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Results in Mathematics

The Existence of Infinitely Many Solutions for the Nonlinear Schrödinger–Maxwell Equations

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Abstract. In this paper, by using variational methods and critical point theory, we shall mainly be concerned with the study of the existence of infinitely many solutions for the following nonlinear Schrödinger–Maxwell equations

$$\begin{cases} -\triangle u + V(x)u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\triangle \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where the potential V is allowed to be sign-changing, under some more assumptions on f, we get infinitely many solutions for the system.

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1. Introduction and Statement of the Main Result

In this paper, we study the Schrödinger–Maxwell equations

$$\begin{cases} -\triangle u + V(x)u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\triangle \phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.1}$$

Such a system is also called Schrödinger–Poisson equations, which arise in an interesting physical context. For a more physical background of system (1.1), we refer the readers to [11,15] and the references therein.

Since it was first introduced by Benci and Fortunato in [11], system (1.1) has been widely studied by many authors. The case $V \equiv 1$ or being radially symmetric, has been studied under various conditions on f in [2,13,15,25]. When $V(x)$ is not a constant, the existence of infinitely many large solutions for (1.1) has been considered in [4,12,22,28] via the fountain theorem (cf.[31,33]. For more results of system (1.1), we refer the reader to [17,20,21,23,26,34] and the references therein, for more results about applying critical point theory to second-order elliptic equations, we refer the reader to [3,6,7,9,10,14,18,19,24, 27,30] and the references therein.

In recent paper [22], the authors studied the existence of infinitely many nontrivial solutions of (1.1) under the following assumptions on V and f:

(V1) $V \in C(\mathbb{R}^3, \mathbb{R}), \inf_{x\in\mathbb{R}^3} V(x) \geq a_1 > 0$, where a_1 is positive constant. Moreover, for any $M > 0$, $meas\{x \in \mathbb{R}^3 \mid V(x) \leq M\} < \infty$.

(f1) $f \in C(\mathbb{R}^3 \times \mathbb{R}.\mathbb{R})$, and there exist $a_2 > 0, p \in (4, 2^*)$ such that

$$\mid f(x,u) \mid \leq a_2(1+ \mid u \mid^{p-1}) \quad \text{for all} \quad x \in \mathbb{R}^3, \quad u \in \mathbb{R},$$

where $2^* = 6$ is the critical exponent for the Sobolev embedding (see in [1]) in dimension 3. $f(x,u)u \geq 0$, for $u \geq 0$.

(f2) $\lim_{|u|\to 0} \frac{f(x,u)}{u} = 0$ uniformly for $x \in \mathbb{R}^3$.

(f3) $\lim_{|u|\to \infty} \frac{f(x,u)u}{u^4} = \infty$ uniformly for $x \in \mathbb{R}^3$.

(f4) For a.e. $x \in \mathbb{R}^3$, we have

$$G(x,s) \leq G(x,t), \qquad \forall (s,t) \in \mathbb{R}^+ \times \mathbb{R}^+, s \leq t,$$

where $G : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}$ is defined by $G(x,s) = \frac{1}{4}f(x,s)s - F(x,s)$, with $F(x,s) = \int_0^s f(x,t)dt$.

(f5) $f(x,-u) = -f(x,u)$ for any $x \in \mathbb{R}^3$, $u \in \mathbb{R}$.

Then the authors established the following theorem:

Theorem 1.1 [22]. *Under the assumptions (V1),(f1)–(f5),system* (1.1) *has infinitely many solutions* $\{(u_k, \phi_k)\}$ *such that when* $k \to \infty$, *we have*

$$\frac{1}{2}\int_{\mathbb{R}^3} (\mid \nabla u_k \mid^2 + V(x)u_k^2)dx - \frac{1}{4}\int_{\mathbb{R}^3} (\mid \nabla \phi_k \mid^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi_k u_k^2 dx - \int_{\mathbb{R}^3} F(x,u)dx \to \infty.$$

In the present paper, motivated by [29], *we shall further study the existence of infinitely many nontrivial solutions of* (1.1) *under the following assumptions.*

(V) $V \in C(\mathbb{R}^3, \mathbb{R}), \inf_{x\in\mathbb{R}^3} V(x) > -\infty$. Moreover, for any $M > 0$, $meas\{x \in \mathbb{R}^3 \mid V(x) \leq M\} < \infty$.

(f1′) $f \in C(\mathbb{R}^3 \times \mathbb{R}.\mathbb{R})$, and there exist $c_1, c_2 > 0, p \in (4, 2^*)$ such that

$$\mid f(x,u) \mid \leq c_1 \mid u \mid + c_2 \mid u \mid^{p-1} \quad \text{for all} \quad x \in \mathbb{R}^3, \quad u \in \mathbb{R},$$

where $2^* = 6$ is the critical exponent for the Sobolev embedding in dimension 3. $f(x,u)u \geq 0$, for $u \geq 0$.

(f2′) $\lim_{|u|\to\infty} \frac{F(x,u)}{u^4} = \infty$ uniformly for $x \in \mathbb{R}^3$, here and subsequently $F(x,u) = \int_0^u f(x,t)dt$.

(f3′) Let $\mathcal{F}(x,u) = \frac{1}{4}f(x,u)u - F(x,u)$, there exist $r_0 > 0$ such that if $\mid u \mid \geq r_0$, then $\mathcal{F}(x,u) \geq 0$ uniformly for $x \in \mathbb{R}^3$.

(f5) $f(x,-u) = -f(x,u)$ for any $x \in \mathbb{R}^3$, $u \in \mathbb{R}$.

Now, we are ready to state the main result of this paper.

Theorem 1.2. *Assume that (V) and (f1′)–(f3′),(f5) satisfy. Then system* (1.1) *possesses infinitely many nontrivial solutions.*

Remark 1.3. (i) It is obvious that condition (V) is weaker than (V1); (ii) There are functions f satisfying the conditions in Theorem 1.2 but not satisfying the assumptions(f2) and (f4), for example $f(x,u) = f(u) = u + |u|^{p-2}u$ where $p \in (4,6)$. Then one can easily check that f satisfies assumptions (f1′)–(f3′),(f5) but not satisfy (f2) and (f4). Meanwhile, we should point out that assumption (f2) is indispensable in many existing results, (see for instant in [12,22,21]).

2. Variational Setting and Proof of Theorem 1.2

Under the assumptions in Theorem 1.2, it is obvious that $F(x,u) \geq 0$ for all $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$. By (f1′) one can easily obtain that for all $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$:

$$F(x,u) \leq \frac{c_1}{2}u^2 + \frac{c_2}{p} \mid u \mid^p . \tag{2.1}$$

This implies that: there exist some $a_0 = a(r_0) > 0$ such that

$$\mid \mathcal{F}(x,u) \mid \leq a_0 \mid u \mid^2 \tag{2.2}$$

for all $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$ with $\mid u \mid \leq r_0$. Actually, for all $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$ with $\mid u \mid \leq r_0$, by (2.1), (f2′) and (f3′)

$$\begin{aligned} |\mathcal{F}(x,u)| &\leq \frac{1}{4}|f(x,u)u| + |F(x,u)| \\ &\leq \frac{1}{4}(u^2 + |u|^p) + \frac{c_1}{2}u^2 + \frac{c_2}{p}|u|^p \\ &\leq \frac{1+2c_1}{4}u^2 + \frac{p+4c_2}{4p}|u|^p \\ &\leq \left(\frac{1+2c_1}{4} + \frac{p+4c_2}{4p}r_0^{p-2}\right)u^2. \end{aligned} \tag{2.3}$$

Let $a_0 = \frac{1+2c_1}{4} + \frac{p+4c_2}{4p}r_0^{p-2}$, then (2.2) holds.

Throughout this section, we make the following assumption instead of (V):

(V′) $V \in C(\mathbb{R}^3, \mathbb{R}), \inf_{x\in\mathbb{R}^3} V(x) \geq a_0 + 1$, where a_0 is the same as in (2.2). Moreover, for any $M > 0$, $meas\{x \in \mathbb{R}^3 \mid V(x) \leq M\} < \infty$.

Now, let's introduce some notations. For any $1 \leq r < \infty$, $L^r(R^3)$ is the usual Lebesgue space with the norm

$$\|u\|_r = \left(\int\limits_{\mathbb{R}^3} |u|^r dx \right)^{\frac{1}{r}}.$$

$H^1(\mathbb{R}^3)$ is the usual Sobolev space with the norm

$$\|u\|_{H^1} = \left(\int\limits_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx \right)^{\frac{1}{2}}.$$

Define the space (see for instance in [31])

$$\mathcal{D}^{1,2} = \{u \in L^{2^*}(\mathbb{R}^3) \mid \quad \nabla u \in L^2(\mathbb{R}^3)\}$$

with the norm

$$\|u\|_{\mathcal{D}^{1,2}} = \left(\int\limits_{\mathbb{R}^3} |\nabla u|^2 \right)^{\frac{1}{2}}.$$

Then $\mathcal{D}^{1,2} \hookrightarrow L^{2^*}$, i.e. there exists some $C_0 > 0$ such that

$$\|u\|_6 \leq C_0 \|u\|_{\mathcal{D}^{1,2}}. \tag{2.4}$$

In the present paper, we work in the Hilbert space

$$E = \left\{ u \in H^1(\mathbb{R}^3) \ : \ \int\limits_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) dx < +\infty \right\}$$

equipped with the inner product

$$(u, v) = \int\limits_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv)\, dx, \qquad u, v \in E,$$

the associated norm

$$\|u\| = \left\{ \int\limits_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) dx \right\}^{1/2}, \qquad u \in E.$$

Evidently, E is continuously embedded into $H^1(\mathbb{R}^3)$ and hence continuously embedded into $L^r(\mathbb{R}^3)$ for $2 \leq r \leq 2^*$, i.e., there exists $S_r > 0$ such that

$$\|u\|_r \leq S_r \|u\|, \quad \forall\ u \in E. \tag{2.5}$$

In fact we further have the following lemma due to [8].

Lemma 2.1 [8]. *Under assumptions* (V') *the embedding from* E *into* $L^r(\mathbb{R}^3)$ *is compact for* $2 \leq r < 2^*$.

Especially, by (V′), we obtain

$$\|u\|_2^2 \leq \frac{1}{a_0+1}\|u\|^2, \quad \forall\, u \in E. \tag{2.6}$$

For every $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ (see [16]) such that

$$-\triangle\phi_u = u^2. \tag{2.7}$$

Moreover, ϕ_u has the following integral expression

$$\phi_u = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}dy.$$

Thus $\phi_u \geq 0$, from (2.4) and (2.7), for any $u \in E$ using Hölder inequality we have

$$\|\phi_u\|_{\mathcal{D}^{1,2}}^2 = \int_{\mathbb{R}^3} \phi_u u^2 dx \leq \|\phi_u\|_6 \|u\|_{\frac{12}{5}}^2 \leq C\|\phi_u\|_{\mathcal{D}^{1,2}}\|u\|_{\frac{12}{5}}^2.$$

Here and subsequently, C denotes an universal positive constant. This implies that

$$\|\phi_u\|_{\mathcal{D}^{1,2}} \leq C\|u\|_{\frac{12}{5}}^2. \tag{2.8}$$

By (2.5) and discussion above, we have

$$\int_{\mathbb{R}^3} \phi_u u^2 dx \leq C\|u\|_{\frac{12}{5}}^4 \leq C\|u\|^4. \tag{2.9}$$

Now we define a functional I on $E \times \mathcal{D}^{1,2}$ by

$$I(u,\phi) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx. \tag{2.10}$$

From the discussion above we know that I is well defined and $I \in C^1(E \times \mathcal{D}^{1,2})$, it is well known that I's critical points are the solutions of system (1.1). Moreover, by discussion above, for every $u \in E$ we obtain

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx = \int_{\mathbb{R}^3} \phi_u u^2 dx.$$

So, (2.9) can be reduced as the following form $\Phi : E \to \mathbb{R}$

$$\Phi(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx. \tag{2.11}$$

Then $\Phi \in C^1(E, \mathbb{R})$ and

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv + \phi_u uv - f(x,u)v)dx, \quad \forall v \in E. \tag{2.12}$$

Moreover if $u \in E$ is a critical point of Φ, then (u, ϕ_u) is a solution of system (1.1). To complete our proof, we have to cite a result in [29].

Lemma 2.2 [29]. *Assume that $p_1, p_2 > 1, r, q \geq 1$ and $\Omega \subseteq \mathbb{R}^N$. Let $g(x,t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$ and satisfy*

$$|g(x,t)| \leq a_1|t|^{(p_1-1)/r} + a_2|t|^{(p_2-1)/r}, \quad \forall\ (x,t) \in \Omega \times \mathbb{R}, \tag{2.13}$$

where $a_1, a_2 \geq 0$. If $u_n \to u$ in $L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$, and $u_n \to u$ a.e. $x \in \Omega$, then for any $v \in L^{p_1 q}(\Omega) \cap L^{p_2 q}(\Omega)$,

$$\lim_{n\to\infty} \int_{\Omega} |g(x,u_n) - g(x,u)|^r |v|^q dx = 0. \tag{2.14}$$

We remark that in the proof of Lemma 2.2, the following inequality (see for instance in [1]) plays an important role: If $1 \leq p < \infty$ and $a, b \geq 0$, then

$$(a+b)^p \leq 2^{p-1}(a^p + b^p). \tag{2.15}$$

We say that $I \in C^1(X, \mathbb{R})$ satisfies the *$(C)_c$-condition* if any sequence $\{u_n\}$ such that

$$I(u_n) \to c, \quad \|I'(u_n)\|(1 + \|u_n\|) \to 0$$

has a convergent subsequence, where X is a Banach space.

Lemma 2.3. *Assume that a sequence $\{u_n\} \subset E$, $u_n \rightharpoonup u$ in E as $n \to \infty$ and $\{\|u_n\|\}$ be a bounded sequence. Then*

$$\left| \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u)dx \right| \to 0, \quad as \quad n \to \infty.$$

Proof. Let $\{u_n\}$ be a sequence satisfying the assumptions $u_n \rightharpoonup u$ in E as $n \to \infty$ and $\{\|u_n\|\}$ is bounded. Lemma 2.1 implies that $u_n \to u$ in $L^r(R^3)$, where $2 \leq r < 6$, and $u_n \to u$ for a.e. $x \in \mathbb{R}^3$. Hence $\sup_{n\in\mathbb{N}} \|u_n\|_r < \infty$ and $\|u\|_r$ is finite. By Hölder inequality, (2.15), (2.4) and (2.8)

$$\begin{aligned}
\left| \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u)dx \right| &\leq \left(\int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} (u_n - u)^2 \right)^{\frac{1}{2}} \\
&\leq \left[2 \int_{\mathbb{R}^3} (|\phi_{u_n} u_n|^2 + |\phi_u u|^2) \right]^{\frac{1}{2}} \|u_n - u\|_2 \\
&\leq C(\|\phi_{u_n}\|_6^2 \|u_n\|_3^2 + \|\phi_u\|_6^2 \|u\|_3^2)^{\frac{1}{2}} \|u_n - u\|_2 \\
&\leq C(\|u_n\|^4 + \|u\|^4)^{\frac{1}{2}} \|u_n - u\|_2 \to 0, \qquad (2.16)
\end{aligned}$$

as $n \to \infty$. □

Lemma 2.4. *Under assumptions (V′), (f5) and (f1′)–(f3′), any sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c > 0, \quad \langle \Phi'(u_n), u_n \rangle \to 0,$$

is bounded in E. *Moreover,* $\{u_n\}$ *contains a converge subsequence.*

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$ as $n \to \infty$. Let $\Omega_n(0, r_0) = \{x \in \mathbb{R}^3 : |u_n(x)| \leq r_0\}$, by (2.2) and (2.6) for sufficiently large $n \in \mathbb{N}$

$$\begin{aligned}
c + 1 &\geq \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n \rangle \\
&= \frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3} \mathcal{F}(x, u_n)dx \\
&= \frac{1}{4}\|u_n\|^2 + \int_{\Omega_n(0,r_0)} \mathcal{F}(x, u_n)dx + \int_{\mathbb{R}^3 \setminus \Omega_n(0,r_0)} \mathcal{F}(x, u_n)dx \\
&\geq \frac{1}{4}\|u_n\|^2 - \int_{\Omega_n(0,r_0)} |\mathcal{F}(x, u_n)|dx \\
&\geq \frac{1}{4}\|u_n\|^2 - a_0\|u_n\|_2^2 \\
&\geq \frac{1}{4}\|u_n\|^2 - \frac{a_0}{4(a_0 + 1)}\|u_n\|^2 \\
&= \frac{1}{4(a_0 + 1)}\|u_n\|^2 \to +\infty.
\end{aligned}$$

Thus $\sup_{n \in \mathbb{N}} \|u_n\| < \infty$. i.e. $\{u_n\}$ is a bounded sequence.

Now we shall prove $\{u_n\}$ contains a subsequence, without loss of generality, by Eberlein–Shmulyan theorem (see for instance in [32]), passing to a subsequence if necessary, there exists a $u \in E$ such that $u_n \rightharpoonup u$ in E, again by Lemma 2.1 , $u_n \to u$ in $L^r(\mathbb{R}^3)$ for $2 \leq r < 6$ and $u_n \to u$ a.e. $x \in \mathbb{R}^3$. By Lemma 2.2

$$\int_{\mathbb{R}^3} |f(x, u_n) - f(x, u)||u_n - u|dx \to 0, \quad as \quad n \to \infty.$$

And it is obvious that

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \to 0 \quad as \quad n \to \infty.$$

This together with Lemma 2.3 implies

$$\|u_n - u\|^2 = \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle - \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u)dx + \int_{\mathbb{R}^3} (f(x,u_n) - f(x,u))(u_n - u)dx \to 0,$$

as $n \to \infty$. That is $u_n \to u$. □

Lemma 2.5. *Under assumptions (V′), (f1′) and (f2′), for any finite dimensional subspace $\tilde{E} \subset E$, there holds*

$$\Phi(u) \to -\infty, \qquad \|u\| \to \infty, \quad u \in \tilde{E}. \tag{2.17}$$

Proof. Arguing indirectly, assume that for some sequence $\{u_n\} \subset \tilde{E}$ with $\|u_n\| \to \infty$, there is $M > 0$ such that $\Phi(u_n) \geq -M$ for all $n \in \mathbb{N}$. Set $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E. Since $\tilde{E}$ is finite dimensional, then $v_n \to v \in \tilde{E}$ in E, $v_n \to v$ a.e. on $\mathbb{R}^N$, and so $\|v\| = 1$. Let $\Omega = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$, then $meas(\Omega) > 0$ and for a.e. $x \in \Omega$, we have $\lim_{n\to\infty} |u_n(x)| \to \infty$. Hence $\Omega \subset \mathbb{R}^3 \backslash \Omega_n(0, r_0)$ for sufficiently large $n \in \mathbb{R}$, where $\Omega_n(0, r_0)$ is the same as in Lemma 2.4. It follows from (2.9), (2.11), that

$$\lim_{n\to\infty} \frac{4\int_{\mathbb{R}^3} F(x,u_n)dx}{\|u_n\|^4} = \lim_{n\to\infty} \frac{2\|u_n\|^2 + \int_{\mathbb{R}^3} \phi_n u_n^2 - 4\Phi(u_n)}{\|u_n\|^4} \leq C. \tag{2.18}$$

But by the non-negative of F, (f2′) and Fadou's Lemma, for large n we have

$$\begin{aligned} \lim_{n\to\infty} \frac{4\int_{\mathbb{R}^3} F(x,u_n)dx}{\|u_n\|^4} &\geq \lim_{n\to\infty} \int_{\Omega} \frac{4F(x,u_n)v_n^4}{u_n^4} dx \\ &\geq \liminf_{n\to\infty} \int_{\Omega} \frac{4F(x,u_n)v_n^4}{u_n^4} dx \\ &\geq \int_{\Omega} \liminf_{n\to\infty} \frac{4F(x,u_n)v_n^4}{u_n^4} dx \\ &= \int_{\Omega} \liminf_{n\to\infty} \frac{4F(x,u_n)}{u_n^4} [\chi_\Omega(x)] v_n^4 dx \to \infty \end{aligned}$$

as $n \to \infty$. This contradicts to (2.18). □

Corollary 2.6. *Under assumptions (V′), (f1′) and (f2′), for any finite dimensional subspace $\tilde{E} \subset E$, there is $R = R(\tilde{E}) > 0$ such that*

$$\Phi(u) \leq 0, \qquad \forall\, u \in \tilde{E}, \quad \|u\| \geq R. \tag{2.19}$$

Let $\{e_j\}$ *is an orthonomormal basis of* E *and define* $X_j = \mathbb{R}e_j$,

$$Y_k = \oplus_{j=1}^{k} X_j, \qquad Z_k = \oplus_{j=k+1}^{\infty} X_j, \qquad k \in \mathbb{N}. \tag{2.20}$$

Lemma 2.7. *Under assumptions (V′), for* $2 \leq r < 2^*$,

$$\beta_k(r) := \sup_{u \in Z_k, \|u\|=1} \|u\|_r \to 0, \qquad k \to \infty. \tag{2.21}$$

Proof. Since the embedding from E into $L^r(\mathbb{R}^3)$ is compact, then Lemma 2.10 can be proved by a similar way as [31, Lemma 3.8].

By Lemma 2.7, we can choose an integer $m \geq 1$ such that

$$\|u\|_2^2 \leq \frac{1}{2c_1}\|u\|^2, \qquad \|u\|_p^p \leq \frac{p}{4c_2}\|u\|^p, \qquad \forall\, u \in Z_m. \tag{2.22}$$

□

Lemma 2.8. *Under assumptions (V′) and (f1′), there exist constants* $\rho, \alpha > 0$ *such that* $\Phi|_{\partial B_\rho \cap Z_m} \geq \alpha$.

Proof. By (2.1), (2.11) and (2.22), we have

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} F(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} F(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{c_1}{2}\|u\|_2^2 - \frac{c_2}{p}\|u\|_p^p \\ &\geq \frac{1}{4}(\|u\|^2 - \|u\|^p).\end{aligned}$$

Hence for any given $0 < \rho < 1$, let $\alpha := \frac{1}{4}(\rho^2 - \rho^p)$, then $\Phi|_{\partial B_\rho \cap Z_m} \geq \alpha > 0$. This complete the proof. □

By (V), there exists a constant $V_0 > 0$ such that $\bar{V}(x) := V(x) + V_0 \geq a_0 + 1 > 0$ for all $x \in \mathbb{R}^3$, where a_0 is the same as (2.2). Let $\bar{f}(x,u) = f(x,u) + V_0 u$. Then it is easy to verify the following lemma.

Lemma 2.9. *Problem* (1.1) *is equivalent to the following problem*

$$\begin{cases} -\triangle u + \bar{V}(x)u + \phi u = \bar{f}(x,u), & \text{in } \mathbb{R}^3, \\ -\triangle \phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{2.23}$$

Lemma 2.10 [5]. *Let* X *be an infinite dimensional Banach space,* $X = Y \oplus Z$, *where* Y *is finite dimensional. If* $I \in C^1(X,\mathbb{R})$ *satisfies* $(C)_c$*-condition for all* $c > 0$, *and*

(I1) $I(0) = 0$, $I(-u) = I(u)$ *for all* $u \in X$;
(I2) *there exist constants* $\rho, \alpha > 0$ *such that* $\Phi|_{\partial B_\rho \cap Z} \geq \alpha$;

(I3) *for any finite dimensional subspace* $\tilde{X} \subset X$*, there is* $R = R(\tilde{X}) > 0$ *such that* $I(u) \leq 0$ on $\tilde{X} \backslash B_R$;

then I *possesses an unbounded sequence of critical values.*

Proof of Theorem 1.2. Let $X = E, Y = Y_m$ and $Z = Z_m$. Obviously, $\bar{f}$ satisfies (f1′)–(f3′) and (f5). By Lemmas 2.4, 2.8 and Corollary 2.6, all conditions of Lemma 2.10 are satisfied. Thus, problem (2.23) possesses infinitely many nontrivial solutions. By Lemma 2.9, problem (1.1) also possesses infinitely many nontrivial solutions. □

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