$$\mathcal{Q}_2^{[1,\frac{1}{3}]}(x;3,e;\lambda;5;4) = 2^{\frac{8}{3}}\,\mathfrak{G}_1^{[1,\frac{1}{3}]}(x;3,e;\lambda),$$

$$\mathcal{Q}_3^{[1,\frac{1}{3}]}(x;3,e;\lambda;5;4) = 12\sqrt[3]{4}\,\mathfrak{G}_2^{[1,\frac{1}{3}]}(x;3,e;\lambda),$$

$$\mathcal{Q}_4^{[1,\frac{1}{3}]}(x;3,e;\lambda;5;4) = \frac{2^{\frac{17}{3}}}{27}\,\mathfrak{G}_3^{[1,\frac{1}{3}]}(x;3,e;\lambda).$$

Example 4. For any $\lambda \in \mathbb{C}\backslash\{-1,1\}$, $m=2$, $c=2$, $a=3$, $\alpha=\frac{1}{2}$, $\mu=2$ and $\nu=5$, the first Apostol-type polynomials and generalized Apostol-type polynomials of level m are:

$$\begin{aligned}
&\mathcal{F}_j^{(\frac{1}{2})}(x;\lambda;2;5)=0, \quad j=0,1,\\
&\mathcal{F}_2^{(\frac{1}{2})}(x;\lambda;2;5)=\frac{32}{\sqrt{1+\lambda}},\\
&\mathcal{F}_3^{(\frac{1}{2})}(x;\lambda;2;5)=\frac{48}{(1+\lambda)^{1+\frac{1}{2}}}(2(\lambda+1)x-\lambda),\\
&\mathcal{F}_4^{(\frac{1}{2})}(x;\lambda;2;5)=\frac{48}{(1+\lambda)^{2+\frac{1}{2}}}[4(\lambda+1)^2x^2-4\lambda(\lambda+1)x-\lambda(2-\lambda)].
\end{aligned}$$

$$\begin{aligned}
&\mathcal{Q}_j^{[1,(\frac{1}{2})]}(x;2,3;\lambda;2;5)=0, \quad j=0,1,\\
&\mathcal{Q}_2^{[1,(\frac{1}{2})]}(x;2,3;\lambda;2;5)=\mathcal{F}_2^{(\frac{1}{2})}(x;\lambda;2;5),\\
&\mathcal{Q}_3^{[1,(\frac{1}{2})]}(x;2,3;\lambda;2;5)=\frac{48}{(1+\lambda)^{1+\frac{1}{2}}}[2\ln(2)(\lambda+1)x-\ln(2)\lambda-\ln(3)],\\
&\mathcal{Q}_4^{[1,(\frac{1}{2})]}(x;2,3;\lambda;2;5)=\frac{2}{(1+\lambda)^{2+\frac{1}{2}}}[4(\ln(2))^2(\lambda+1)^2x^2\\
&\qquad -4\ln(2)(\lambda+1)(\ln(2)\lambda+\ln(3))x\\
&\qquad -2(\ln(2))^2\lambda+6\ln(2)\lambda\ln(3)+3(\ln(3))^2\\
&\qquad +(\ln(2))^2\lambda^2].
\end{aligned}$$

The following theorem summarizes some elementary properties of the polynomials $\mathcal{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$, which are a straightforward consequence of (13). Therefore, we will omit the details of its proof.

Theorem 1. *For a fixed $m \in \mathbb{N}$, let $\{\mathcal{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\}_{n\geq 0}$ be the sequence of generalized Apostol-type polynomials in the variable x, parameters $\lambda,\mu,\nu \in \mathbb{C}$, $a,c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level m. Then the following statements hold.*

1. *Special values. For every $n \in \mathbb{N}_0$,*
$$\mathcal{Q}_n^{[m-1,0]}(x;c,a;\lambda;\mu;\nu)=(x\ln c)^n. \tag{19}$$
2. *Summation formulas.*
$$\mathcal{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{Q}_{n-k}^{[m-1,\alpha]}(c,a;\lambda;\mu;\nu)(x\ln c)^k, \tag{20}$$
$$\begin{aligned}\mathcal{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)=\sum_{k=0}^{n}\binom{n}{k}&\mathcal{Q}_{n-k}^{[m-1,\alpha-1]}(c,a;\lambda;\mu;\nu)\\ &\times\mathcal{Q}_k^{[m-1,1]}(x;c,a;\lambda;\mu;\nu).\end{aligned} \tag{21}$$

3. *Differential relations (Appell polynomial sequences). For* $m \in \mathbb{N}$, *fixed* α, λ, a, c, *and* $n, j \in \mathbb{N}_0$ *with* $0 \leq j \leq n$, *we have*

$$[\mathcal{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)]^{(j)} = \frac{n!}{(n-j)!}(\ln c)^j \, \mathcal{Q}_{n-j}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu). \tag{22}$$

4. *Difference equations. For* $m \in \mathbb{N}$, *fixed* $\alpha, \lambda, a, c, \mu, \nu$. *If* $a = c$ *and* $\nu \in \mathbb{N}_0$, *we have*

$$\lambda \mathcal{Q}_n^{[m-1,\alpha]}(x+1;c,c;\lambda;\mu;\nu) + \sum_{l=0}^{m-1} \frac{1}{l!}[\mathcal{Q}_n^{[m-1,\alpha]}(x;c,c;\lambda;\mu;\nu)]^{(l)}$$
$$= \frac{2^{\mu m}}{(\ln c)^{\nu m}} [\mathcal{Q}_n^{[m-1,\alpha-1]}(x;c,c;\lambda;\mu;\nu)]^{(\nu m)}.$$

5. *Integral formulas. For fixed* α, λ, a, c *and* $m \in \mathbb{N}$, *we have*

$$\begin{aligned}\int_{x_0}^{x_1} \mathcal{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\,dx &= \frac{\ln c}{n+1}[\mathcal{Q}_{n+1}^{[m-1,\alpha]}(x_1;c,a;\lambda;\mu;\nu) \\ &\quad -\mathcal{Q}_{n+1}^{[m-1,\alpha]}(x_0;c,a;\lambda;\mu;\nu)] \\ &= \ln c \sum_{k=0}^{n} \frac{1}{n-k+1}\binom{n}{k} \mathcal{Q}_k^{[m-1,\alpha]} \\ &\quad \times (c,a;\lambda;\mu;\nu)((x_1 \ln c)^{n-k+1} \\ &\quad -(x_0 \ln c)^{n-k+1}).\end{aligned}$$

6. *Addition theorem of the argument.*

$$\begin{aligned}\mathcal{Q}_n^{[m-1,\alpha+\beta]}(x+y;c,a;\lambda;\mu;\nu) &= \sum_{k=0}^{n} \binom{n}{k} \mathcal{Q}_k^{[m-1,\alpha]} \\ &\quad \times (x;c,a;\lambda;\mu;\nu) \mathcal{Q}_{n-k}^{[m-1,\beta]}(y;c,a;\lambda;\mu;\nu),\end{aligned} \tag{23}$$

$$\mathcal{Q}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{Q}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(x \ln c)^k. \tag{24}$$

The use of the Stirling numbers of second kind (6) and a straightforward calculation involving binomial coefficient identities [2,3,10–12,33,45], allows us to get an alternative representation for a particular case of (23) as follows.

Proposition 3. *For* $m \in \mathbb{N}$, $\alpha, \lambda, \mu, \nu \in \mathbb{C}$, $a, c \in \mathbb{R}^+$ *and* $n \in \mathbb{N}_0$, *we have*

$$\begin{aligned}\mathcal{Q}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) &= \sum_{k=0}^{n} k! \binom{x}{k} \sum_{j=0}^{n-k} \binom{n}{j} \mathcal{Q}_j^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu) \\ &\quad \times (\ln c)^{n-j} S(n-j,k)\end{aligned}$$

$$= \sum_{k=0}^{n} k! \binom{x}{k} \sum_{j=k}^{n} \binom{n}{n-j} \mathcal{Q}_{n-j}^{[m-1,\alpha]} \times (y;c,a;\lambda;\mu;\nu)(\ln c)^{j} S(j,k). \tag{25}$$

Consequently, taking into account the relations (15)–(16) and making the corresponding modifications in (25), we get

Corollary 1. *For $m \in \mathbb{N}$, the generalized Apostol–Bernoulli polynomials, generalized Apostol–Euler polynomials and generalized Apostol–Genocchi polynomials with parameters λ, a, c, order α and level m are related with the Stirling numbers of second kind by means of the following identities.*

$$\mathfrak{B}_{n}^{[m-1,\alpha]}(x+y;c,a;\lambda) = \sum_{k=0}^{n} k! \binom{x}{k} \sum_{j=k}^{n} \binom{n}{j} \mathfrak{B}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda)(\ln c)^{j} S(j,k). \tag{26}$$

$$\mathfrak{E}_{n}^{[m-1,\alpha]}(x+y;c,a;\lambda) = \sum_{k=0}^{n} k! \binom{x}{k} \sum_{j=k}^{n} \binom{n}{j} \mathfrak{E}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda)(\ln c)^{j} S(j,k). \tag{27}$$

$$\mathfrak{G}_{n}^{[m-1,\alpha]}(x+y;c,a;\lambda) = \sum_{k=0}^{n} k! \binom{x}{k} \sum_{j=k}^{n} \binom{n}{j} \mathfrak{G}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda)(\ln c)^{j} S(j,k). \tag{28}$$

Remark 2. The substitution $a = 1$, $c = e$ into (26)–(28) yields the identities

$$\mathfrak{B}_{n}^{[m-1,\alpha]}(x+y;\lambda) = \sum_{k=0}^{n} k! \binom{x}{k} \sum_{j=k}^{n} \binom{n}{j} \mathfrak{B}_{n-j}^{[m-1,\alpha]}(y;\lambda) S(j,k), \tag{29}$$

$$\mathfrak{E}_{n}^{[m-1,\alpha]}(x+y;\lambda) = \sum_{k=0}^{n} k! \binom{x}{k} \sum_{j=k}^{n} \binom{n}{j} \mathfrak{E}_{n-j}^{[m-1,\alpha]}(y;\lambda) S(j,k), \tag{30}$$

$$\mathfrak{G}_{n}^{[m-1,\alpha]}(x+y;\lambda) = \sum_{k=0}^{n} k! \binom{x}{k} \sum_{j=k}^{n} \binom{n}{j} \mathfrak{G}_{n-j}^{[m-1,\mu]}(y;\lambda) S(j,k). \tag{31}$$

Remark 3. Notice that (30) is just [9, Equation (4.3)]. If $m = 1$ in (29) and (30) then we recover [26, Equations (75), (76)], respectively.

4. Some Connection Formulas for the Polynomials $\mathcal{Q}_{n}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$

From the identity (8) and Proposition 2 it is possible to deduce some interesting algebraic relations connecting the polynomials $\mathcal{Q}_{n}^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$ and other families of polynomials such as Jacobi polynomials, generalized Bernoulli polynomials of level m, Genocchi polynomials and Apostol–Euler polynomials. In this section, we will prove these relations following the ideas of [9].

Theorem 2. *For $m \in \mathbb{N}$, the generalized Apostol-type polynomials of level m $\mathcal{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$, are related with the Jacobi polynomials $P_n^{(\kappa,\beta)}(x)$, by means of the following identity.*

$$\mathcal{Q}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) = \sum_{k=0}^{n}(-1)^k \sum_{j=k}^{n} j!(\ln c)^j \binom{j+\kappa}{j-k}\binom{n}{j} \times \frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{j+1}} \mathcal{Q}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu))P_k^{(\kappa,\beta)}(1-2x). \tag{32}$$

Proof. We can proceed as the proof of [9, Theorem 4.4]. By substituting (8) into the right-hand side of (24) and using appropriate binomial coefficient identities (see, for instance [3,10,12]), we see that

$$\begin{aligned}
&\mathcal{Q}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu)\\
&= \sum_{j=0}^{n}\binom{n}{j}\mathcal{Q}_j^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(n-j)!(\ln c)^{n-j}\sum_{k=0}^{n-j}(-1)^k\binom{n-j+\kappa}{n-j-k}\\
&\quad\times\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{n-j+1}}P_k^{(\kappa,\beta)}(1-2x)\\
&= \sum_{j=0}^{n}\sum_{k=0}^{n-j}\binom{n}{j}\mathcal{Q}_j^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(n-j)!(\ln c)^{n-j}(-1)^k\binom{n-j+\kappa}{n-j-k}\\
&\quad\times\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{n-j+1}}P_k^{(\kappa,\beta)}(1-2x)\\
&= \sum_{k=0}^{n}(-1)^k\sum_{j=0}^{n-k}\binom{n}{j}\binom{n-j+\kappa}{n-j-k}\mathcal{Q}_j^{[m-1,\mu]}(y;c,a;\lambda;\mu;\nu)(n-j)!(\ln c)^{n-j}\\
&\quad\times\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{n-j+1}}P_k^{(\kappa,\beta)}(1-2x)\\
&= \sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!(\ln c)^j\binom{j+\kappa}{j-k}\binom{n}{j}\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{j+1}}\\
&\quad\times\mathcal{Q}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)P_k^{(\kappa,\beta)}(1-2x).
\end{aligned}$$

Therefore, (32) holds. □

Consequently, taking into account the relations (15)–(16) and making the corresponding modifications in (32), we get

Corollary 2. *For $m \in \mathbb{N}$, the generalized Apostol–Bernoulli polynomials, generalized Apostol–Euler polynomials and generalized Apostol–Genocchi polynomials with parameters λ, a, c, order α and level m are related with the Jacobi* *polynomials by means of the following identities.*

$$\begin{aligned}&\mathfrak{B}_n^{[m-1,\alpha]}(x+y;c,a;\lambda)\\&\quad=\sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!(\ln c)^j\binom{j+\kappa}{j-k}\binom{n}{j}\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{j+1}}\mathfrak{B}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda)\\&\qquad\times P_k^{(\kappa,\beta)}(1-2x).\end{aligned}\tag{33}$$

$$\begin{aligned}&\mathfrak{E}_n^{[m-1,\alpha]}(x+y;c,a;\lambda)\\&\quad=\sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!(\ln c)^j\binom{j+\kappa}{j-k}\binom{n}{j}\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{j+1}}\mathfrak{E}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda)\\&\qquad\times P_k^{(\kappa,\beta)}(1-2x).\end{aligned}\tag{34}$$

$$\begin{aligned}&\mathfrak{G}_n^{[m-1,\alpha]}(x+y;c,a;\lambda)\\&\quad=\sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!(\ln c)^j\binom{j+\kappa}{j-k}\binom{n}{j}\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{j+1}}\mathfrak{G}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda)\\&\qquad\times P_k^{(\kappa,\beta)}(1-2x).\end{aligned}\tag{35}$$

Remark 4. The substitution $a=1$, $c=e$ into (33)–(35) yields the identities

$$\begin{aligned}\mathfrak{B}_n^{[m-1,\alpha]}(x+y;\lambda)&=\sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!\binom{j+\kappa}{j-k}\binom{n}{j}\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{j+1}}\\&\quad\times\mathfrak{B}_{n-j}^{[m-1,\alpha]}(y;\lambda)P_k^{(\kappa,\beta)}(1-2x),\end{aligned}\tag{36}$$

$$\begin{aligned}\mathfrak{E}_n^{[m-1,\alpha]}(x+y;\lambda)&=\sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!\binom{j+\kappa}{j-k}\binom{n}{j}\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{j+1}}\\&\quad\times\mathfrak{E}_{n-j}^{[m-1,\alpha]}(y;\lambda)P_k^{(\kappa,\beta)}(1-2x),\end{aligned}\tag{37}$$

$$\begin{aligned}\mathfrak{G}_n^{[m-1,\alpha]}(x+y;\lambda)&=\sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!\binom{j+\kappa}{j-k}\binom{n}{j}\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{j+1}}\\&\quad\times\mathfrak{G}_{n-j}^{[m-1,\alpha]}(y;\lambda)P_k^{(\kappa,\beta)}(1-2x).\end{aligned}\tag{38}$$

Notice that (37) corresponds to [9, Equation (4.7)]. If $\alpha=1$ in (36), then we have

$$\begin{aligned}B_n^{[m-1]}(x+y;\lambda)&=\sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!\binom{j+\kappa}{j-k}\binom{n}{j}\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{j+1}}\\&\quad B_{n-j}^{[m-1]}(y;\lambda)P_k^{(\kappa,\beta)}(1-2x).\end{aligned}$$

Theorem 3. *For $m\in\mathbb{N}$, the generalized Apostol-type polynomials of level m $\mathcal{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$, are related with the generalized Bernoulli polynomials* *of level m* $B_n^{[m-1]}(x)$*, by means of the following identity.*

$$\mathcal{Q}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) = \sum_{k=0}^{n}\sum_{j=k}^{n} \frac{k!(\ln c)^j}{(k+m)!}\binom{n}{j}\binom{j}{k} \times \mathcal{Q}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)B_{j-k}^{[m-1]}(x). \quad (39)$$

Proof. By substituting (9) into the right-hand side of (24), it suffices to follow the proof given in Theorem 2, making the corresponding modifications (see also, [9, Theorem 4.6]). □

Consequently, taking into account the relations (15)–(16) and making the corresponding modifications in (39), we get

Corollary 3. *For* $m \in \mathbb{N}$*, the generalized Apostol–Bernoulli polynomials, Apostol–Euler polynomials and Apostol–Genocchi polynomials of level m are related with the generalized Bernoulli polynomials of level m polynomials by means of the following identities.*

$$\mathfrak{B}_n^{[m-1,\alpha]}(x+y;c,a;\lambda) = \sum_{k=0}^{n}\sum_{j=k}^{n} \frac{k!(\ln c)^j}{(k+m)!}\binom{n}{j}\binom{j}{k} \mathfrak{B}_{n-j}^{[m-1,\mu]}(y;c,a;\lambda)B_{j-k}^{[m-1]}(x). \quad (40)$$

$$\mathfrak{E}_n^{[m-1,\alpha]}(x+y;c,a;\lambda) = \sum_{k=0}^{n}\sum_{j=k}^{n} \frac{k!(\ln c)^j}{(k+m)!}\binom{n}{j}\binom{j}{k} \mathfrak{E}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda)B_{j-k}^{[m-1]}(x). \quad (41)$$

$$\mathfrak{G}_n^{[m-1,\alpha]}(x+y;c,a;\lambda) = \sum_{k=0}^{n}\sum_{j=k}^{n} \frac{k!(\ln c)^j}{(k+m)!}\binom{n}{j}\binom{j}{k} \mathfrak{G}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda)B_{j-k}^{[m-1]}(x). \quad (42)$$

Remark 5. The substitution $a = 1$, $c = e$ into (40)–(42) yields the identities

$$\mathfrak{B}_n^{[m-1,\alpha]}(x+y;\lambda) = \sum_{k=0}^{n}\sum_{j=k}^{n} \frac{k!}{(k+m)!}\binom{n}{j}\binom{j}{k} \mathfrak{B}_{n-j}^{[m-1,\mu]}(y;\lambda)B_{j-k}^{[m-1]}(x),$$

$$\mathfrak{B}_n^{[m-1,\alpha]}(x+y;\lambda) = \sum_{k=0}^{n}\sum_{j=k}^{n} \frac{k!}{(k+m)!}\binom{n}{j}\binom{j}{k} \mathfrak{B}_{n-j}^{[m-1,\alpha]}(y;\lambda)B_{j-k}^{[m-1]}(x),$$

$$\mathfrak{G}_n^{[m-1,\alpha]}(x+y;\lambda) = \sum_{k=0}^{n}\sum_{j=k}^{n} \frac{k!}{(k+m)!}\binom{n}{j}\binom{j}{k} \mathfrak{G}_{n-j}^{[m-1,\alpha]}(y;\lambda)B_{j-k}^{[m-1]}(x). \quad (43)$$

Notice that (43) is just [9, Equation (4.11)].

Theorem 4. *For $m \in \mathbb{N}$, the generalized Apostol-type polynomials of level m $\mathcal{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$, are related with the Genocchi polynomials $G_n(x)$, by means of the following identity.*

$$\begin{aligned}
&\mathcal{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\
&\quad= \frac{1}{2}\sum_{k=0}^{n}\frac{(\ln c)^k}{k+1}\Bigg[\binom{n}{k}\mathcal{Q}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)\\
&\qquad+\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k}\mathcal{Q}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(\ln c)^{j-k}\Bigg]G_{k+1}(x). \qquad (44)
\end{aligned}$$

Proof. We can proceed as the proof of [9, Theorem 4.1]. By substituting (10) into the right-hand side of (24), we see that

$$\begin{aligned}
&\mathcal{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)\\
&= \sum_{j=0}^{n}\binom{n}{j}\mathcal{Q}_j^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)\frac{(\ln c)^{n-j}}{2(n-j+1)}\\
&\quad\times\left[\sum_{k=0}^{n-j}\binom{n-j+1}{k+1}\binom{n-j+1}{k+1}G_{k+1}(x)+G_{n-j+1}(x)\right]\\
&= \sum_{j=0}^{n}\binom{n}{j}\mathcal{Q}_j^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)\frac{(\ln c)^{n-j}}{2(n-j+1)}\sum_{k=0}^{n-j}\binom{n-j+1}{k+1}G_{k+1}(x)\\
&\quad+\sum_{j=0}^{n}\binom{n}{j}\mathcal{Q}_j^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)\frac{(\ln c)^{n-j}}{2(n-j+1)}G_{n-j+1}(x).
\end{aligned}$$

Then, using appropriate combinational identities and summations (see, for instance [3,10,12]), we obtain

$$\begin{aligned}
&\mathcal{Q}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu)\\
&\quad= \frac{1}{2}\sum_{k=0}^{n}\frac{(\ln c)^k}{k+1}\Bigg[\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k}\mathcal{Q}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(\ln c)^{j-k}\\
&\qquad+\binom{n}{k}\mathcal{Q}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)\Bigg]G_{k+1}(x).
\end{aligned}$$

Therefore, (44) holds. □

Again, taking into account the relations (15)–(16) and making the corresponding modifications in (44), we get

Corollary 4. *For $m \in \mathbb{N}$, the generalized Apostol–Bernoulli polynomials of level m and generalized Apostol–Genocchi polynomials of level m, are related with* *the Genocchi polynomials, by means of the following identities.*

$$\mathfrak{B}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) = \frac{1}{2}\sum_{k=0}^{n}\frac{(\ln c)^k}{k+1}\left[\binom{n}{k}\mathfrak{B}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)\right.$$
$$\left.+\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k}\mathfrak{B}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(\ln c)^{j-k}\right]G_{k+1}(x). \tag{45}$$

$$\mathfrak{E}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) = \frac{1}{2}\sum_{k=0}^{n}\frac{(\ln c)^k}{k+1}\left[\binom{n}{k}\mathfrak{E}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)\right.$$
$$\left.+\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k}\mathfrak{E}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(\ln c)^{j-k}\right]G_{k+1}(x), \tag{46}$$

$$\mathfrak{G}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) = \frac{1}{2}\sum_{k=0}^{n}\frac{(\ln c)^k}{k+1}\left[\binom{n}{k}\mathfrak{G}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)\right.$$
$$\left.+\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k}\mathfrak{G}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(\ln c)^{j-k}\right]G_{k+1}(x). \tag{47}$$

Remark 6. The substitution $a = 1$, $c = e$ into (45)–(47) yields the identities

$$\mathfrak{B}_n^{[m-1,\alpha]}(x+y;\lambda;\mu;\nu) = \frac{1}{2}\sum_{k=0}^{n}\frac{1}{k+1}$$
$$\times\left[\binom{n}{k}\mathfrak{B}_{n-k}^{[m-1,\alpha]}(y;\lambda;\mu;\nu)+\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k}\mathfrak{B}_{n-j}^{[m-1,\alpha]}(y;\lambda;\mu;\nu)\right]G_{k+1}(x),$$
$$\mathfrak{E}_n^{[m-1,\alpha]}(x+y;\lambda;\mu;\nu) = \frac{1}{2}\sum_{k=0}^{n}\frac{1}{k+1}$$
$$\times\left[\binom{n}{k}\mathfrak{E}_{n-k}^{[m-1,\alpha]}(y;\lambda;\mu;\nu)+\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k}\mathfrak{E}_{n-j}^{[m-1,\alpha]}(y;\lambda;\mu;\nu)\right]G_{k+1}(x),$$
$$\mathfrak{G}_n^{[m-1,\alpha]}(x+y;\lambda;\mu;\nu) = \frac{1}{2}\sum_{k=0}^{n}\frac{1}{k+1}$$
$$\times\left[\binom{n}{k}\mathfrak{G}_{n-k}^{[m-1,\alpha]}(y;\lambda;\mu;\nu)+\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k}\mathfrak{G}_{n-j}^{[m-1,\alpha]}(y;\lambda;\mu;\nu)\right]G_{k+1}(x). \tag{48}$$

Notice that (48) is just [9, Equation (4.1)].

For the case of connection formulas involving the Apostol–Euler polynomials, we have the following result.

Theorem 5. *For $m \in \mathbb{N}$, the generalized Apostol-type polynomials of level m $\mathcal{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda;\mu;\nu)$, are related with the Apostol–Euler polynomials $\mathcal{E}_n(x;\lambda)$, by means of the following identity.*

$$\begin{aligned}
&\mathcal{Q}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu)\\
&\quad= \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}[\lambda\mathcal{Q}_n^{[m-1,\alpha]}(y+1;c,a;\lambda;\mu;\nu)+(\ln c)^j\mathcal{Q}_n^{[m-1,\alpha]}\\
&\quad\times(y;c,a;\lambda;\mu;\nu)]\mathcal{E}_{n-j}(x;\lambda). \qquad (49)
\end{aligned}$$

Proof. By substituting (11) into the right-hand side of (24), we can see that

$$\begin{aligned}
&\mathcal{Q}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu)\\
&= \sum_{k=0}^{n}\binom{n}{k}\mathcal{Q}_k^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(\ln c)^{n-k}\left(\frac{1}{2}\right)\\
&\quad\times\left[\lambda\sum_{j=0}^{n-k}\binom{n-k}{j}\mathcal{E}_j(x;\lambda)+\mathcal{E}_{n-k}(x;\lambda)\right]\\
&= \sum_{k=0}^{n}\binom{n}{k}\mathcal{Q}_k^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(\ln c)^{n-k}\left(\frac{\lambda}{2}\right)\sum_{j=0}^{n-k}\binom{n-k}{j}\mathcal{E}_j(x;\lambda)\\
&\quad+\sum_{k=0}^{n}\binom{n}{k}\mathcal{Q}_k^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(\ln c)^{n-k}\left(\frac{1}{2}\right)\mathcal{E}_{n-k}(x;\lambda). \qquad (50)
\end{aligned}$$

Upon inverting the order of summation and using the following elementary binomial coefficient identities (see [12, Equations (5.4) and (5.21)]):

$$\binom{n}{k}=\binom{n}{n-k}, \quad k,n\in\mathbb{N}_0,$$
$$\binom{r}{m}\binom{m}{k}=\binom{r}{k}\binom{r-k}{m-k}, \quad k,m,r\in\mathbb{N}_0,$$

the first sum in (50) becomes

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\mathcal{Q}_k^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(\ln c)^{n-k}\left(\frac{\lambda}{2}\right)\sum_{j=0}^{n-k}\binom{n-k}{j}\mathcal{E}_j(x;\lambda)\\
&= \sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}(\ln c)^{n-k}\left(\frac{\lambda}{2}\right)\binom{n-k}{j}\mathcal{Q}_k^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)\mathcal{E}_j(x;\lambda)
\end{aligned}$$

$$= \sum_{j=0}^{n} \left(\frac{\lambda}{2}\right) \binom{n}{j} \mathcal{E}_j(x;\lambda) \sum_{k=0}^{n-j} \binom{n-j}{k} \mathcal{Q}_k^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(\ln c)^{n-k}$$

$$= \sum_{j=0}^{n} \left(\frac{\lambda}{2}\right) \binom{n}{j} \mathcal{E}_j(x;\lambda) \mathcal{Q}_{n-j}^{[m-1,\alpha]}(y+1;c,a;\lambda;\mu;\nu). \tag{51}$$

For the second sum in (50), we obtain

$$\sum_{k=0}^{n} \binom{n}{k} \mathcal{Q}_k^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(\ln c)^{n-k} \left(\frac{1}{2}\right) \mathcal{E}_{n-k}(x;\lambda)$$

$$= \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \mathcal{Q}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(\ln c)^{k} \mathcal{E}_k(x;\lambda). \tag{52}$$

Combining (51) and (52) we get

$$\begin{aligned}
&\mathcal{Q}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) \\
&\quad = \left(\frac{\lambda}{2}\right) \sum_{j=0}^{n} \binom{n}{j} \mathcal{E}_j(x;\lambda) \mathcal{Q}_{n-j}^{[m-1,\alpha]}(y+1;c,a;\lambda;\mu;\nu) \\
&\qquad + \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} \mathcal{Q}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)(\ln c)^{j} \mathcal{E}_j(x;\lambda) \\
&\quad = \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} [\lambda \mathcal{Q}_n^{[m-1,\alpha]}(y+1;c,a;\lambda;\mu;\nu) + (\ln c)^{j} \\
&\qquad \times \mathcal{Q}_n^{[m-1,\alpha]}(y;c,a;\lambda;\mu,\nu)] \mathcal{E}_{n-j}(x;\lambda).
\end{aligned}$$

Therefore, (49) holds. □

Finally, as a consequence of Theorem 5 we get

Corollary 5. *For $m \in \mathbb{N}$, the generalized Apostol–Bernoulli polynomials of level m and generalized Apostol–Genocchi polynomials of level m, are related with the Apostol–Euler polynomials, by means of the following identities.*

$$\begin{aligned}
\mathfrak{B}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) &= \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} [\lambda \mathfrak{B}_n^{[m-1,\alpha]}(y+1;c,a;\lambda;\mu;\nu) \\
&\quad + (\ln c)^{j} \mathfrak{B}_n^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)] \mathcal{E}_{n-j}(x;\lambda), \\
\mathfrak{G}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;\mu;\nu) &= \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} [\lambda \mathfrak{G}_n^{[m-1,\alpha]}(y+1;c,a;\lambda;\mu;\nu) \\
&\quad + (\ln c)^{j} \mathfrak{G}_n^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu)] \mathcal{E}_{n-j}(x;\lambda).
\end{aligned}$$

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Received: August 23, 2014.
Accepted: December 21, 2014.