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Results in Mathematics

Some Remarks on the Eigenvalue Multiplicities of the Laplacian on Infinite Locally Finite Trees

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Abstract. We consider the continuous Laplacian on an infinite uniformly locally finite network under natural transition conditions as continuity at the ramification nodes and the classical Kirchhoff flow condition at all vertices in a L^∞-setting. The characterization of eigenvalues of infinite multiplicity for trees with finitely many boundary vertices (von Below and Lubary, Results Math 47:199–225, 2005, 8.6) is generalized to the case of infinitely many boundary vertices. Moreover, it is shown that on a tree, any eigenspace of infinite dimension contains a subspace isomorphic to $\ell^\infty(\mathbb{N})$. As for the zero eigenvalue, it is shown that a locally finite tree either is a Liouville space or has infinitely many linearly independent bounded harmonic functions if the edge lengths do not shrink to zero anywhere. This alternative is shown to be false on graphs containing circuits.

Mathematics Subject Classifications. 34B45, 05C50, 05C10, 35J05, 34L10, 35P10.

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1. Introduction

We consider the continuous Laplacian on a uniformly locally finite network under natural transition conditions as continuity at the ramification nodes and the classical Kirchhoff flow condition at all vertices in a L^∞-setting. The main concern of the present paper is the occurrence of eigenvalues of infinite

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geometric multiplicity on trees, in particular their common occurrence and their corresponding combinatorial constraints.

The spectrum, especially the point spectrum, of the Laplacian on finite networks has been considered by many authors, see e.g. [1,2,4,5,17–19,22] and the references therein. For the infinite case we can refer to [7–10,13,20], for the finite algebraic graph theory to the monographs [12,14,15], while for the ℓ^2-setting in the infinite case we can refer to [21,24] and the references therein, and for the ℓ^∞-setting to [3,6–10].

The present paper is organized as follows. Some graph theoretical preliminaries, the basic node transition conditions for the Laplacian and some results about harmonic functions are summarized in Sects. 2 and 3. The following main result of Sect. 4 is the solution of a conjecture from [7] under an edge length constraint.

Theorem 4.1. *A uniformly locally finite tree satisfying* $\inf\{\ell_j \,|\, j \in \mathbb{N}\} > 0$ *either is a Liouville network or has infinitely many linearly independent bounded harmonic functions.*

Moreover, it is shown that the alternative is false for graphs containing circuits. In fact, for each $M \in \mathbb{N}$ there exists an infinite uniformly locally finite graph that has exactly M linearly independent bounded harmonic functions.

Section 5 is devoted to eigenvalues of the Laplacian of infinite multiplicity, in particular to the occurrence of inseparable eigenspaces containing a subspace isomorphic to $\ell^\infty(\mathbb{N})$. In fact, the following result will be shown.

Theorem 5.2. *Let T be an infinite uniformly locally finite tree and $\lambda \in [0,\infty)$ an eigenvalue of the Laplacian in $\mathcal{C}^2_K(T) \cap L^\infty(T)$. Then the geometric multiplicity of λ is infinite iff the corresponding eigenspace in $\mathcal{C}^2_K(T) \cap L^\infty(T)$ contains a subspace isomorphic to $\ell^\infty(\mathbb{N})$.*

It has been shown in [8] that the point spectrum is exactly $[0,\infty)$ for a uniformly locally finite infinite tree with all edge lengths equal to 1 and with finitely many boundary vertices. Without the latter condition, this assertion does no longer hold, but it can be shown that the point spectrum lies between $[0,\infty)$ and $[0,\infty)\backslash\left(\frac{\pi}{2}+\pi\mathbb{Z}\right)^2$ (Theorem 5.3), and that these bounds are optimal. Furthermore, the equivalence [8, 8.6] can be conceivably extended to the general case as follows.

Theorem 5.5. *Let T be a uniformly locally finite tree with all edge lengths equal to* 1*. Then the following conditions are equivalent:*

(a) There exists an eigenvalue $\lambda \in (0,\infty)\backslash\left(\frac{\pi}{2}+\pi\mathbb{Z}\right)^2$ of infinite geometric multiplicity.

(b) All $\lambda \in (0,\infty)\backslash\left(\frac{\pi}{2}+\pi\mathbb{Z}\right)^2$ are eigenvalues of infinite geometric multiplicity.

(c) The tree T has infinitely many vertices of internal degree at least 3*.*

Again this result is shown to be optimal by means of an example.

2. Preliminaries, Vertex Transition and Laplacian

For any graph $\Gamma = (V, E, \in)$, the vertex set is denoted by $V = V(\Gamma)$, the edge set by $E = E(\Gamma)$ and the incidence relation by $\in \subset V \times E$. The valency of each vertex v is denoted by $\gamma(v) = \mathrm{card}\{e \in E | \, v \in e\}$.

Unless otherwise stated, all graphs considered in this paper are assumed to be nonempty, simple, connected and *uniformly locally finite*, i.e.

$$\max_{v \in V(\Gamma)} \gamma(v) =: \gamma_{\max} < \infty. \tag{1}$$

The simplicity property means that Γ contains no loops, and at most one edge can join two vertices in Γ. Moreover, the conditions imply that Γ is countable. For a given numbering of the vertices v_i, $i \in \mathbb{N}$, set $\gamma_i = \gamma(v_i)$ and define the *adjacency matrix* or *adjacency operator* by

$$\mathcal{A}(\Gamma) = (e_{ih})_{i,h \in \mathbb{N}} : \mathbb{R}^{V(\Gamma)} \longrightarrow \mathbb{R}^{V(\Gamma)} \tag{2}$$

where

$$e_{ih} = \begin{cases} 1 & \text{if } v_i \text{ and } v_h \text{ are adjacent in } \Gamma \\ 0 & \text{else} \end{cases}$$

Note that $\mathcal{A}(\Gamma)$ is indecomposable iff Γ is connected. By simplicity, any two adjacent vertices v_i and v_h determine uniquely the edge e_s joining them, and we can set

$$s(i,h) = \begin{cases} s & \text{if } e_s \cap V = \{v_i, v_h\}, \\ 1 & \text{otherwise.} \end{cases}$$

The sequences or vectors with constant entries equal to 1 are denoted by $\mathbf{e}$. For a subgraph Θ in Γ let $\bar{\Theta} = (V(\Theta), E(\bar{\Theta}), \in)$ denote the subgraph of Γ spanned by the vertices in Θ with

$$E(\bar{\Theta}) = \{e | \, e \in E(\Gamma), \; e \cap V(\Gamma) \subset V(\Theta)\}.$$

The subgraph Θ is called *induced* if $\bar{\Theta} = \Theta$. The (combinatorial) distance between two vertices v_1 and v_2 is defined to be the minimal number of edges of all paths joining v_1 and v_2. For further graph theoretical terminology we refer to [16,23,25], and for the algebraic graph theory to [12,15].

Moreover, without loss of generality, we consider each graph as a connected *topological graph* in $\mathbb{R}^m$, i.e. $V(\Gamma) \subset \mathbb{R}^m$ and the edge set consists of a collection of Jordan curves

$$E(\Gamma) = \{\pi_j : [0, \ell_j] \to \mathbb{R}^m | \, j \in \mathbb{N}\}$$

with the following properties: each support $e_j := \pi_j\left([0, \ell_j]\right)$ has its endpoints in the set $V(\Gamma)$, any two vertices in $V(\Gamma)$ can be connected by a path with arcs in $E(\Gamma)$, and any two edges $e_j \neq e_h$ satisfy $e_j \cap e_h \subset V(\Gamma)$ and $\mathrm{card}(e_j \cap e_h) \leq 1$.

The arc length parameter of an edge e_j is denoted by t_j. Unless otherwise stated, we identify the graph $\Gamma = (V, E, \in)$ with its associated network

$$G = \bigcup_{j \in \mathbb{N}} \pi_j\left([0, \ell_j]\right),$$

especially each edge π_j with its support e_j. G is called a $\mathcal{C}^\nu$-network, if all $\pi_j \in \mathcal{C}^\nu([0, \ell_j], \mathbb{R}^m)$. We shall distinguish the *boundary vertices* $V_b = \{v_i \in V \,|\, \gamma_i = 1\}$ from the *ramification nodes* $V_r = \{v_i \in V \,|\, \gamma_i \geq 2\}$, especially, we define the *essential ramification nodes* by $V_{\text{ess}} = \{v_i \in V \,|\, \gamma_i \geq 3\}$. By definition, a *boundary edge* is one being incident to a boundary vertex. Moreover, introduce the *internal degree* of a vertex v by

$$\gamma^\circ(v) = \gamma(v) - \#\left\{w \in V_b \,|\, v \text{ is adjacent to } w\right\}.$$

The orientation of Γ or the network is given by the incidence factors

$$d_{ij} = \begin{cases} 1 & \text{if } \pi_j(\ell_j) = v_i, \\ -1 & \text{if } \pi_j(0) = v_i, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The *two-sided unbounded path* Γ_1 is the graph with $V(\Gamma_1) = \mathbb{Z}$ and the adjacency relation

$$e_{ik} = 1 \iff |i - k| = 1, \tag{4}$$

while the *one-sided unbounded path* Γ_0 is the induced subgraph for $V(\Gamma_0) = \mathbb{N}$. Numbering each edge by its left node, in both cases the edges are oriented by $d_{ii} = -1$ and $d_{i\ i+1} = 1$.

For a function $u : G \to \mathbb{R}$ we set $u_j := u \circ \pi_j : [0, \ell_j] \to \mathbb{R}$ and use the abbreviations

$$u_j(v_i) := u_j\left(\pi_j^{-1}(v_i)\right), \quad \partial_j u_j(v_i) := \frac{\partial}{\partial t_j} u_j(t_j)\Big|_{\pi_j^{-1}(v_i)} \quad \text{etc.}$$

As the basic geometric transition condition at ramification nodes we impose the *continuity condition*

$$\forall v_i \in V_r \,:\; e_j \cap e_s = \{v_i\} \implies u_j(v_i) = u_s(v_i), \tag{5}$$

that clearly is contained in the condition $u \in \mathcal{C}(G)$. Moreover, at all vertices we impose the classical Kirchhoff condition

$$\forall i \in \mathbb{N} \,:\; \sum_{j \in \mathbb{N}} d_{ij} \partial_j u_j(v_i) = 0, \tag{6}$$

that includes the Neumann boundary condition at boundary vertices. Note that Condition (6) does not depend on the orientation.

In the present context we consider the Laplacian Δ on a $\mathcal{C}^2$-network G defined as the operator

$$\Delta = \Delta_G^K = \left(u \mapsto \left(\partial_j^2 u_j\right)_{j \in \mathbb{N}}\right),$$

with the domain

$$\mathcal{C}^2_K(G) = \{u \in \mathcal{C}(G) \cap L^\infty(G) \,|\, \forall j \in \mathbb{N} : u_j \in \mathcal{C}^2([0, \ell_j]),\, u \text{ satisfies } (6)\}.$$

We might also consider Δ in a weighted Sobolev space setting as e.g. $W^{2,\infty}_{K,c}(G)$, but, due to classical regularity results in one dimension, working in spaces of continuous functions does not constitute a restriction. Including the vertex transition conditions, the eigenvalue problem for $\Delta = \Delta^K_G$ in question reads

$$0 \neq u \in \mathcal{C}^2_K(G) \cap L^\infty(G) \quad \text{and} \quad \partial^2_j u_j = -\lambda u_j \quad \text{for} \quad j \in N. \tag{7}$$

For the sake of simplicity, we shall use the following notations for the point spectra and the geometric multiplicities.

Definition 2.1.

$$\begin{aligned} \ell^\infty(\Gamma) &= \ell^\infty(V(\Gamma)) \\ \mathbf{S}(G) &= \sigma_p\left(-\Delta^K_G, \mathcal{C}^2_K(G) \cap L^\infty(G)\right) \\ M(\lambda) = M(\lambda; G) &= m_g\left(\lambda, -\Delta^K_G, \mathcal{C}^2_K(G) \cap L^\infty(G)\right) \end{aligned}$$

Let us recall some basic results for infinite graphs from [8–10]. All the eigenvalues of (7) are nonnegative. As in the finite case [2], the eigenvalues and their multiplicities can be determined in terms of the eigenvalues of the weighted transition operator $\mathrm{Diag}\left(\mathcal{L}(\Gamma)\ \mathbf{e}\right)^{-1} \mathcal{L}(\Gamma)$ resulting from the length adjacency operator $\mathcal{L}$, where $\mathrm{Diag}\left(\mathcal{L}(\Gamma)\ \mathbf{e}\right)$ denotes the diagonal matrix of the row sums of $\mathcal{L}$. In the case of equal edge lengths, the eigenvalues and their multiplicities are directly obtained from those of the transition operator $\mathrm{Diag}_i\left(\gamma_i\right)^{-1} \mathcal{A}(\Gamma)$ stemming from the adjacency operator $\mathcal{A}$, except for the ones satisfying $\sin\sqrt{\lambda} = 0$, whose multiplicities depend only on the corank and the parity of the graph.

3. Harmonic Functions

In this section we recall some basic facts and results about harmonic functions on networks from [7]. On networks with rectifiable edges that are not necessarily parametrized in a $\mathcal{C}^2$-manner, harmonic functions can be defined intrinsically without referring to the Laplacian. Conceivably, a continuous function $u : G \to \mathbb{R}$ is called *harmonic* if on each edge e_j it is of the form $t_j \mapsto \alpha_j t_j + b_j$ and satisfies the Kirchhoff condition (6). Of course, in the $\mathcal{C}^2$-case, they are just the functions satisfying $\Delta = 0$ on G. The set of all harmonic functions in G will be denoted by $\mathcal{H}(G)$ or $\mathcal{H}(\Gamma)$. Any element $u \in \mathcal{H}(\Gamma)$ is uniquely determined by its values in the vertices

$$\mathbf{x} = (x_i)_{i \in \mathbb{N}}, \quad x_i = u(v_i).$$

Using the slopes on the edges $\alpha_j = \frac{u_j(\ell_j)-u_j(0)}{\ell_j}$, the Kirchhoff condition (6) reads

$$\forall i \in \mathbb{N}\,:\, \sum_{j\in\mathbb{N}} d_{ij}\,\alpha_j = 0. \tag{8}$$

In turn, using the inverse length adjacency operator $\mathcal{P}(\Gamma) = (p_{ih})_{i,h\in\mathbb{N}} : \mathbb{R}^{V(\Gamma)} \longrightarrow \mathbb{R}^{V(\Gamma)}$ defined by $p_{ih} = e_{ih}\ell^{-1}_{s(i,h)}$, the latter condition becomes the mean value property

$$\forall i \in \mathbb{N}\,:\, \sum_{h\in\mathbb{N}} p_{ih}\,x_h = x_i \sum_{h\in\mathbb{N}} p_{ih}, \tag{9}$$

or, equivalently, $\mathcal{Z}\mathbf{x} := \mathrm{Diag}\,(\mathcal{P}(\Gamma)\,\mathbf{e})^{-1}\,\mathcal{P}(\Gamma)\,\mathbf{x} = \mathbf{x}$. It is easy to see that

$$m_g(1, \mathcal{Z}, \ell^\infty(V(\Gamma))) = \dim \mathcal{H}(G) \cap L^\infty(G) = m_g(0, \Delta, \mathcal{C}^2_K(G) \cap L^\infty(G)). \tag{10}$$

Definition 3.1. $M(0;\Gamma) = \dim \mathcal{H}(G) \cap L^\infty(G)$

A network G is called a *Liouville network*, if $M(0;\Gamma) = 1$. A *viaduct* in a graph Γ is a path π of length at least 2 in Γ joining two distinct vertices u and v such that there is no other walk in Γ joining u and v having a vertex in the set $V(\pi)\backslash\{u, v\}$.

Definition 3.2. The *reduced graph* Γ_{red} of a given graph Γ is constructed as follows. Introduce the operations

(I) Withdraw all edges in Γ incident to boundary vertices.
(II) Withdraw each one-sided unbounded path π in Γ whose ramification nodes $V_r(\pi)$ are all nodes of valency 2 in Γ.
(III) Replace any viaduct π in Γ by a single edge of length l, where l is the sum of the lengths of all edges of π.

Repeat (I) and (II) successively until there are no more vertices of valency 1 and no more one-sided unbounded paths as in (II) in the remaining graph. Then apply (III) such that there are no more vertices of valency 2. The resulting graph is called the *reduced graph* Γ_{red} of Γ.

Note that, in general, the reduced graph can display multiple edges, but for trees it is always a simple graph. If $\#V\,(\Gamma_{\mathrm{red}}) < \infty$, then Γ is a Liouville space, [7, 4.10].

4. Simplicity Versus Infinite Multiplicity for Bounded Harmonic Functions

It has been conjectured in [7, 5.10] that under the hypothesis

$$\inf\{\ell_j \,|\, j \in \mathbb{N}\} > 0, \tag{11}$$

a uniformly locally finite tree either is a Liouville space or has infinitely many linearly independent bounded harmonic functions. This will be confirmed. The

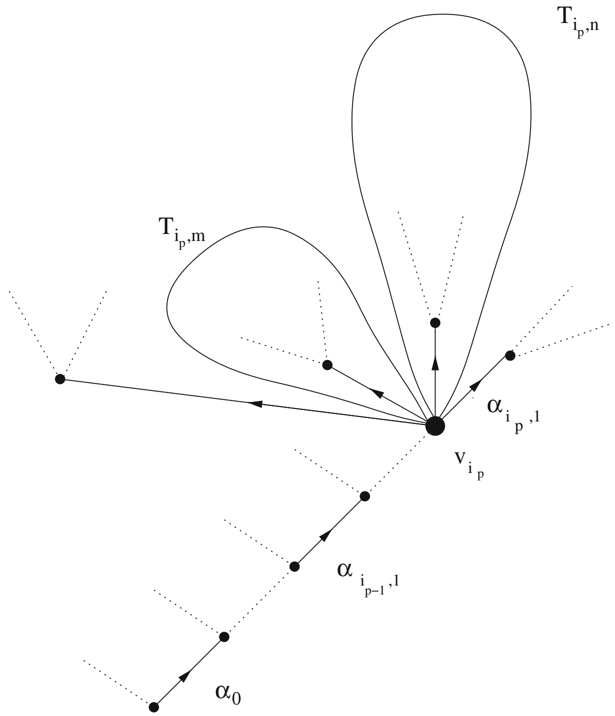

FIGURE 1. Construction of an independent harmonic at v_{i_p}

alternative holds in particular for trees with equal edge lengths. If the edge lengths shrink to 0, then the alternative can fail to hold. Take e.g. an infinite star graph Σ formed by $K \geq 3$ one-sided unbounded paths Γ_0 having their endpoints as common essential ramification node v_0. Now, let the edge lengths on each of the paths shrink successively by a given factor $0 < \lambda < 1$, starting with the K edges incident to v_0. The resulting graph is not a Liouville network and satisfies $M(0;\Sigma) = K - 1$, since the graph Σ corresponds to a finite star graph without transition condition at all boundary vertices.

Theorem 4.1. *A uniformly locally finite tree T satisfying Condition* (11) *either is a Liouville network or satisfies $M(0;T) = \infty$.*

Proof. Let $\mathbf{1}$ denote the constant harmonic function equal to 1 and suppose that $M(0;T) \geq 2$. We shall show that $M(0;T) = \infty$ by constructing a sequence $(u_k)_{k\in\mathbb{N}}$ of linearly independent nonconstant functions in $\mathcal{H}(T) \cap L^\infty(T)$ and $\mathcal{H}(T_{\text{red}}) \cap L^\infty(T_{\text{red}})$. By hypothesis, there exists such a function on T and T_{red}, say u_0. Let k_0 be an edge in T_{red} on which u_0 has non zero slope α_0 and set $v_0 := \pi_0(0)$. W.l.o.g. endow T_{red} with the orientation given by choosing v_0 as a source and by choosing the indegree to be 1 for all the remaining vertices:

$$\gamma_i^+ = \#\{j \in \mathbb{N} \mid d_{ij} = 1\} = 1, \quad \gamma_i^- = \#\{j \in \mathbb{N} \mid d_{ij} = -1\} = \gamma_i - 1.$$

By (6), there exists a one-sided unbounded path $\pi \cong \Gamma_0$ in T_{red} with initial node v_0 such that on each edge of π the slope α_i of u_0 does not vanish. Condition (11) and the boundedness of u_0 imply

$$\lim_{i\to\infty} \alpha_i = 0.$$

Set $i_1 := \min\{j > 0 \,|\, \alpha_j \neq \alpha_0\}$ and $\alpha_{i_1,1} := \alpha_{i_1}$. Let $k_{i_1,1}$ be the edge in π on which u_0 has slope $\alpha_{i_1,1}$ and set $v_{i_1} := \pi_{i_1,1}(0)$. The remaining outgoing incident edges at v_{i_1} will be denoted by $k_{i_1,2}, \ldots, k_{i_1,\gamma(v_{i_1})-1}$. Then (6) implies

$$\sum_{j=1}^{\gamma(v_{i_1})-1} \alpha_{i_1,j} = \alpha_0 \neq 0.$$

Thus, there exists $n, m \in \{1, \ldots, \gamma(v_{i_1}) - 1\}$ such that

$$n \neq m, \quad \alpha_{i_1,n} \neq 0, \ \text{ and } \ \alpha_{i_1,m} \neq 0.$$

Let $T_{i_1,n}$ and $T_{i_1,m}$ denote the subtrees obtained by cutting T_{red} at v_{i_1} that contain $k_{i_1,n}$ and $k_{i_1,m}$ respectively. Now we can define $u_1 \in \mathcal{H}(T_{\mathrm{red}}) \cap L^\infty(T_{\mathrm{red}})$ by setting

$$u_1\Big|_{T_{i_1,n}} := u_0\Big|_{T_{i_1,n}} - u_0(v_{i_1})\mathbf{1}$$

$$u_1\Big|_{T_{i_1,m}} := -\frac{\alpha_{i_1,n}}{\alpha_{i_1,m}} u_0\Big|_{T_{i_1,m}} + \frac{\alpha_{i_1,n}\ u_0(v_{i_1})}{\alpha_{i_1,m}}\mathbf{1}$$

and by extending u_1 by 0 outside $T_{i_1,n}$ and $T_{i_1,m}$. Then $\mathbf{1}, u_0$, and u_1 are linearly independent since $\lambda\mathbf{1} + \lambda_0 u_0 + \lambda_1 u_1 = 0$ with $\lambda, \lambda_i \in \mathbb{R}$ implies $\lambda = \lambda_0 = 0$ as u_1 vanishes on k_0. Thus, $\lambda_1 = 0$.

Next suppose that for $p \geq 2$, $u_1, \ldots, u_{p-1} \in \mathcal{H}(T_{\mathrm{red}}) \cap L^\infty(T_{\mathrm{red}})$ are non constant such that $\mathbf{1}, u_0, \ldots, u_{p-1}$ are linearly independent with indices i_κ and supports $T_{i_\kappa,n}$ and $T_{i_\kappa,m}$ as above for $1 \leq \kappa \leq p-1$. As indicated in Fig. 1, we can repeat the above construction involving u_0 and $\pi \cong \Gamma_0$ and set $i_p := \min\{j > i_{p-1} : \alpha_j \neq \alpha_{i_{p-1}}\}$, $\alpha_{i_p,1} := \alpha_{i_p}$. Let $k_{i_p,1}$ be the edge on which u_0 has slope $\alpha_{i_p,1}$ and set $v_{i_p} := \pi_{i_p,1}(0)$. The remaining outgoing incident edges at v_{i_p} will be denoted by $k_{i_p,2}, \ldots, k_{i_p,\gamma(v_{i_p})-1}$. Again (6) implies

$$\sum_{j=1}^{\gamma(v_{i_p})-1} \alpha_{i_p,j} = \alpha_{i_{p-1}} \neq 0.$$

Thus, there exists $n, m \in \{1, \ldots, \gamma(v_{i_p}) - 1\}$ such that $n \neq m$, $\alpha_{i_p,n} \neq 0$ and $\alpha_{i_p,m} \neq 0$. Let $T_{i_p,n}$ and $T_{i_p,m}$ denote the subtrees obtained by cutting T_{red} at v_{i_p} that contain $k_{i_p,n}$ and $k_{i_p,m}$ respectively. Now we can define

$u_p \in \mathcal{H}(T_{\mathrm{red}}) \cap L^\infty(T_{\mathrm{red}})$ by setting

$$u_p\Big|_{T_{i_p,n}} := u_0\Big|_{T_{i_p,n}} - u_0(v_{i_p}), \mathbf{1}$$

$$u_p\Big|_{T_{i_p,m}} := -\frac{\alpha_{i_p,n}}{\alpha_{i_p,m}} u_0\Big|_{T_{i_p,m}} + \frac{\alpha_{i_p,n}\, u_0(v_{i_p})}{\alpha_{i_p,m}}\mathbf{1}$$

and by extending u_p by 0 outside $T_{i_p,n}$ and $T_{i_p,m}$. It remains to show that $\mathbf{1}, u_0, \ldots, u_p$ are linearly independent. For $\lambda, \lambda_0, \ldots, \lambda_p \in \mathbb{R}$ such that $\lambda\mathbf{1} + \lambda_0 u_0 + \ldots + \lambda_p u_p = 0$, by construction and induction hypothesis, we obtain successively $\lambda = \lambda_0 = \ldots \lambda_{p-1} = 0$, by evaluating on the edge $k_{i_{\kappa-1}}$. Finally, $\lambda_p = 0$. This permits to conclude. □

Next, we shall show that the assertion of Theorem 4.1 is false for graphs with circuits. In fact, for each given $m \geq 2$ we shall construct a uniformly locally finite graph G satisfying $M(0;T) = m$, see Fig. 2. First, we use infinitely many copies of the complete bipartite graph $K_{4,2}$ to construct a graph G satisfying $M(0;G) = 2$. Define a graph H as follows. Number the vertices of H by $V(H) = \bigcup_{k\geq 1} V^{(k)}$, where the *k-th floor* is denoted by

$$V^{(k)} = \{v_i^{(k)} \,|\, 1 \leq i \leq 2^k\}.$$

The adjacencies in H are defined by the following rules:

(a) There is no edge between two vertices on the same floor or between two floors of level difference at least 2.

(b) For $k \geq 1$, the adjacencies between the k-th floor and the $(k+1)$-th floor are defined by 2^{k-1} copies of $K_{4,2}$ between $v_i^{(k)}, v_{2^k-i+1}^{(k)}$ and $v_{4i-3}^{(k+1)}, v_{4i-2}^{(k+1)}, v_{4i-1}^{(k+1)}, v_{4i}^{(k+1)}$ for $1 \leq i \leq 2^{k-1}$, see Fig. 2.

(c) The outgoing incident edges at $v_i^{(k)}$ will be denoted by $e_{i,j}^{(k)}$ for $1 \leq j \leq 4$.

Label the vertices of $K_{4,2}$ by the numbers 2 to 7 as displayed by bold edges in Fig. 2. Then glue two copies of H to a $K_{4,2}$ by identifying 4 and 5 with $v_1^{(1)}$ and $v_2^{(1)}$ respectively and, again, by identifying 6 and 7 with $v_1^{(1)}$ and $v_2^{(1)}$ respectively. Finally, add the node 1 and an edge between 1 and 2 and another one between 1 and 3 in order to obtain the graph G, see Fig. 2. All edge lengths in G are supposed to be equal to 1. Each edge is oriented by choosing the incident vertex of lower number or index as initial node. The edges between the vertices $\{1, 2, 3, 4, 5, 6, 7, 8\}$ are denoted by a_{ij}. Note that, by construction, $\gamma_{\max}(G) = 6$.

Lemma 4.2. $M(0;G) = 2$.

Proof. For a presumed harmonic function on G, let α_{ij} denote the slope on the edge a_{ij}, and $\alpha_{i,j}^{(k)}$ denote the slope on the edge $e_{i,j}^{(k)}$. We show that the harmonic functions vanishing in 1 are bounded and form a vector space of

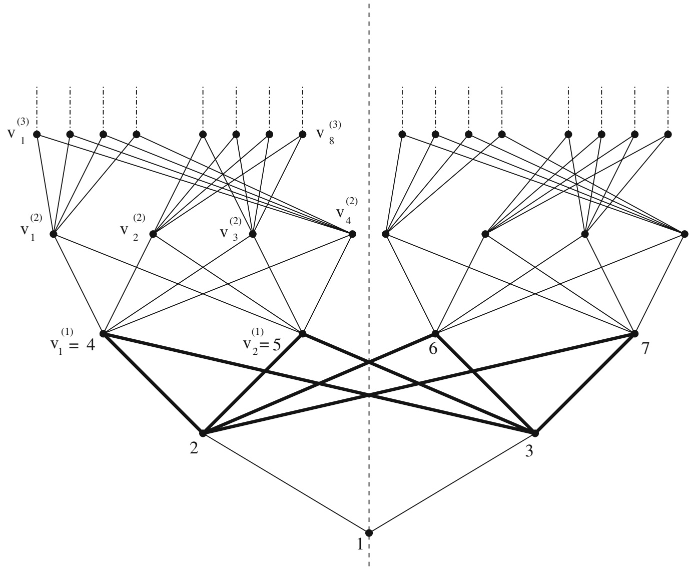

FIGURE 2. The graph G satisfying $M(0;G) = 2$

dimension 1. The Kirchhoff conditions at $1, 2, 3$ and the continuity at $4, 5, 6, 7$ yield

(i) $\alpha_{12} + \alpha_{13} = 0$,
(ii) $\alpha_{24} + \alpha_{25} + \alpha_{26} + \alpha_{27} = \alpha_{12}$ and $\alpha_{34} + \alpha_{35} + \alpha_{36} + \alpha_{37} = \alpha_{13}$,
(iii) $\alpha_{12} + \alpha_{2i} = \alpha_{13} + \alpha_{3i}$ for $4 \leq i \leq 7$.

Thus, summing up yields $\alpha_{12} = \alpha_{13} = 0$. By continuity $\alpha_{2i} = \alpha_{3i}$ for $4 \leq i \leq 7$. The Kirchhoff conditions at $4, 5$ and the continuity at $v_i^{(2)}$ yield

(iv) $\sum_{j=1}^{4} \alpha_{1,j}^{(1)} = 2\alpha_{24}$ and $\sum_{j=1}^{4} \alpha_{2,j}^{(1)} = 2\alpha_{25}$,
(v) $\alpha_{1,j}^{(1)} + \alpha_{24} = \alpha_{2,j}^{(1)} + \alpha_{25}$ for $1 \leq j \leq 4$.

Summing up yields $\alpha_{24} = \alpha_{25}$. Together with $\alpha_{24} + \alpha_{25} + \alpha_{26} + \alpha_{27} = 0$, we get

$$\alpha := \alpha_{24} = \alpha_{25} = -\alpha_{26} = -\alpha_{27},$$

and, by continuity and symmetry of G,

$$\alpha = \alpha_{34} = \alpha_{35} = -\alpha_{36} = -\alpha_{37},$$

We shall show that α determines a unique element in $\mathcal{H}(G)$ vanishing in 1. Using the above formulae and the symmetry of G, it suffices to consider only the graph H. As the essential step, we shall show by induction on k that the slopes between the floors $V^{(k)}$ and $V^{(k+1)}$ are all equal to $\frac{\alpha}{2^k}$. For that purpose, we first note that, by summing up all Kirchhoff conditions in each k-th floor, all sums of slopes of the same floor are equal:

$$\sum_{i=1}^{2}\sum_{j=1}^{4}\alpha_{i,j}^{(1)} = \sum_{i=1}^{2^k}\sum_{j=1}^{4}\alpha_{i,j}^{(k)} = 4\alpha.$$

For $k = 1$, using continuity at $v_i^{(2)}$ we have $\alpha_{1,j}^{(1)} = \alpha_{2,j}^{(1)}$ for $1 \leq j \leq 4$. By the Kirchhoff conditions at $v_1^{(2)}$ and $v_4^{(2)}$

$$\sum_{j=1}^{4}\alpha_{1,j}^{(2)} = 2\alpha_{1,1}^{(1)}, \quad \sum_{j=1}^{4}\alpha_{4,j}^{(2)} = 2\alpha_{1,4}^{(1)},$$

and by continuity at $v_i^{(3)}$ we have $\alpha_{1,j}^{(2)} + \alpha_{1,1}^{(1)} = a_{4,j}^{(2)} + \alpha_{1,4}^{(1)}$ for $1 \leq j \leq 4$. Summing up yields $\alpha_{1,1}^{(1)} = \alpha_{1,4}^{(1)}$ and, in the same way, $\alpha_{1,2}^{(1)} = \alpha_{1,3}^{(1)}$. This shows also $\alpha_{1,j}^{(2)} = \alpha_{4,j}^{(2)}$ and $\alpha_{2,j}^{(2)} = \alpha_{3,j}^{(2)}$ for $1 \leq j \leq 4$.

It remains to show that $\alpha_{1,1}^{(1)} = \alpha_{1,2}^{(1)}$. Summing up the Kirchhoff conditions in all $v_1^{(3)}, \ldots, v_4^{(3)}$ yields

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\alpha_{i,j}^{(3)} = \sum_{j=1}^{4}\alpha_{1,j}^{(2)} + \alpha_{4,j}^{(2)} = 2\sum_{j=1}^{4}\alpha_{1,j}^{(2)} = 4\alpha_{1,1}^{(1)},$$

and in all $v_5^{(3)}, \ldots, v_8^{(3)}$,

$$\sum_{i=5}^{8}\sum_{j=1}^{4}\alpha_{i,j}^{(3)} = 2\sum_{j=1}^{4}\alpha_{2,j}^{(2)} = 4\alpha_{1,2}^{(1)}.$$

The continuity condition in the 4-th floor gives for $1 \leq j \leq 4$:

$$\begin{aligned}
\alpha_{1,j}^{(3)} + \alpha_{1,1}^{(2)} + \alpha_{1,1}^{(1)} &= \alpha_{8,j}^{(3)} + \alpha_{2,4}^{(2)} + \alpha_{1,2}^{(1)}\\
\alpha_{2,j}^{(3)} + \alpha_{1,2}^{(2)} + \alpha_{1,1}^{(1)} &= \alpha_{7,j}^{(3)} + \alpha_{2,3}^{(2)} + \alpha_{1,2}^{(1)}\\
\alpha_{3,j}^{(3)} + \alpha_{1,3}^{(2)} + \alpha_{1,1}^{(1)} &= \alpha_{6,j}^{(3)} + \alpha_{2,2}^{(2)} + \alpha_{1,2}^{(1)}\\
\alpha_{4,j}^{(3)} + \alpha_{1,4}^{(2)} + \alpha_{1,1}^{(1)} &= \alpha_{5,j}^{(3)} + \alpha_{2,1}^{(2)} + \alpha_{1,2}^{(1)}
\end{aligned}$$

Thus, summing up these 16 equations yields

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\alpha_{i,j}^{(3)} + 4\sum_{j=1}^{4}\alpha_{1,j}^{(2)} + 16\alpha_{1,1}^{(1)} = \sum_{i=5}^{8}\sum_{j=1}^{4}\alpha_{i,j}^{(3)} + 4\sum_{j=1}^{4}\alpha_{2,j}^{(2)} + 16\alpha_{1,2}^{(1)},$$

and shows $\alpha_{1,1}^{(1)} = \alpha_{1,2}^{(1)}$. Thus, the assertion is shown for $k = 1$.

By induction, suppose that the slopes between the κ-th floor and the $(\kappa + 1)$-th floor are all equal for $1 \leq \kappa \leq k - 1$. Thus, all slopes between the floors $V^{(k-1)}$ and $V^{(k)}$ amount to $2^{-k+1}\alpha$, and by the Kirchhoff condition at $v_i^{(k)}$, the sum of the four outgoing slopes amounts to $2^{-k+2}\alpha$. Thus, it suffices to show that at each $v_i^{(k)}$, all the four outgoing slopes are the same. In fact, we can follow the proof of the case $k = 1$. By continuity at $v_i^{(k+2)}$ for $1 \le i \le 4$, we have

$$\alpha_{1,1}^{(k)} + \alpha_{1,j}^{(k+1)} = \alpha_{2^{k-1},4}^{(k)} + \alpha_{2^{k+1},j}^{(k+1)}.$$

Summing up yields $\alpha_{1,1}^{(k)} = \alpha_{2^{k-1},4}^{(k)}$, and correspondingly, $\alpha_{1,j}^{(k)} = \alpha_{2^{k-1},5-j}^{(k)}$ for $1 \le j \le 4$. Now we only have to sum up the Kirchhoff laws at $v_1^{(k+2)}, \ldots, v_4^{(k+2)}$ and proceed as above in order to obtain

$$\alpha_{1,j}^{(k)} = \alpha_{2^{k-1},5-j}^{(k)} = \frac{\alpha}{2^k} \quad \text{for} \quad 1 \le j \le 4.$$

This shows that each $u \in \mathcal{H}(G)$ is completely determined by its value at the node 1 and the slope $\alpha = \alpha_{24}$. Thus, $M(0; G) = 2$. $\square$

Now we take $m \ge 2$ copies $H_1, \ldots, H_m$ of the graph H, identify the nodes $v_1^{(1)}$ and $v_1^{(2)}$ of each H_i with two boundary vertices of a 3-star with ramification node $v_{i,1}$ and of edge length 1 and identify the remaining boundary vertex of each 3-star to one ramification node v_0 of valency m, that will be considered as a source. The resulting graph will be denoted by H^m and satisfies $\gamma_{\max}(H^m) = \max\{6, m\}$, see Figs. 3 and 4. It will be shown that $M(0; H^m) = m$.

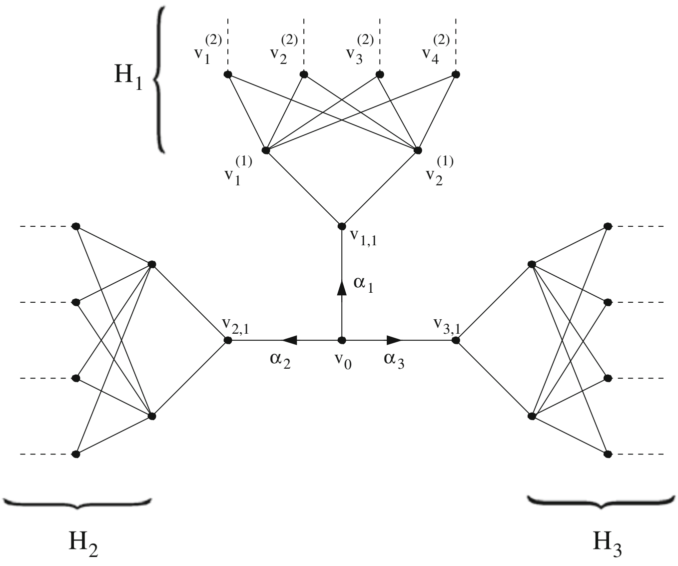

FIGURE 3. The graph H^3

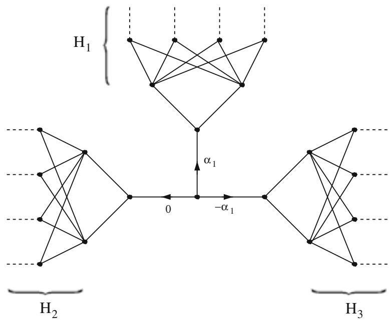

FIGURE 4. Nonconstant independent harmonic functions on H^3

Corollary 4.3. $M(0; H^m) = m$.

Proof. We show that a harmonic $u \in \mathcal{H}(H^m)$ is completely defined by its value in v_0 and $m-1$ slopes on the edges incident with v_0. Let $\alpha_1, \dots, \alpha_m$ denote the slopes of u on these edges. Thus, $\alpha_m = -\sum_{j=1}^{m-1} \alpha_j$. In H_1, we use the same notations for edges, vertices and slopes as for H above. Let α_{1j} denote the slope of u on the edge between $v_{1,1}$ and $v_j^{(1)}$, with $v_{1,1}$ as indicated in Fig. 3. Then the Kirchhoff law at $v_{1,1}, v_1^{(1)}, v_2^{(1)}$ and the continuity at $v_1^{(2)}, \dots, v_4^{(2)}$ yield

(i) $\alpha_1 = \alpha_{11} + \alpha_{12}$,

(ii) $\alpha_{11} = \sum_{j=1}^{4} \alpha_{1,j}^{(1)}$ and $\alpha_{12} = \sum_{j=1}^{4} \alpha_{2,j}^{(1)}$,

(iii) $\alpha_{11} + \alpha_{1,j}^{(1)} = \alpha_{12} + \alpha_{2,j}^{(1)}$ for $1 \le j \le 4$.

Thus, summing up yields $\alpha_{11} = \alpha_{12} = \frac{\alpha_1}{2}$. As in the proof of Lemma 4.2, the slopes of u between the k-th floor and $(k+1)$-th floor in H_1 are all equal to $\frac{\alpha_1}{2^{k+2}}$. The same holds on each H_i. For $1 \le i \le m-1$, let $w^{(i)}$ denote the unique harmonic bounded function on H^m with $w^{(i)}(v_0) = 0$ and having the slopes $\alpha_i = 1$, $\alpha_m = -1$ and $\alpha_j = 0$ for $j \notin \{i, m\}$. Then

$$u = u(v_0)\mathbf{1} + \sum_{i=1}^{m-1} \alpha_i w^{(i)} \in \left\langle \mathbf{1}, w^{(1)}, \dots, w^{(m-1)} \right\rangle_{\mathbb{R}},$$

which permits to conclude that $M(0; H^m) = m$, since the functions $\mathbf{1}, w^{(1)}, \dots, w^{(m-1)}$ are linearly independent. □

Theorem 4.1 can be extended to graphs with a finite number of circuits.

Corollary 4.4. *A uniformly locally finite graph* Γ *having only finitely many finite circuits and satisfying Condition* (11) *either is a Liouville network or satisfies* $M(0; \Gamma) = \infty$.

Proof. By hypothesis, Γ consists of a finite graph F containing all the circuits of Γ and of finitely many trees $T_1, \ldots, T_m$ with $\{v_0^i\} = V(T_i) \cap V(F)$. Increasing the parameter m if necessary, we can assume w.l.o.g. that for each $i \in \{1, \ldots, m\}$, the vertex v_0^i is a boundary vertex in T_i. If $m = 0$ then $M(0;\Gamma) = 1$. Thus, we can suppose $m \geq 1$. If one of the trees, say T_k, satisfies $M(0;T_k) = \infty$, then $M(0;\Gamma) = \infty$, as the harmonic functions on T_k define such functions on the whole graph by constant extension. Using Theorem 4.1, we can assume that all the T_k are Liouville graphs. If all $u \in \mathcal{H}(\Gamma) \cap L^\infty(\Gamma)$ are constant on each edge incident with v_0^i in T_i for all $i \in \{1, \ldots, m\}$, then $M(0;\Gamma) = 1$. If there is a function $u \in \mathcal{H}(\Gamma) \cap L^\infty(\Gamma)$ with non zero slope on the edge in T_k incident with v_0^k for some $k \in \{1, \ldots, m\}$, then the same construction as in the proof of Theorem 4.1 shows that there are non constant harmonic bounded functions on T_k though the restriction of u to T_k does not satisfy the Kirchhoff condition at $v_0^k \in V(T_k)$ in T_k. This is impossible and permits to conclude that Γ is a Liouville graph. $\square$

5. Infinite Eigenvalue Multiplicities on Trees

According to [9, Thm. 7.5] all eigenvalues on a uniformly locally finite tree are nonnegative and, according to [9, Thm. 7.3], a uniformly locally finite infinite tree T with finitely many boundary vertices satisfies $\mathbf{S}(T) = [0, \infty)$. On the other hand, *medusas* G, i.e. graphs with $\#V_{\text{ess}}(G) < \infty$, can only have finite eigenvalue multiplicities using the recurrences corresponding to homogeneous or inhomogeneous eigenvalue equations on the respective one-sided unbounded paths in G. In [8, Thm. 8.6] the following result has been shown for equal edge lengths, but the proof given there is readily extended to trees with arbitrary edge lengths.

Theorem 5.1. *Let T be a uniformly locally finite tree with at most a finite number of boundary vertices. Then the following conditions are equivalent:*

(a) $\exists \lambda \in (0, \infty) : M(\lambda; T) = \infty$
(b) $\forall \lambda \in (0, \infty) : M(\lambda; T) = \infty$
(c) $\#V_{\text{ess}}(T) = \infty$

Furthermore, in this class of trees, all trees with $V_{\text{ess}} = V_{\text{r}}$ are isospectral and have each nonnegative real number as *black hole* eigenvalue, cf. [10]. In fact, any eigenvalue of infinite multiplicity on an infinite uniformly locally finite tree is a *black hole* eigenvalue. This is the contents of the following theorem.

Theorem 5.2. *Let T be an infinite uniformly locally finite tree and $\lambda \in [0, \infty)$. Then $M(\lambda; T) = \infty$ iff the corresponding eigenspace in $\mathcal{C}_K^2(T) \cap L^\infty(T)$ contains a subspace isomorphic to $\ell^\infty(\mathbb{N})$.*

Proof. The proof consists of the following steps. Clearly, if the eigenspace contains some $\ell^\infty(\mathbb{N})$, then $M(\lambda; T) = \infty$. Suppose $M(\lambda; T) = \infty$.

(a) If T contains only finitely many essential ramification nodes, then all multiplicities are finite as stated above.

(b) If $V_{\mathrm{ess}} = V_{\mathrm{r}}$ and $\#V_{\mathrm{b}}(T) < \infty$, then the assertion follows from [10, Thm. 3.2]. Note that here, all eigenvalues are black holes.

(c) If $\#\{v \in V_r(T) \,|\, \gamma^\circ(v) \geq 3\} < \infty$, then T has a very special form: T consists of a finite tree F connected to finitely many one-sided unbounded paths Γ_0 that at each node allow additional boundary edges. Let us call these subtrees outside F, $\tau_1, \ldots, \tau_m$. If $\cos\sqrt{\lambda} \neq 0$, then the value and the slope at the initial node on each τ_k determines completely the eigensolution thereon which permits to conclude that $M(\lambda; T) < \infty$. Thus, we can assume that $\cos\sqrt{\lambda} = 0$. Then an eigenfunction vanishes at all nodes adjacent to boundary vertices, especially at the nodes of valency ≥ 3 on all τ_k. By hypothesis, one of the subtrees τ_k contains infinitely many essential ramification nodes that will be denoted by w_i following the Γ_0-orientation on τ_k. Then there are infinitely many eigenfunctions of compact support on τ_k. This can be seen as follows. If $\#\{w_i \,|\, \gamma(w_i) \geq 4\} = \infty$, then the assertion follows readily by using pairs of boundary edges as an appropriate cosinus support. Thus, by enlarging F to a suitable finite subgraph containing finitely many w_i with $\gamma(w_i) \geq 4$ if necessary, we can suppose that τ_k contains only nodes of degree 3, 2 or 1. But then the support of an eigenfunction cannot contain pairs of subsequent nodes w_i and w_{i+1} of odd distance. Thus, by infinite multiplicity, there must be infinitely many different pairs of subsequent ramification nodes w_i and w_{i+1} of even distance. These lead to infinitely many linearly independent eigenfunctions u_k of compact support as indicated in Fig. 5. Then a linear injection $\Phi : \ell^\infty \to \mathcal{C}^2_K(T) \cap L^\infty(T)$ is given by $\Phi(\alpha) = \sum_{k\in\mathbb{N}} \alpha_k\, u_k$ for $\alpha = (\alpha_k)_{k\in\mathbb{N}} \in \ell^\infty$.

(d) If $\#\{v \in V_r(T) \,|\, \gamma^\circ(v) \geq 3\} = \infty$ and $\cos\sqrt{\lambda} \neq 0$, then the same recursive construction as in the proof of [8, Thm. 8.4] or [9, Thm. 7.3] leads to the assertion.

(e) It remains to show the assertion in the case $\#\{v \in V_r(T) \,|\, \gamma^\circ(v) \geq 3\} = \infty$ and $\cos\sqrt{\lambda} = 0$. If there are three or more linearly independent eigenfunctions on a subtree Σ of T, then there is an eigenfunction having its support in Σ and vanishing on a given edge in Σ, since the multiplicity on each edge is at most 2. This leads to *irreducible* supports Σ of eigenfunctions that have at most two linearly independent eigenfunctions. Under the hypothesis $M(\lambda; T) = \infty$, there exists a family $\{\Sigma_k \,|\, k \in \mathbb{N}\}$ of irreducible subtrees and corresponding linearly independent eigenfunctions $\left\{u_k \in \mathcal{C}^2_K(T) \cap L^\infty(T) \,\middle|\, k \in \mathbb{N}\right\}$ such that each u_k has its support in Σ_k and $\|u_k\|_\infty \leq \frac{1}{1+k^2}$, and such that each Σ_k possesses an edge that all the others of the family do not have. Then, for each $\alpha = (\alpha_k)_{k\in\mathbb{N}} \in \ell^\infty$

$$\Phi(\alpha) = \sum_{k\in\mathbb{N}} \alpha_k\, u_k$$

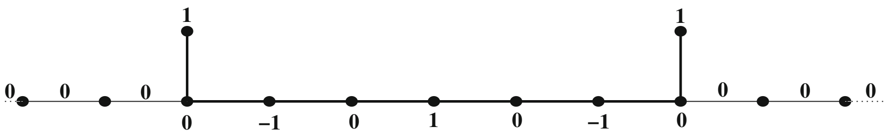

Figure 5. Defining an eigenfunction of compact support for $\cos\sqrt{\lambda} = 0$

defines an element of $\mathcal{C}^2_K(T) \cap L^\infty(T)$. Moreover, $\Phi : \ell^\infty \to \mathcal{C}^2_K(T) \cap L^\infty(T)$ is linear and injective by construction. This permits to conclude. $\square$

In the presence of different edge lengths, the eigenvalues depend strongly on the edge lengths in general. But in the case of equal edge lengths, all point spectra of uniformly locally finite infinite trees coincide outside of a common countable set of nonnegative numbers. This is the contents of the following

Theorem 5.3. *Let T be a uniformly locally finite infinite tree with all edge lengths equal to* 1*. Then*

$$[0, \infty) \supset \mathbf{S}(T) \supset [0, \infty) \backslash \left(\frac{\pi}{2} + \pi\mathbb{Z}\right)^2.$$

Proof. The first inclusion follows from [8,9]. For the second one and $\lambda \in [0, \infty) \backslash \left(\frac{\pi}{2} + \pi\mathbb{Z}\right)^2$, i.e. $\cos\sqrt{\lambda} \neq 0$, we can follow the construction of an eigenfunction from [8, Thm. 8.4]. By hypothesis, on each edge k_j incident to some boundary vertex, prescribing the value at the ramification node and the zero slope at the boundary vertex defines a unique eigenfunction restriction u_j on this edge. Moreover, at each ramification node v, the infinite and connected character of T ensures the existence of at least one adjacent ramification node. Thus, the cited iteration procedure works also in the present case. $\square$

Note that both inclusions are optimal. For uniformly locally finite infinite trees with finitely many boundary vertices $[0, \infty) = \mathbf{S}(T)$ holds, while for the infinite comb Z_1, i.e. $\tilde{Z}_2$ below without the boundary edge incident to the valencies 1 and 2, $\mathbf{S}(Z_1) = [0, \infty) \backslash \left(\frac{\pi}{2} + \pi\mathbb{Z}\right)^2$, see [8, Ex. 8.2]. In fact, adding a single edge, or doubling exactly one edge length can change the point spectrum from the minimal case to the maximal one.

Example 5.4. Let Z_2 be the infinite tree obtained by adding in the two-sided unbounded Γ_1 to each vertex one boundary edge such that all vertices of the resulting tree have valency 1 or 3 as depicted in Fig. 6, and such that one boundary edge,say at 0, has length 2, while all the other lengths amount to 1. Let $\tilde{Z}_2$ be the tree obtained by dividing in Z_2 the edge of length 2 into two edges of lengths 1. Then

$$\mathbf{S}(Z_2) = \mathbf{S}(\tilde{Z}_2) \supset [0, \infty) \backslash \left(2^{-1}\pi + \pi\mathbb{Z}\right)^2,$$

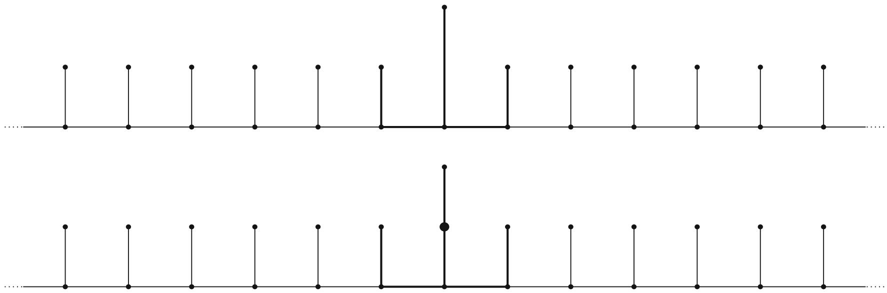

FIGURE 6. The trees Z_2 and $\tilde{Z}_2$

where the second inclusion follows from Theorem 5.3. But for $\cos\sqrt{\lambda} = 0$, Z_2 admits eigenfunctions of compact support in the subgraph that is displayed by bold edges in Fig. 6. This permits to conclude that $\mathbf{S}(Z_2) = \mathbf{S}(\tilde{Z}_2) = [0, \infty)$ while $\mathbf{S}(Z_1) = [0, \infty) \setminus \left(\frac{\pi}{2} + \pi\mathbb{Z}\right)^2$.

Let us come back to Theorem 5.1 under equal edge lengths that can be generalized to the case of infinitely many boundary nodes as follows, by taking into account the special role of the eigenvalues of the form $\cos\sqrt{\lambda} = 0$. In fact, these exceptional numbers stem from the iteration procedure [8, Thm. 8.4] and from the equation $\partial_j^2 u_j + \lambda u_j = 0$ on a boundary edge e_j of length 1 under inhomogeneous Dirichlet condition at the ramification node and Neumann condition at the boundary vertex. For $\cos\sqrt{\lambda} = 0$ it is not always possible to extend a partially defined eigenfunction to e_j.

Theorem 5.5. *Let T be a uniformly locally finite tree with all edge lengths equal to* 1. *Then the following conditions are equivalent:*

(a) $\exists \lambda \in (0, \infty) \setminus \left(\frac{\pi}{2} + \pi\mathbb{Z}\right)^2 : \ M(\lambda; T) = \infty$

(b) $\forall \lambda \in (0, \infty) \setminus \left(\frac{\pi}{2} + \pi\mathbb{Z}\right)^2 : \ M(\lambda; T) = \infty$

(c) $\#\{v \in V_r(T) \,|\, \gamma^\circ(v) \geq 3\} = \infty$

Proof. Note first that $\mathbf{S}(T) \supset [0, \infty) \setminus \left(\frac{\pi}{2} + \pi\mathbb{Z}\right)^2$ by Theorem 5.3. In order to conclude (a) $\Rightarrow$ (c) we observe that for trees T with finitely many internal nodes satisfying $\gamma^\circ(v) \geq 3$ the eigenvalue λ has finite multiplicity, since $\cos\sqrt{\lambda} \neq 0$ and since, again, T has the special form described already under c) in the proof on Theorem 5.2. Now we can follow the above argument.

It remains to show (c) $\Rightarrow$ (b). In fact, we can follow the construction in the proof of [8, Thm. 8.4]. The hypothesis on $\#\{v \in V_r(T) \,|\, \gamma^\circ(v) \geq 3\} = \infty$ guarantees that the recursive construction in the cited proof leads to infinitely many linearly independent bounded eigensolutions. $\square$

Note that this result can be extended neither to eigenvalues of the form $\cos\sqrt{\lambda} = 0$, nor to trees violating Condition (c). The same example as in [8] illustrates this.

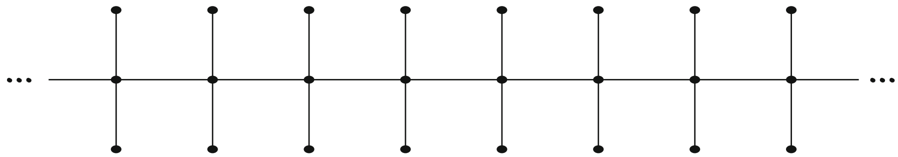

FIGURE 7. Example 5.6

Example 5.6. Let Γ be the infinite tree as depicted in Fig. 7. For $\lambda = 0$, at all boundary vertices, the slopes vanish, so any bounded harmonic function on Γ leads to such a function on Γ_1 and has to be constant, see also [7]. But in the general case, we readily deduce

$$M(\lambda) = \begin{cases} 1 & \text{if } \lambda = 0, \\ 2 & \text{if } \cos\sqrt{\lambda} \neq 0, \\ \infty & \text{if} \cos\sqrt{\lambda} = 0. \end{cases}$$

The example shows also that there are trees with eigenvalues of infinite multiplicity though $\{v \in V_r(T) \,|\, \gamma^\circ(v) \geq 3\} = \emptyset$.

Remark 5.7. We note in passing that for another common edge length, say $\ell > 0$, the exceptional values read $\left(\frac{1}{\ell}\left(\frac{\pi}{2} + \pi\mathbb{Z}\right)\right)^2$. Admitting different edge lengths can cause the impossibility of many real numbers to be eigenvalues. As explained above, the iteration procedure for the construction of a bounded eigenfunction might fail if $\lambda = \left(\frac{1}{\ell_j}\left(\frac{\pi}{2} + \pi k\right)\right)^2$ for some edge e_j incident to a boundary vertex with $k \in \mathbb{Z}$. Thus, avoiding all these numbers leads to eigenvalues for $-\Delta$. Thus, Theorem 5.3 can be generalized to the case of different edge lengths by the formula

$$[0,\infty) \supset \mathbf{S}(T) \supset [0,\infty) \backslash \bigcup_{j\in\mathbb{N}} \frac{1}{\ell_j^2}\left(\frac{\pi}{2} + \pi\mathbb{Z}\right)^2.$$

Nevertheless, the possible exception form a countable set. If $-\Delta$ is replaced by the canonical Laplacian

$$\Delta_G^K = \left(u \mapsto \left(\ell_j^2 \partial_j^2 u_j\right)_{j\in\mathbb{N}}\right),$$

under continuity in V_r and the Kirchhoff condition

$$\forall i \in \mathbb{N} : \sum_{j\in\mathbb{N}} d_{ij}\ell_j^2 \partial_j u_j(v_i) = 0,$$

then the assertions of Theorems 5.3 and 5.5 remain valid. Here the zeros of $\cos\sqrt{\lambda}$ play the same role as above.

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