Results. Math. 66 (2014), 213–225

1422-6383/14/010213-13
published online March 1, 2014
DOI 10.1007/s00025-014-0373-7

Results in Mathematics

Contact Pseudo-Metric Manifolds of Constant Curvature and CR Geometry

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Dedicated to my grand-daughter Elisa

Abstract. In this paper, we show that if an integrable contact pseudo-metric manifold of dimension $2n+1$, $n \geq 2$, has constant sectional curvature κ, then the structure is Sasakian and $\kappa = \varepsilon = g(\xi, \xi)$, where ξ is the Reeb vector field. We note that the notion of contact pseudo-metric structure is equivalent to the notion of non-degenerate almost CR manifold, then an equivalent statement of this result holds in terms of CR geometry. Moreover, we study the pseudohermitian torsion τ of a non-degenerate almost CR manifold.

Mathematics Subject Classification (2000). 53D10, 53C50, 53C15.

Keywords. Contact pseudo-metric structures, pseudo-Riemannian metrics, sectional curvature, non-degenerate CR structure, pseudohermitian torsion.

1. Introduction

In [5] was introduced a systematic study of *contact pseudo-metric manifolds* $(M\varphi, \xi, \eta, g)$, that is, contact manifolds equipped with an associated pseudo-Riemannian metric. In such study the tensor $h = (1/2)\mathcal{L}_\xi\varphi$ plays a fundamental role. In particular, while K-contact Riemannian manifolds of dimension $2n+1$ are characterized by the Ricci curvature condition $Ric(\xi, \xi) = 2n$ [1], we showed that a corresponding characterization fails for general contact pseudo-metric structures because in the pseudo-Riemannian case the condition trace $h^2 = 0$ does not imply $h = 0$ ([6], Example 1.1). In Section 4 of [5]

Supported by funds of the Universitá del Salento and of the M.I.U.R. (within PRIN 2010–2011).

we dealt with contact pseudo-metric manifolds of constant sectional curvature in dimension ≥ 5, proving that the value of the constant sectional curvature is determined by the casual character of the Reeb vector field ξ. In particular, there are not flat contact pseudo-metric manifolds of dimension ≥ 5. In [12] we showed that any conformally flat K-contact pseudo-metric manifold is Sasakian and of constant sectional curvature $\kappa = \varepsilon = \pm 1$. The main result of Section 4 in [5] was the following generalization in pseudo-Riemannian settings of the classification obtained by Olszak [11] in the Riemannian case: if $(M, \eta, g, \xi, \varphi)$ is a contact pseudo-metric manifold, $\dim M = 2n+1$, $n \geq 2$, of constant sectional curvature κ, then $\kappa = \varepsilon = g(\xi, \xi)$ and the structure (η, g) is Sasakian. The proof of this result we gave in [5] is not correct in order to conclude that (M, η, g) is Sasakian. As a consequence, the above result must be replaced by the following weaker version.

Theorem 1.1 [**6**]. *Let $(M, \eta, g, \xi, \varphi)$ be a contact pseudo-metric manifold,* $\dim M = 2n+1$*, $n \geq 2$. If (M, g) is of constant sectional curvature κ, then $\kappa = \varepsilon = g(\xi, \xi)$ and $h^2 = 0$.*

In the present paper we show the following

Theorem 1.2. *Let $(M, \eta, g, \xi, \varphi)$ be an integrable contact pseudo-metric manifold,* $\dim M = 2n+1$*, $n \geq 2$. If (M, g) is of constant sectional curvature κ, then $\kappa = \varepsilon = g(\xi, \xi)$ and the structure (η, g) is Sasakian.*

We note that the notion of contact pseudo-metric structure is equivalent to the notion of non-degenerate almost CR manifold (see Proposition 2.1). Then, an equivalent statement of Theorem 1.2 in terms of CR geometry is the following :

Theorem 1.3. *Let $(M, \mathcal{H}(M), J, \theta)$ be a non-degenerate CR manifold,* $\dim M = 2n+1$*, $n \geq 2$. If the Webster metric g_θ is of constant sectional curvature κ, then $\kappa = \varepsilon = g_\theta(\xi, \xi)$ and the pseudohermitian torsion $\tau = 0$, i.e. (M, θ, g_θ) is Sasakian.*

In the above Theorem 1.1 we can not conclude that M is Sasakian, because in the pseudo-Riemannian case the condition $h^2 = 0$ does not imply $h = 0$. In the Riemannian case, due to the fact that h is diagonalizable, these two conditions are equivalent. On the other hand, the papers [5] and [6] do not contain an example of contact pseudo-metric manifold satisfying the conditions $h^2 = 0$ and $h \neq 0$. In Sect. 4 of this paper we give examples (in dimension five) of such contact pseudo-metric manifolds.

In Sect. 5 we study the pseudohermitian torsion τ of a non-degenerate almost CR manifold, because it is related to some interesting geometric properties (see Theorem 5.1).

2. Contact Pseudo-Metric Manifolds and Non-Degenerate CR Manifolds

2.1. Contact Pseudo-Metric Manifolds

In this Section we collect some basic facts about contact pseudo-metric manifolds. All manifolds are assumed to be connected and smooth. A $(2n+1)$-dimensional manifold M is said to be a *contact manifold* if it admits a global 1-form η, such that $\eta \wedge (d\eta)^n \neq 0$. Given such a form η, there exists a unique vector field ξ, called the *characteristic vector field* or the *Reeb vector field*, such that $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. A pseudo-Riemannian metric g is said to be an *associated metric* if there exists a tensor φ of type $(1,1)$, such that

$$\eta = \varepsilon g(\xi, \cdot), \qquad d\eta(\cdot, \cdot) = g(\cdot, \varphi \cdot), \qquad \varphi^2 = -I + \eta \otimes \xi,$$

where $\varepsilon = g(\xi, \xi) = \pm 1$. Then, (η, g, ξ, φ) (more briefly, (η, g)) is called a *contact pseudo-metric* (or *pseudo-Riemannian*) structure, and $(M, \eta, g, \xi, \varphi)$ a *contact pseudo-metric* (or *pseudo-Riemannian*) manifold [5]. The associated pseudo-Riemannian metric g satisfies

$$g(\varphi X, \varphi Y) = -(d\eta)(Y, \varphi X) = g(X, Y) - \varepsilon \eta(X)\eta(Y).$$

We denote by ∇ the Levi-Civita connection and by R the corresponding Riemann curvature tensor, given by

$$R_{XY} = -[\nabla_X, \nabla_Y] + \nabla_{[X,Y]}$$

The tensor $h = \frac{1}{2}\mathcal{L}_\xi \varphi$, where $\mathcal{L}$ denotes the Lie derivative, is self-adjoint and satisfies

$$\nabla \xi = -\varepsilon \varphi - \varphi h, \quad h\varphi = -\varphi h, \quad h\xi = 0, \tag{2.1}$$

Since h is self-adjoint and $h\varphi = -\varphi h$, then we get $\mathrm{trace}_g h = \mathrm{trace}_g h\varphi = 0$.

A contact pseudo-metric manifold (M, η, g) is said to be *K-contact* if ξ is a Killing vector field, or equivalently, $h = 0$. It is said to be *Sasakian* if the contact pseudo-Riemannian structure (η, g, ξ, φ) is *normal*, that is, satisfies $[\varphi, \varphi] + 2\mathrm{d}\eta \otimes \xi = 0$. This condition is equivalent to

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \varepsilon \eta(Y)X.$$

Any Sasakian manifold is K-contact, and the converse also holds when $n = 1$, that is, for three-dimensional spaces. Moreover by a result of [12], a K-contact pseudo-Riemannian manifold of dimension $2n+1$, is Sasakian if and only if

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X. \tag{2.2}$$

We may refer to [5–7,12], for more information about contact pseudo-Riemannian geometry.

2.2. Almost CR Structures

We proceed by recalling a few notions of CR and pseudohermitian geometry (cf., for example, [9]). Let M be a real $(2n+1)$-dimensional (connected) smooth manifold and let $(\mathcal{H}(M), J)$ be an almost CR structure on M. An almost CR structure is called *CR structure* if it is *integrable*, that is the following two conditions are satisfied

$$[JX, Y]+[X, JY] \in \mathcal{H}(M) \quad \text{and} \quad J\big([JX, Y]+[X, JY]\big) = [JX, JY]-[X, Y], \tag{2.3}$$

for any $X, Y \in \mathcal{H}(M)$. A *pseudohermitian structure* on M is a differentiable 1-form θ on M such that $\ker\theta = \mathcal{H}(M)$. A pseudohermitian almost CR structure $(\mathcal{H}(M), J, \theta)$ is called to be *nondegenerate* if the *Levi form* $L_\theta(X, Y) = (\mathrm{d}\theta)(X, JY), \quad X, Y \in \mathcal{H}(M)$ is a nondegenerate Hermitian form. In such case the pseudohermitian structure θ is a *contact form*, and the integrability condition (2.3) is equivalent to

$$J\big([JX, Y] + [X, JY]\big) = [JX, JY] - [X, Y], \quad X, Y \in \mathcal{H}(M). \tag{2.4}$$

Let $(M, \mathcal{H}(M), J, \theta)$ be a non-degenerate almost CR manifold. It is customary to extend J (the complex structure along $\mathcal{H}(M)$) to an endomorphism φ of the tangent bundle by requesting that $\varphi_{|\mathcal{H}(M)} = J$ and $\varphi(T) = 0$, where T is the Reeb vector field of θ. Then $\varphi^2 = -I + \theta \otimes T$. The *Webster metric* g_θ is given by

$$g_\theta(X, Y) = (d\theta)(X, JY), \;\; g_\theta(X, T) = 0, \;\; g_\theta(T, T) = \varepsilon(= \pm 1),$$

for any $X, Y \in \mathcal{H}(M)$. g_θ is a pseudo-Riemannian metric on M. In this case the synthetic object $(\varphi, \xi = -T, \eta = -\theta, g = g_\theta)$ is a contact pseudo-metric structure on M. Vice versa, a contact pseudo-metric structure (φ, ξ, η, g) defines a non-degenerate almost CR structure on M given by $(\mathcal{H}(M), J, \theta)$, where $\mathcal{H}(M) = \ker\eta$, $\theta = -\eta$ and $J = \varphi_{|\mathcal{H}(M)}$. So, we have the following

Proposition 2.1. *The notion of non-degenerate almost CR structure* $(\mathcal{H}(M), J, \theta)$ *is equivalent to the notion of contact pseudo-metric structure* (φ, ξ, η, g).

If the Levi-form L_θ is positive definite, the Webster metric g_θ (with $\varepsilon = 1$) is a Riemannian metric and "non-degenerate" is replaced by "strictly pseudocovexity", and then in such case $(\varphi, \xi = T, \eta = \theta, g = g_\theta)$ is a contact metric structure. However, the nondegeneracy is more natural in CR geometry because it is a *CR invariant* property, i.e. it is invariant under a transformation $\tilde{\theta} = f\theta$, where $f : M \to R - \{0\}$ is a smooth function. In fact, a simple calculation gives $L_{\tilde{\theta}} = fL_\theta$. Clearly, strictly pseudocovexity is not a CR invariant property (if L_θ is positive definite and $\tilde{\theta} = -\theta$, then $L_{\tilde{\theta}}$ is negative definite). Moreover, we note that the contact pseudo-metric structure (φ, ξ, η, g) and the reversed contact pseudo-metric structure $(\tilde{\varphi} = \varphi, \tilde{\xi} = -\xi, \tilde{\eta} = -\eta, \tilde{g} = -g)$ induce the same almost CR structure $(\mathcal{H}(M), J)$.

Tanaka [13] defined the canonical linear connection, called the *Tanaka-Webster* connection, on a non-degenerate CR manifold, that is, on an integrable non-degenerate almost CR manifold. For a contact pseudo-metric manifold $(M, \varphi, \xi, \eta, g)$, equivalently for a non-degenerate almost CR structure, the *generalized Tanaka-Webster* connection $\hat{\nabla}$ is given by

$$\hat{\nabla}_X Y = \nabla_X Y + \varepsilon\,\eta(X)\,\varphi(Y) - \eta(Y)\,\nabla_X\xi + \{(\nabla_X\eta)Y\}\,\xi \tag{2.5}$$

for any $X, Y \in \mathfrak{X}(M)$, where ∇ is the Levi-Civita connection of (M, g). The definition of $\hat{\nabla}$ by (2.5) is due to S. Tanno [14] (though confined to the positive definite case). In particular, from (2.5), one gets

$$(\hat{\nabla}_X\varphi)Y = (\nabla_X\varphi)Y + \{(\nabla_X\eta)\varphi Y\}\,\xi + \eta(Y)\,\varphi(\nabla_X\xi). \tag{2.6}$$

3. Proof of Theorem 1.2

Let $(M, \eta, g, \xi, \varphi)$ be an integrable contact pseudo-metric manifold, $\dim M = 2n+1$, $n \geq 2$, of constant sectional curvature κ. Let $(\mathcal{H}(M), J, \theta)$ be the associated non-degenerate CR structure. By Theorem 1.1, M has constant sectional curvature $\kappa = \varepsilon$. By a result in [14], the integrability condition (2.4) of the underlying almost CR structure $(\mathcal{H}(M), J)$ is equivalent to the vanishing of $\hat{\nabla}\varphi$. Then by (2.6), we have

$$(\nabla_X\varphi)Y = -\{(\nabla_X\eta)\varphi Y\}\,\xi - \eta(Y)\,\varphi(\nabla_X\xi), \tag{3.7}$$

where

$$\{(\nabla_X\eta)\varphi Y\} = -g(X, Y) + \varepsilon\eta(X)\eta(Y) - \varepsilon g(hX, Y)$$

and

$$\varphi(\nabla_X\xi) = \varepsilon X - \varepsilon\eta(X)\xi + hX.$$

Thus, (3.7) becomes

$$(\nabla_X\varphi)Y = g(X + \varepsilon hX, Y)\xi - \varepsilon\eta(Y)(X + \varepsilon hX).$$

Consequently, for any $X, Y, Z \in \mathfrak{X}(M)$, we have

$$\begin{aligned}(\nabla^2_{X,Y}\varphi)Z &= (\nabla_X\nabla\varphi)(Y, Z)\\ &= \nabla_X\left((\nabla_Y\varphi)Z\right) - \left(\nabla_{\nabla_X Y}\varphi\right)(Z) - (\nabla_Y\varphi)\nabla_X Z\\ &= \nabla_X\left(g(Y + \varepsilon hY, Z)\xi\right) - g\left(\nabla_X Y + \varepsilon h\nabla_X Y, Z\right)\xi\\ &\quad - \left(g(Y + \varepsilon hY, \nabla_X Z)\xi - \varepsilon\eta(\nabla_X Z)(Y + \varepsilon hY)\right).\end{aligned}$$

For $Z \in \mathcal{H}(M)$, the above formula gives

$$\begin{aligned}(\nabla^2_{X,Y}\varphi)Z = g\left(\varepsilon(\nabla_X h)Y, Z\right)\xi - g(Y + \varepsilon hY, Z)(\varepsilon\varphi X + \varphi hX)\\ + g(\varepsilon\varphi X + \varphi hX, Z)(Y + \varepsilon hY).\end{aligned}$$

Then, we have

$$\begin{aligned}-(\nabla^2_{X,Y}\varphi)Z+(\nabla^2_{Y,X}\varphi)Z&=\varepsilon g\left((\nabla_Y h)X-(\nabla_X h)Y,Z\right)\xi\\&\quad+g(Y+\varepsilon hY,Z)(\varepsilon\varphi X+\varphi hX)-g(\varepsilon\varphi X+\varphi hX,Z)(Y+\varepsilon hY)\\&\quad-g(X+\varepsilon hX,Z)(\varepsilon\varphi Y+\varphi hY)+g(\varepsilon\varphi Y+\varphi hY,Z)(X+\varepsilon hX).\end{aligned} \tag{3.8}$$

On the other hand, the curvature tensor satisfies the identity

$$R(X,Y)\varphi=-(\nabla^2_{X,Y}\varphi)+(\nabla^2_{Y,X}\varphi) \tag{3.9}$$

where

$$(R(X,Y)\varphi)Z:=R(X,Y)\varphi Z-\varphi R(X,Y)Z.$$

Consequently, for $X,Y,Z\in\mathcal{H}(M)$, since the curvature tensor is given by $R(X,Y)Z=\varepsilon(g(X,Z)Y-g(Y,Z)X)$, (3.8) and (3.9) imply

$$g\left((\nabla_Y h)X-(\nabla_X h)Y,Z\right)=g(R(X,Y)\varphi Z,\xi)=0$$

and thus

$$\begin{aligned}&g(Y+\varepsilon hY,Z)(\varepsilon\varphi X+\varphi hX)-g(\varepsilon\varphi X+\varphi hX,Z)(Y+\varepsilon hY)\\&\quad-g(X+\varepsilon hX,Z)(\varepsilon\varphi Y+\varphi hY)+g(\varepsilon\varphi Y+\varphi hY,Z)(X+\varepsilon hX)\\&=R(X,Y)\varphi Z-\varphi R(X,Y)Z\\&=\varepsilon g(X,\varphi Z)Y-\varepsilon g(Y,\varphi Z)X-\varepsilon g(X,Z)\varphi Y+\varepsilon g(Y,Z)\varphi X.\end{aligned}$$

This last equation is equivalent to

$$\begin{aligned}&g(Y,Z)\varphi hX+g(hY,Z)\varphi X+\varepsilon g(hY,Z)\varphi hX\\&\quad-g(\varphi X,Z)hY-g(\varphi hX,Z)Y-\varepsilon g(\varphi hX,Z)hY\\&=+g(X,Z)\varphi hY+g(hX,Z)\varphi Y+\varepsilon g(hX,Z)\varphi hY\\&\quad-g(\varphi Y,Z)hX-g(\varphi hY,Z)X-\varepsilon g(\varphi hY,Z)hX\end{aligned} \tag{3.10}$$

for any $X,Y,Z\in\mathcal{H}(M)$. Now, we consider the tensor S of type $(1,3)$ on M defined, for any $X,Y,Z\in\mathfrak{X}(M)$, by

$$\begin{aligned}S(X,Y,Z):=&\,g(Y,Z)\varphi hX+g(hY,Z)\varphi X+\varepsilon g(hY,Z)\varphi hX\\&-g(\varphi X,Z)hY-g(\varphi hX,Z)Y-\varepsilon g(\varphi hX,Z)hY\\&-g(X,Z)\varphi hY-g(hX,Z)\varphi Y-\varepsilon g(hX,Z)\varphi hY\\&+g(\varphi Y,Z)hX+g(\varphi hY,Z)X+\varepsilon g(\varphi hY,Z)hX.\end{aligned} \tag{3.11}$$

From (3.10), we get $S(X,Y,Z)=0$ for any $X,Y,Z\in\mathcal{H}(M)$. In particular, fixed $X\in\mathcal{H}(M)$, the tensor $S_X(Y,Z):=S(X,Y,Z)$ satisfies

$$\mathrm{trace}_g\pi_{\mathcal{H}}S_X=0,$$

where $\pi_{\mathcal{H}}S_X$ is the restriction of S_X to $\mathcal{H}\times\mathcal{H}$. On the other hand, from (3.11) we obtain

$$\begin{aligned}\operatorname{trace}_g\pi_{\mathcal{H}}S_X &= 2n\varphi hX + (\operatorname{trace}_g h)\varphi X + \varepsilon(\operatorname{trace}_g h)\varphi hX\\ &\quad -h\varphi X - \varphi hX - \varepsilon h\varphi hX - \varphi hX - \varphi hX - \varepsilon\varphi h^2X\\ &\quad +(\operatorname{trace}_g\varphi)hX + (\operatorname{trace}_g(\varphi h))X + \varepsilon(\operatorname{trace}_g(\varphi h))hX.\end{aligned}$$

Since $h\varphi = -\varphi h$, we have $\operatorname{trace}_g h = \operatorname{trace}_g(\varphi h) = \operatorname{trace}_g\varphi = 0$. Moreover, by Theorem 1.1, $h^2 = 0$. Then

$$\operatorname{trace}_g\pi_{\mathcal{H}}S_X = 2(n-1)\varphi hX.$$

Therefore, we get $hX = 0$ for any $X \in \mathcal{H}(M)$, that is, the contact pseudo-metric manifold is K-contact. Moreover, because (M, g) has constant sectional curvature ε, $R(X, Y)\xi = \varepsilon g(X, \xi)Y - \varepsilon g(Y, \xi)X$, that is, (2.2) is satisfied. On the other hand, in [12] was proved that a K-contact pseudo-Riemannian manifold is Sasakian if and only if (2.2) is satisfied. This concludes the proof of Theorem 1.2.

4. Examples of Contact Pseudo-Metric Manifolds with $h^2 = 0$ and $h \neq 0$

The papers [5] and [6] do not contain an example of contact pseudo-metric manifold satisfying the conditions $h^2 = 0$ and $h \neq 0$. Now, we give examples (in dimension five) of such contact pseudo-metric manifolds.

Consider the space $M = \mathbb{R}^5(x_1, x_2, x_3, x_4, z)$ and two smooth functions $\alpha, \beta \in C^\infty(\mathbb{R}^5)$. We put $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, 3, 4$, and $\partial_z = \frac{\partial}{\partial z}$. Define the vector fields $V_i, = 1, \dots, 4, V_5 = \xi$, by

$$\sqrt{2}V_1 = \alpha\partial_1 + \partial_2 + \partial_3 - 2x_1\partial_z, \quad \sqrt{2}V_2 = \partial_1 - \beta\partial_2 - \partial_4 + 2x_2\partial_z,$$
$$\sqrt{2}V_3 = -\alpha\partial_1 + \partial_2 - \partial_3 + 2x_1\partial_z, \quad \sqrt{2}V_4 = \partial_1 + \beta\partial_2 + \partial_4 - 2x_2\partial_z, \quad \xi = \partial_z.$$

Such vector fields define a frame of vector fields on $\mathbb{R}^5$. We define a pseudo-Riemannian metric g of signature $(- - + + \pm)$ by

$$g(V_1, V_1) = g(V_2, V_2) = -1,\ g(V_3, V_3) = g(V_4, V_4) = 1,$$
$$g(V_i, V_j) = 0,\ i \neq j,\ g(\xi, \xi) = \varepsilon = \pm 1.$$

The 1-form defined by

$$\eta = \varepsilon g(\xi, \cdot),$$

satisfies $\eta(\partial_1) = \eta(\partial_2) = 0$, $\eta(\partial_3) = 2x_1$, $\eta(\partial_4) = 2x_2$, $\eta(\xi) = 1$. Thus η can be expressed in the form

$$\eta = 2x_1\mathrm{d}x_3 + 2x_2\mathrm{d}x_4 + \mathrm{d}z.$$

Now, define the tensor φ of type $(1, 1)$ by

$$\varphi(V_1) = -V_2, \quad \varphi(V_2) = V_1, \quad \varphi(V_3) = -V_4, \quad \varphi(V_4) = V_3, \quad \varphi(\xi) = 0,$$

equivalently,

$$\varphi(E_1) = E_2, \quad \varphi(E_2) = -E_1, \quad \varphi(E_3) = E_4, \quad \varphi(E_4) = -E_3, \quad \varphi(\xi) = 0,$$

where

$$E_1 = \frac{(V_2 + V_4)}{\sqrt{2}}, \quad E_2 = \frac{(V_1 + V_3)}{\sqrt{2}}, \quad E_3 = \frac{(V_1 - V_3)}{\sqrt{2}}, \quad E_4 = \frac{(V_4 - V_2)}{\sqrt{2}},$$

are null vector fields. We note that $\{E_1, E_2, E_3, E_4, \xi\}$ defines a frame of vector fields on $\mathbb{R}^5$ with

$$\eta(E_i) = 0,\ i = 1, \ldots, 4.$$

With respect to the frame of vector fields $\{E_1, \ldots, E_4, E_5 = \xi\}$, the 2-form $\mathrm{d}\eta = 2\mathrm{d}x_1 \wedge \mathrm{d}x_3 + 2\mathrm{d}x_2 \wedge \mathrm{d}x_4$ can be expressed in the form

$$\mathrm{d}\eta = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and the pseudo-Riemannian metric g by the matrix.

$$G = (g(E_i, E_j)) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon \end{pmatrix}.$$

Since

$$G\varphi = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{d}\eta,$$

we get that η is a contact form and (ξ, φ, η, g) is a contact pseudo-metric structure. Moreover, the tensor $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ satisfies

$$\begin{aligned}
2hE_1 &= [\xi, \varphi E_1] - \varphi\,[\xi, E_1] = [\partial_z, \partial_2] - \varphi\,[\partial_z, \partial_1] = 0,\\
2hE_2 &= [\xi, \varphi E_2] - \varphi\,[\xi, E_2] = [\partial_z, -\partial_1] - \varphi\,[\partial_z, \partial_2] = 0,\\
2hE_3 &= [\xi, \varphi E_3] - \varphi\,[\xi, E_3] = [\partial_z, \beta\partial_2 + \partial_4 - 2x_2\partial_z] - \varphi\,[\partial_z, \alpha\partial_1 + \partial_3 - 2x_1\partial_z]\\
&= (\beta - \alpha)_z\, E_2,\\
2hE_4 &= [\xi, \varphi E_4] - \varphi\,[\xi, E_4] = [\partial_z, -\alpha\partial_1 - \partial_3 + 2x_1\partial_z] - \varphi\,[\partial_z, \beta\partial_2 + \partial_4 + 2x_2\partial_z]\\
&= (\beta - \alpha)_z\, E_1.
\end{aligned}$$

Therefore, $h^2 = 0$. Moreover, $h = 0$, that is the structure is K-contact, if and only if $(\beta - \alpha)_z = 0$. So, taking the functions α, β such that $\alpha_z \neq \beta_z$, we obtain a contact pseudo-metric structure with $h^2 = 0$ and ξ not Killing.

5. Remarks on the Pseudohermitian Torsion

Let $(M, \mathcal{H}(M), J, \theta)$ be a non-degenerate almost CR manifold. Denote by $(M, \eta, g, \xi, \varphi)$ the associated contact pseudo-metric structure. The *pseudohermitian torsion* of $\hat{\nabla}$ is the vector valued 1-form τ defined by

$$\tau(X) = \hat{T}(\xi, X) \quad \text{for any } X \in \mathfrak{X}(M), \tag{5.1}$$

where $\hat{T}$ is the torsion of $\hat{\nabla}$ (cf., for example, [13], and [9] p. 36). If the integrability condition (2.3) of the underlying almost CR structure $(\mathcal{H}, J)$ is satisfied, the linear connection $\hat{\nabla}$ is the ordinary Tanaka-Webster connection of $(M, \mathcal{H}(M), J, \theta, g_\theta)$ with $\varepsilon = +1$.

Let (M, g) be a pseudo-Riemannian manifold, $p \in M$ and X a lightlike vector of T_pM. A plane P of T_pM is called *lightlike plane* (or, *null plane*) directed by X if $X \in P$, $g(X, Y) = 0$ for any $Y \in P$, and there exists $Y_0 \in P$ such that $g(Y_0, Y_0) \neq 0$. In such case, following [10] p. 95, define the *lightlike sectional curvature* (or, the *null sectional curvature*) of P with respect to X by $K_X(P) = g(Y, Y)R(X, Y, X, Y)$, where Y is an arbitrary non-null vector of P. Now, given a contact pseudo-metric structure (η, g, ξ, φ) on M, in the sequel, for $X \in \ker\eta$, by $K(\xi, X)$ we denote the usual sectional curvature if $P = span(\xi, X)$ is a non-degenerate plane, and the lightlike sectional curvature $K_X(\xi) = \varepsilon R(\xi, X, \xi, X)$ if X is lightlike.

The following Theorem extends Corollary 1 of [8] (see p. 174) and Theorem 1.4 of [9] (see p. 37).

Theorem 5.1. *Let $(M, \mathcal{H}(M), J, \theta)$ be a non-degenerate almost CR manifold. Then,*

1) *the Levi distribution $\mathcal{H}(M)$ is minimal in (M, g_θ); moreover $\mathcal{H}(M)$ is totally geodesic in (M, g_θ) if and only if the pseudohermitian torsion τ vanishes, that is, the Reeb vector field ξ is Killing;*

2) *the following properties are equivalent:*
   a) $\nabla_\xi \tau = 0$;
   b) $\nabla_\xi h = 0$;
   c) $\nabla_\xi \hat{T} = 0$;
   d) $(\nabla_\xi \hat{T})(\xi, \cdot) = 0$;
   e) $K(\xi, X) = K(\xi, \varphi X)$ *for any* $X \in \mathcal{H}(M)$.

Proof. We recall that given a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$ and a smooth distribution $\mathcal{D} : p \mapsto \mathcal{D}_p \subset T_p\bar{M}$ on $\bar{M}$, then D is called minimal if $\text{trace}_{\bar{g}}(B) = 0$, where $B(X,Y) = (\bar{\nabla}_X Y)^\perp$ for any $X, Y \in \mathcal{D}$, $\bar{\nabla}$ denotes the Levi-Civita connection of $\bar{M}$ and $(\bar{\nabla}_X Y)^\perp$ is the natural projection on $\mathcal{D}^\perp$. Moreover, the distribution $\mathcal{D}$ is totally geodesic if the symmetrized second fundamental form $B_s(X,Y) := (1/2)\big(B(X,Y) + B(Y,X)\big)$ vanishes. Now, consider the non-degenerate almost CR manifold $(M, \mathcal{H}(M), J, \theta)$ and the associated contact pseudo-metric structure $(\theta, g = g_\theta, \xi = T, \varphi)$. Since we have the orthogonal decomposition $\mathfrak{X}(M) = \mathcal{H}(M) \oplus span(\xi)$, $B(X,Y)$ is the component of $\nabla_X Y$ on $span(\xi)$ for any $X, Y \in \mathcal{H}(M)$. Then

$$B(X,Y) = \varepsilon\, g(\nabla_X Y, \xi)\xi,$$

and by using (2.1) and a local orthonormal basis $\{E_i\}$, we get

$$\begin{aligned}
\text{trace}_g(B) &= \varepsilon \sum_{i=1}^{2n} \varepsilon_i\, g(\nabla_{E_i} E_i, \xi)\xi = -\varepsilon \sum_{i=1}^{2n} \varepsilon_i g(\nabla_{E_i}\xi, E_i)\xi \\
&= -\varepsilon \sum_{i=1}^{2n} \varepsilon_i\, g(-\varepsilon\varphi E_i - \varphi h E_i, E_i)\xi \\
&= \varepsilon(\text{trace}_g \varphi h)\xi.
\end{aligned}$$

Since $\text{trace}_g(\varphi h) = 0$, we get $\text{trace}_g(B) = 0$.

Next, we compute the torsion tensor $\hat{T}$. By using (2.5), we get

$$\hat{T} = \varphi h \otimes \eta + \eta \otimes h\varphi + 2(\mathrm{d}\eta) \otimes \xi. \tag{5.2}$$

Then (5.2) implies $\tau(X) = \hat{T}(\xi, X) = h\varphi X$, and consequently $\text{trace}_g(\tau) = \varepsilon g(\text{trace}_g(B), \xi) = 0$. Moreover, by using (2.1), the symmetrized second fundamental form is given by

$$\begin{aligned}
B_s(X,Y) &= \frac{\varepsilon}{2}\left\{g(\nabla_X Y, \xi) + g(\nabla_Y X, \xi)\right\}\xi \\
&= -\frac{\varepsilon}{2}\left\{g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)\right\}\xi \\
&= \varepsilon g(\varphi h X, Y)\xi \\
&= -\varepsilon g(\tau X, Y)\xi.
\end{aligned}$$

So, B_s vanishes if and only if the pseudohermitian torsion τ vanishes. To prove the second part in Theorem, since $\nabla_\xi \varphi = 0$, $\nabla_\xi \eta = 0$ and $\nabla_\xi(\mathrm{d}\eta) = 0$, from (5.2) we obtain

$$\nabla_\xi \hat{T} = \varphi \nabla_\xi h \otimes \eta + \eta \otimes (\nabla_\xi h)\varphi.$$

Then, $(\nabla_\xi \hat{T})(\xi, \xi) = 0$, $(\nabla_\xi \hat{T})(X, Y) = 0$ for any $X, Y \in \mathcal{H}(M)$, and

$$(\nabla_\xi \hat{T})(\xi, \cdot) = (\nabla_\xi h)\varphi = \nabla_\xi \tau.$$

So,

$$\nabla_\xi \hat{T} = 0 \Longleftrightarrow (\nabla_\xi \hat{T})(\xi, \cdot) = 0 \Longleftrightarrow \nabla_\xi h = 0 \Longleftrightarrow \nabla_\xi \tau = 0.$$

Next, consider the Jacobi operator

$$\ell(X) := R(X, \xi)\xi = -\nabla_X \nabla_\xi \xi + \nabla_\xi \nabla_X \xi + \nabla_{[X,\xi]}\xi.$$

Using (2.1), we get the following

$$\ell = -\varphi \nabla_\xi h + \varphi^2 + h^2 = \nabla_\xi \tau + \varphi^2 + \tau^2.$$

Then, for any $X \in \mathcal{H}(M)$:

$$\ell X = (\nabla_\xi \tau)(X) - X + \tau^2 X \quad \text{and} \quad \ell \varphi X = (\nabla_\xi \tau)(\varphi X) - \varphi X + \tau^2 \varphi X. \quad (5.3)$$

Thus, $\nabla_\xi \tau = 0$ implies $g(\ell X, X) = g(\ell \varphi X, \varphi X)$ and so the sectional curvature $K(\xi, X) = K(\xi, \varphi X)$ for any $X \in \mathcal{H}(M)$, where if X_p is null, $K(\xi_p, X_p)$ is a lightlike sectional curvature. Conversely, suppose $K(\xi, X) = K(\xi, \varphi X)$ for any $X \in \mathcal{H}(M)$. Then $g(\ell X, X) = g(\ell \varphi X, \varphi X)$ and hence, by using (5.3), $g((\nabla_\xi \tau)X, X) = 0$ for any $X \in \mathcal{H}(M)$. This gives $\nabla_\xi \tau = 0$. □

The above Theorem 5.1 and Theorem 4.1 of [12] give the following

Theorem 5.2. *Let $(M, \mathcal{H}(M), J, \theta)$ be a non-degenerate almost CR manifold. Then, the Webster metric g_θ is conformally flat and the Levi distribution $\mathcal{H}(M)$ is totally geodesic if and only if g_θ is of constant sectional curvature $c = g_\theta(\xi, \xi) = \varepsilon$ and the structure (θ, g_θ) is Sasakian.*

Now, we recall that there is a canonical way to associate a contact Lorentzian structure to a contact metric structure (and conversely). Let (η, g, ξ, φ) a contact metric structure on a smooth manifold M. Then,

$$g_L = g - 2\eta \otimes \eta$$

is a Lorentzian metric, and is still compatible with the same contact structure (η, ξ, φ), where the Reeb vector field ξ is time-like: $g_L(\xi, \xi,) = -1$. The Levi-Civita connection $\bar{\nabla}$ of g_L can be easily deduced from the Levi-Civita connection ∇ of g. More precisely, we have the following:

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + 2g(hX, \varphi Y)\xi + 2\{\eta(X)\varphi Y + \eta(Y)\varphi X\} \\ &= \nabla_X Y + 2g(\tau X, Y)\xi + 2\{\eta(X)\varphi Y + \eta(Y)\varphi X\}. \end{aligned} \quad (5.4)$$

Since $\tau = \frac{1}{2}(\mathcal{L}_\xi \varphi)\varphi$ is the pseudohermitian torsion for both the structures, from (5.4) we get

$$\bar{\nabla}_\xi \tau = \nabla_\xi \tau - 4\tau\varphi. \quad (5.5)$$

Example 5.3. Let (N, G) be a Riemannian manifold of dimension $n \geq 2$. Denote by (η, g, ξ, φ) the standard contact metric structure on the unit tangent sphere bundles T_1N, and by $(\eta, g_L, \xi, \varphi)$ the corresponding Lorentzian structure on T_1N. Suppose that $(N(-2), G)$ is a Riemannian manifold of constant sectional curvature $c = -2$, then by using the proof of Theorem 1 of [4] we easily deduce that

$$\nabla_\xi \tau = 4\tau\varphi.$$

Therefore, by using (5.5), we get that $(T_1N(-2), \eta, g_L, \xi, \varphi)$ is an example of contact pseudo-metric manifold satisfying the condition $\bar{\nabla}_\xi \tau = 0$

Example 5.4. Let $(M, \eta, g, \xi, \varphi)$ be a contact metric manifold which is a (κ, μ)-space, that is, its curvature tensor satisfies (cf. [2])

$$R(X, Y)\xi = \kappa(\eta(X)Y - \eta(Y)X) + \mu(\eta(X)hY - \eta(Y)hX),$$

for all tangent vector fields X, Y, where $\kappa, \mu \in \mathbb{R}$, $\kappa \leq 1$, and $\kappa = 1$ if and only if the space is Sasakian. For a (κ, μ)-space, we have ([2], Lemma 3.8)

$$\begin{aligned}(\nabla_X h)Y = &\{(1-\kappa)g(X, \varphi Y) + g(X, h\varphi Y)\}\,\xi + \eta(Y)h(\varphi X + \varphi hX) \\ &-\mu\eta(X)\varphi hY.\end{aligned}$$

Thus, the pseudohermitian torsion $\tau = h\varphi$ satisfies

$$\nabla_\xi \tau = \mu\,\tau\varphi.$$

Then, by using (5.5), for $\mu = 4$, $(M, \eta, g_L, \xi, \varphi)$ is an example of contact pseudo-metric manifold satisfying the condition $\bar{\nabla}_\xi \tau = 0$. Boeckx [3] gave explicit examples of (κ, μ)-spaces on Lie groups of dimension ≥ 5, with the contact metric structure left invariant, where

$$\kappa = 1 - \frac{(\beta^2 - \alpha^2)^2}{16}, \quad \mu = 2 + \frac{(\beta^2 + \alpha^2)}{2},$$

and α, β are real numbers satisfying $\beta^2 > \alpha^2$. So, for an appropriate choice of α and β we get $\mu = 4$.

References

[1] Blair, D.E.: Riemannian geometry of contact and symplectic manifolds. In: Progress in Math. vol. 203, 2nd edn. Birkhäuser, Boston (2010)

[2] Blair, D.E., Koufogiorgos, T., Papantoniou, B.J.: Contact metric manifolds satisfying a nullity condition. Israel J. Math. **91**, 189–214 (1995)

[3] Boeckx, E.: A full classification of contact metric (κ, μ)-spaces. Ill. J. Math. **44**, 212–219 (2000)

[4] Boeckx, E., Perrone, D., Vanhecke, L.: Unit tangent sphere bundles and two-point homogeneous spaces. Periodica Mathematica Hungarica **36**(2–3), 79–95 (1998)

[5] Calvaruso, G., Perrone, D.: Contact pseudo-metric manifolds. Differ. Geom. Appl. **28**, 615–634 (2010)

[6] Calvaruso, G., Perrone, D.: Erratum to: "Contact pseudo-metric manifolds" [Differential Geom. Appl. 28 (2010), 615–634], Differential Geom. Appl. (2013), doi:10.1016/j.difgeo.2013.05.010.

[7] Calvaruso, G., Perrone, D.: H-contact semi-Riemannian manifolds. J. Geom. Phys. **71**, 11–21 (2013)

[8] Dragomir, S.: On pseudohermitian immersion between strictly pseudoconvex manifolds. Am. J. Math. **117**(1), 169–202 (1995)

[9] Dragomir, S., Tomassini, G.: Differential geometry and analysis on CR manifolds. In: Progress in Math., vol. 246. Birkhäuser, Basel

[10] Duggal, K.L., Bejancu, A.: Lightlike submanifolds of semi-Riemannian manifolds and applications, vol. 364. Kluwer Academic Publishers, Dordrecht (1996)

[11] Olszak, Z.: On contact metric manifolds. Tôhoku Math. J. **31**, 247–253 (1979)

[12] Perrone, D.: Curvature of K-contact semi-Riemannian manifolds. Can. Math. Bull. (2014) doi:10.4153/CMB-2013-016-7

[13] Tanaka, N.: On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections. Jpn. J. Math. **2**, 131–190 (1976)

[14] Tanno, S.: Variational problems on contact Riemannian manifolds. Trans. Am. Math. Soc. **314**, 349–379 (1989)

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Received: July 15, 2013.
Accepted: February 10, 2014.