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Results in Mathematics

Uniqueness Theorems for Sturm–Liouville Operators with Boundary Conditions Polynomially Dependent on the Eigenparameter from Spectral Data

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Abstract. In this paper, we discuss the inverse problems for Sturm–Liouville operators with boundary conditions polynomially dependent on the spectral parameter. We establish some uniqueness theorems on the potential $q(x)$ for the half inverse problem and the interior inverse problem from spectral data, respectively.

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Key words. Half inverse problem, interior inverse problem, boundary condition polynomially dependent on the spectral parameter, spectrum.

1. Introduction

Consider the following Sturm–Liouville operator $L := L(q, U_0, U_1)$ defined by

$$Ly = -y'' + q(x)y = \lambda y, \tag{1.1}$$

with boundary conditions

$$U_0(y) := R_{01}(\lambda)y'(0, \lambda) + R_{00}(\lambda)y(0, \lambda) = 0 \tag{1.2}$$

and

$$U_1(y) := R_{11}(\lambda)y'(\pi, \lambda) + R_{10}(\lambda)y(\pi, \lambda) = 0, \tag{1.3}$$

where q is a complex-value function and $q \in L^2(0, \pi)$,

$$R_{\xi k}(\lambda) = \Sigma_{j=0}^{r_{\xi k}} R_{\xi kj}\lambda^{r_{\xi k}-j}, r_{\xi 1} = r_{\xi 0} \geq 0, R_{\xi 10} = 1(\xi, k = 0, 1)$$

are arbitrary polynomials of degree $r_{\xi k}$ with complex coefficients such that $R_{\xi 1}(\lambda)$ and $R_{\xi 0}(\lambda)(\xi = 0, 1)$ have no common zeros.

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For the Sturm–Liouville problem (1.1)–(1.3), Chernozhukova and Freiling [1] established a uniqueness theorem on the potential $q(x)$. They showed that if coefficient functions $R_{0k}(\lambda)(k = 0, 1)$ of the boundary condition at $x = 0$ are known a priori, then the potential $q(x)$ and coefficient functions $R_{1k}(\lambda)(k = 0, 1)$ of the boundary condition at $x = \pi$ can be uniquely determined by Weyl function of this operator. By using the method of spectral mappings, Freiling and Yurko [2] discussed three inverse problems for the Sturm–Liouville problem (1.1)–(1.3) from the Weyl function, or from discrete spectral data or from two spectra and provided procedures for reconstructing this differential operator from the above spectral data, respectively. Later, using nodal points (zeros of eigenfunctions) as spectral data, Yang (C. F.) and Yang (X. P.) [3] reconstructed the potential $q(x)$ and coefficient functions $\frac{R_{\xi 0}(\lambda)}{R_{\xi 1}(\lambda)}(\xi = 0, 1)$ of the boundary conditions. In 1977, Fulton [4] considered the Sturm–Liouville equation (1.1) with one boundary condition dependent on the spectral parameter and obtained asymptotic estimates of eigenvalues or eigenfunctions. Since 1977, one of such Sturm–Liouville equation (1.1) with boundary conditions dependent on the spectral parameter was discussed by a number of authors (see [1–11]). Sturm–Liouville problem with eigenparameter dependent boundary conditions has many applications in engineering, physics, mathematics, etc (see [1–11]).

Inverse problem for differential operators consists in reconstructing operators from its spectral data (see [1–28,30,31]). Hochstadt and Lieberman [12] first considered the half inverse problem for the Sturm–Liouville operator with separated boundary conditions and showed that if $q(x)$ is prescribed on $[\frac{\pi}{2}, \pi]$, then the potential $q(x)$ on the interval $[0, \pi]$ can be uniquely determined by one spectrum. Later, Castillo [13] also discussed the half inverse problem for the Sturm–Liouville operator, and by an example, Castillo showed that the necessity of the boundary condition (1.3) for $R_{11}(\lambda) = 1, R_{10}(\lambda) = H$ is given. Then, one of such half inverse problems for differential operators was addressed by many authors (see, [11–19]). By using Weyl m-function techniques, Gesztesy and Simon [20] established a uniqueness theorem (see [20, Theorem 1.3]) by partial spectra and information on the potential, which is a generalization of Hochstadt–Lieberman's theorem [12]. While Mochizuki and Trooshin [21] explored the inverse problem for interior spectral data of Sturm–Liouville operators on the finite interval [0,1] and showed that a set of values of eigenfunctions at some interior point and parts of two spectra can uniquely determine the potential $q(x)$. Then, Yang(C. F.) and Yang(X. P.) [22] discussed the inverse problem for Sturm–Liouville operators with discontinuous boundary conditions and proved that the spectral data of parts of two spectra and some information on eigenfunctions at some interior point of the interval $(0, \pi)$ is sufficient to determine the potential $q(x)$. Later, Wang [10] established a uniqueness theorem for Sturm–Liouville operators with eigenparameter dependent boundary conditions from a set of values of eigenfunctions at some interior point and parts of two spectra. To the best of my knowledge, half inverse problem and interior inverse problem for the Sturm–Liouville problem (1.1)–(1.3) are not considered. In this paper, we discuss the half inverse problem and the interior inverse problem for the Sturm–Liouville problem (1.1)–(1.3), respectively. We always assume that coefficient functions $R_{00}(\lambda)$ and $R_{01}(\lambda)$ of the boundary condition at $x = 0$ are given a priori.

The aim of this article is to establish some uniqueness theorems for Sturm–Liouville equations with boundary conditions polynomially dependent on the spectral parameter on the finite interval $[0, \pi]$. From Lemma 2.1 in Sect. 2 (the result of Refs. [1] and [2]), we show that if $q(x)$ is prescribed on $[0, \frac{\pi}{2}]$, then one spectrum is sufficiently to determine the potential $q(x)$ on the finite interval $[0, \pi]$ and coefficient functions $R_{1k}(\lambda)(k = 0, 1)$ of the boundary condition. By improving Mochizuki–Trooshin's method and using Lemma 2.1, we prove that the potential $q(x)$ and coefficient functions $R_{1k}(\lambda)(k = 0, 1)$ of the boundary condition are uniquely determined by a set of values of eigenfunctions at some interior point and parts of two spectra.

This article is organized as follows. In Sect. 2, we present some preliminaries. In Sect. 3, we show that the uniqueness theorem of the half inverse problem for the Sturm–Liouville problem (1.1)–(1.3) holds. In Sect. 4, we prove some uniqueness theorems for the Sturm–Liouville problem (1.1)–(1.3) from a set of values of eigenfunctions at some interior point and parts of two spectra.

2. Preliminaries

Let $S_1(x, \lambda), S_2(x, \lambda), \varphi(x, \lambda)$ and $\psi(x, \lambda)$ be solutions of Eq. (1.1) under the initial conditions (see [1,2])

$$\begin{aligned} &S_1(0, \lambda) = S_2'(0, \lambda) = 0, S_1'(0, \lambda) = S_2(0, \lambda) = 1 \\ &\varphi(0, \lambda) = R_{01}(\lambda),\ \varphi'(0, \lambda) = -R_{00}(\lambda), \\ &\psi(\pi, \lambda) = R_{11}(\lambda),\ \psi'(\pi, \lambda) = -R_{10}(\lambda). \end{aligned}$$

Denote $\Delta_j(\lambda) = U_1(S_j)$. Clearly, $U_0(\varphi) = U_1(\psi) = 0$, and

$$\varphi(x, \lambda) = R_{01}(\lambda)S_2(x, \lambda) - R_{00}(\lambda)S_1(x, \lambda), \tag{2.1}$$

$$\psi(x, \lambda) = \Delta_1(\lambda)S_2(x, \lambda) - \Delta_2(\lambda)S_1(x, \lambda). \tag{2.2}$$

Let

$$\Delta(\lambda) =< \psi(x, \lambda), \varphi(x, \lambda) >, \tag{2.3}$$

where $< y(x), z(x) >:= yz' - y'z$ is the Wronskian of y and z. Then

$$\begin{aligned} \Delta(\lambda) &= R_{01}(\lambda)\Delta_2(\lambda) - R_{00}(\lambda)\Delta_1(\lambda) \\ &= U_1(\varphi) = -U_0(\psi), \end{aligned} \tag{2.4}$$

which is called the characteristic function of L. Let $\{\lambda_n\}_0^\infty$ be the zeros (counting with multiplicities) of the entire functions $\Delta(\lambda)$, when n sufficiently large, λ_n is simile.

Denote $\lambda=\rho^2$, then $S_1(x,\lambda)$ and $S_2(x,\lambda)$ can be rewritten as (see [30,31])

$$\begin{aligned} S_1(x,\lambda) &= \frac{\sin\rho x}{\rho}+\frac{1}{\rho}\int_0^x A(x,t)\sin(\rho t)dt,\\ S_2(x,\lambda) &= \cos\rho x+\int_0^x B(x,t)\cos(\rho t)dt, \end{aligned} \tag{2.5}$$

where the kernels $A(x,t)$ and $B(x,t)$ do not depend on λ and satisfy

$$\frac{\partial^2 A(x,t)}{\partial x^2}-q(x)A(x,t)=\frac{\partial^2 A(x,t)}{\partial t^2},$$

where $q(x)=2\frac{d}{dx}A(x,x), A(x,0)=0$ and

$$\frac{\partial^2 B(x,t)}{\partial x^2}-q(x)B(x,t)=\frac{\partial^2 B(x,t)}{\partial t^2},$$

where $q(x)=2\frac{d}{dx}B(x,x), B(0,0)=1, \frac{\partial B(x,t)}{\partial t}|_{t=0}=0$. Therefore,

$$\begin{aligned} S_1'(x,\lambda) &= \cos\rho x+A(x,x)\frac{\sin\rho x}{\rho}+\int_0^x \frac{\partial A(x,t)}{\partial x}\frac{\sin(\rho t)}{\rho}dt,\\ S_2'(x,\lambda) &= -\rho\sin\rho x+B(x,x)\cos\rho x+\int_0^x \frac{\partial B(x,t)}{\partial x}\cos(\rho t)dt. \end{aligned} \tag{2.6}$$

For sufficiently large $|\lambda|$, by virtue of (2.1), (2.2), (2.5), (2.6), this yields

$$\begin{aligned} \varphi(x,\lambda) &= \lambda^{r_{01}}\left(\cos\rho x+O\left(\frac{e^{\tau x}}{\rho}\right)\right),\\ \varphi'(x,\lambda) &= \lambda^{r_{01}}(-\rho\sin\rho x+O(e^{\tau x})), \end{aligned} \tag{2.7}$$

$$\begin{aligned} \psi(x,\lambda) &= \lambda^{r_{11}}\left(\cos\rho(\pi-x)+O\left(\frac{e^{\tau(\pi-x)}}{\rho}\right)\right),\\ \psi'(x,\lambda) &= \lambda^{r_{11}}(-\rho\sin\rho(\pi-x)+O(e^{\tau(\pi-x)})), \end{aligned} \tag{2.8}$$

where $\tau=|Im\rho|$.

By using (2.1), (2.4), (2.7), we can calculate

$$\Delta(\lambda)=\lambda^{r_{01}+r_{11}}(-\rho\sin\rho\pi+\omega\cos\rho\pi+\kappa_0(\rho)), \tag{2.9}$$

where

$$\kappa_0(\rho)=\int_0^\pi f_0(t)\cos(\rho t)dt+O\left(\frac{e^{\tau\pi}}{\rho}\right),\qquad f_0(t)\in L^2(0,\pi),$$

$$\omega=q_0-R_{000}+R_{100},\qquad q_0=\int_0^\pi q(t)dt.$$

Denote $G_\delta:=\{\rho||\rho-k|\geq\delta,k\in\mathbf{Z}\}$ for fixed $\delta>0$. From Ref. [2], we have

$$\rho_n=\sqrt{\lambda_n}=n-r_{01}-r_{11}+\frac{\omega}{n\pi}+\frac{\kappa_n}{n},\qquad \{\kappa_n\}\in l^2, \tag{2.10}$$

$$\Delta(\lambda)\geq C_\delta|\rho\lambda^{(r_{01}+r_{11})}|e^{\tau\pi},\qquad \lambda\in G_\delta,\qquad |\lambda|\geq\lambda^*. \tag{2.11}$$

Let $\Phi(x, \lambda)$ be the solution of Eq. (1.1) satisfying the boundary conditions $U_0(\Phi) = 1$ and $U_1(\Phi) = 0$. Then

$$\Phi(x, \lambda) = -\frac{\psi(x, \lambda)}{\Delta(\lambda)}. \tag{2.12}$$

Denote

$$M(\lambda) := \Phi(0, \lambda) = -\frac{\Delta_1(\lambda)}{\Delta(\lambda)}, \tag{2.13}$$

which is called the Weyl function of the Sturm–Liouville problem (1.1)–(1.3).

In virtue of the result of Refs. [1] and [2], we present Lemma 2.1, which is important for us to prove main results.

Lemma 2.1. ([**1**, **2**]) *Let $M(\lambda)$ be the Weyl function of the Sturm–Liouville problem (1.1)–(1.3) and $\tilde{M}(\lambda)$ be the Weyl function of the Sturm–Liouville problem (3.1), (1.2) and (3.2)(see below), respectively. If coefficient functions $R_{0k}(\lambda)(k = 0, 1)$ of the boundary condition are given a priori and $M(\lambda) = \tilde{M}(\lambda),\ \forall\lambda \in \mathbf{C}$, then*

$$q(x) = \tilde{q}(x) \textit{ a.e. on } [0, \pi] \textit{ and } R_{1k}(\lambda) = \tilde{R}_{1k}(\lambda)(k = 0, 1).$$

3. Half Inverse Problem for the Sturm–Liouville Problem (1.1)–(1.3)

In this section, we discuss the half inverse problem for the Sturm–Liouville operators with boundary conditions polynomially dependent on the spectral parameter on the finite interval $[0, \pi]$ and prove Hochstadt–Lieberman type theorem for the Sturm–Liouville problem (1.1)–(1.3). Consider the following Sturm–Liouville operator $\tilde{L} := \tilde{L}(\tilde{q}, U_0, \tilde{U}_1)$ defined by

$$\tilde{L}y = -y'' + \tilde{q}(x)y = \lambda y, \tag{3.1}$$

with boundary conditions $U_0(y)$ and

$$\tilde{U}_1(y) := \tilde{R}_{11}(\lambda)y'(\pi, \lambda) + \tilde{R}_{10}(\lambda)y(\pi, \lambda) = 0. \tag{3.2}$$

where $\tilde{q}$ is a complex-value function and $\tilde{q} \in L^2(0, \pi)$,

$$\tilde{R}_{1k}(\lambda) = \Sigma_{j=0}^{\tilde{r}_{1k}} \tilde{R}_{1kj}\lambda^{\tilde{r}_{1k}-j},\ \tilde{r}_{11} = \tilde{r}_{10} = r_{10} \geq 0, \tilde{R}_{11j} = 1\ (k = 0, 1)$$

are arbitrary polynomials of degree r_{10} with complex coefficients such that $\tilde{R}_{11}(\lambda)$ and $\tilde{R}_{10}(\lambda)$ have no common zeros.

We establish the following uniqueness theorem for the half inverse problem of the Sturm–Liouville problem (1.1)–(1.3).

Theorem 3.1. *Let $\{\lambda_n(q, U_0, U_1)\}_0^\infty$ be spectrum of the Sturm–Liouville problem (1.1)–(1.3) and $\{\tilde{\lambda}_n(\tilde{q}, U_0, \tilde{U}_1)\}_0^\infty$ be spectrum of the Sturm–Liouville problem*

(3.1), (1.2), (3.2), *respectively. If coefficient functions* $R_{0k}(\lambda)(k=0,1)$ *of the boundary condition are known a priori,* $q(x)=\tilde{q}(x)$ *on* $[0,\frac{\pi}{2}], r_{01}\geq r_{11}$ *and*

$$\lambda_n(q,U_0,U_1)=\tilde{\lambda}_n(\tilde{q},U_0,\tilde{U}_1)\ (\forall n\in\mathbf{N}_0), \tag{3.3}$$

then

$$q(x)=\tilde{q}(x)\quad a.e.\quad on\ [0,\pi]$$

and

$$R_{1k}(\lambda)=\tilde{R}_{1k}(\lambda)\ (k=0,1),$$

where $\mathbf{N}_0=\{0,1,2,\ldots\}$.

Proof. Let $y_1(x,t)$ be the solution of Eq.(1.1) satisfying $y_1(\pi,t)=R_{11}(\lambda)$, $y_1'(\pi,t)=-R_{10}(\lambda)$ and $y_2(x,t)$ be the solution of the Eq. (1.1) satisfying $y_2(\pi,t)=\tilde{R}_{11}(\lambda), y_2'(\pi,t)=-\tilde{R}_{10}(\lambda)$, respectively. By multiplying (2.1) by y_1 and (1.1) by y_2, and subtracting and integrating from 0 to π, we have

$$\begin{aligned}\int_0^\pi & Q(x)y_1(x,\lambda)y_2(x,\lambda)dx\\ &=[y_1(x,\lambda){y_2}'(x,\lambda)-y_2(x,\lambda){y_1}'(x,\lambda)]_0^\pi\\ &=F(\pi,\lambda)-F(0,\lambda),\end{aligned} \tag{3.4}$$

where $Q(x)=\tilde{q}(x)-q(x)$ and

$$F(x,\lambda)=y_1(x,\lambda)y_2'(x,\lambda)-y_1'(x,\lambda)y_2(x,\lambda). \tag{3.5}$$

From $Q(x)=0$ on $[0,\frac{\pi}{2}]$, we get

$$F(0,\lambda)=F(\pi,\lambda)-\int_{\frac{\pi}{2}}^{\pi}Q(x)y_1(x,\lambda)y_2(x,\lambda)dx. \tag{3.6}$$

In addition

$$\begin{aligned}F(0,\lambda)&=y_1(0,\lambda)y_2'(0,\lambda)-y_1'(0,\lambda)y_2(0,\lambda)\\ &=y_2(0,\lambda)\frac{U_0(y_1)}{R_{01}(\lambda)}-y_1(0,\lambda)\frac{U_0(y_2)}{R_{01}(\lambda)}.\end{aligned} \tag{3.7}$$

From Eq. (26) and Lemma 1 in Ref. [2], we can see that $U_0(y_j)(j=1,2)$ and $R_{01}(\lambda)$ have no common zeros. Therefore,

$$F(0,\lambda_n)=0,\quad \forall\lambda_n\in\sigma(L) \tag{3.8}$$

and for all $\lambda=\lambda_n$, we obtain that multiplicity of zero of $F(0,\lambda)$ is not less than multiplicity of zero of $\Delta(\lambda)$.

Let

$$K(\lambda)=\frac{F(0,\lambda)}{\Delta(\lambda)}. \tag{3.9}$$

Then, $K(\lambda)$ is an entire function in λ. In virtue of (2.8) and (3.6), this yields

$$|F(0,\lambda)| \leq C|\lambda|^{r_{11}+\tilde{r}_{11}} e^{\tau\pi}, \tag{3.10}$$

where C is a constant.

From (2.11) and (3.10), we have

$$|K(\lambda)| = O\left(\frac{1}{|\rho|^{[1+2(r_{01}-r_{11})]}}\right), \quad \lambda \in G_\delta, \quad |\lambda| \geq \lambda^*. \tag{3.11}$$

By the maximum modulus principle, we obtain

$$|K(\lambda)| = O\left(\frac{1}{|\rho|^{[1+2(r_{01}-r_{11})]}}\right), \quad \forall \lambda \in \mathbf{C}. \tag{3.12}$$

From Liouville theorem together with (3.12), this yields

$$K(\lambda) = 0, \quad \forall \lambda \in \mathbf{C}. \tag{3.13}$$

Therefore

$$F(0,\lambda) = y_1(0,\lambda)y_2'(0,\lambda) - y_1'(0,\lambda)y_2(0,\lambda) = 0, \quad \forall \lambda \in \mathbf{C}. \tag{3.14}$$

From (3.14), we get

$$\begin{aligned} &y_1(0,\lambda)(R_{01}(\lambda)y_2'(0,\lambda) + R_{00}(\lambda)y_2(0,\lambda)) \\ &\quad = (R_{01}(\lambda)y_1'(0,\lambda) + R_{00}(\lambda)y_1(0,\lambda))y_2(0,\lambda). \end{aligned} \tag{3.15}$$

By virtue of (3.15), this yields

$$M(\lambda) = \tilde{M}(\lambda). \tag{3.16}$$

From Lemma 2.1 together with (3.16), we have

$$q(x) = \tilde{q}(x) \ a.e. \ on \ [0,\pi]$$

and

$$R_{1k}(\lambda) = \tilde{R}_{1k}(\lambda)(k = 0,1).$$

By now, this completes the proof of Theorem 3.1. □

4. Inverse Problem for Sturm–Liouville Problem (1.1)–(1.3) from Interior Spectral Data

In this section, we discuss the interior inverse problem for the Sturm–Liouville problem (1.1)–(1.3) and show that the potential $q(x)$ and coefficient functions $R_{1k}(\lambda)(k = 0,1)$ of the boundary condition are uniquely determined by a set of values of eigenfunctions at some interior point and parts of two spectra.

When $b = \frac{\pi}{2}$, the following uniqueness theorem is established.

Theorem 4.1. *Let $\{\lambda_n\}_0^\infty$ be spectrum of the Sturm–Liouville problem (1.1)–(1.3) and $\{\tilde{\lambda}_n\}_0^\infty$ be spectrum of the Sturm–Liouville problem (3.1), (1.2) and (3.2), respectively. If coefficient functions $R_{0k}(\lambda)(k=0,1)$ of the boundary condition are given a priori, $r_{01}=r_{11}$ and for any $n(n\in\mathbf{N}_0)$,*

$$\lambda_n=\tilde{\lambda}_n \textit{ and } \frac{y_n'(\frac{\pi}{2},\lambda_n)}{y_n(\frac{\pi}{2},\lambda_n)}=\frac{\tilde{y}'_n(\frac{\pi}{2},\tilde{\lambda}_n)}{\tilde{y}_n(\frac{\pi}{2},\tilde{\lambda}_n)}, \tag{4.1}$$

then

$$q(x)=\tilde{q}(x) \text{ a.e. on } [0,\pi]$$

and

$$R_{1k}(\lambda)=\tilde{R}_{1k}(\lambda)(k=0,1),$$

where $y_n(x,\lambda_n)$ is an eigenfunction of λ_n and $\tilde{y}_n(x,\tilde{\lambda}_n)$ is an eigenfunction of $\tilde{\lambda}_n$.

Let $l(n)$ and $r(n)$ be a subsequence of natural numbers such that

$$l(n)=\frac{n}{\sigma_1}(1+\varepsilon_{1,n}), \quad 0<\sigma_1\le 1,\ \varepsilon_{1,n}\to 0, \tag{4.2}$$

$$r(n)=\frac{n}{\sigma_2}(1+\varepsilon_{2,n}), \quad 0<\sigma_2\le 1,\ \varepsilon_{2,n}\to 0 \tag{4.3}$$

and let μ_n be the eigenvalues of the boundary-value problem (1.1), (1.2) and (4.4)(see below) and $\tilde{\mu}_n$ be the eigenvalues of the boundary-value problem (3.1), (1.2) and (4.4), where the boundary condition (4.4) is defined as follows

$$U_2(y):=R_{21}(\lambda)y'(\pi,\lambda)+R_{20}(\lambda)y(\pi,\lambda)=0, \tag{4.4}$$

where $R_{2k}(\lambda)(k=0,1)$ are arbitrary polynomials of degree r_{2k} with complex coefficients such that $R_{21}(\lambda)$ and $R_{20}(\lambda)$ have no common zeros.

When $b\in(\frac{\pi}{2},\pi)$, from a part of the two spectra and some information on eigenfunctions at the point b, we obtain the following uniqueness theorem on the potential $q(x)$, which is a generalization of Mochizuki and Trooshin's theorem.

Theorem 4.2. *Let $l(n)$ and $r(n)$ be subsequence of natural numbers satisfying (4.2) and (4.3), respectively, and $b\in(\frac{\pi}{2},\pi)$ be such that $\sigma_1>\frac{2b}{\pi}-1, \sigma_2>2-\frac{2b}{\pi}$. Suppose that $\{\lambda_n\}_0^\infty$ and $\{\tilde{\lambda}_n\}_0^\infty$ be spectrum of both Sturm–Liouville problem (1.1)–(1.3) and Sturm–Liouville problem (3.1), (1.2) and (3.2), respectively. If coefficient functions $R_{0k}(\lambda)(k=0,1)$ of the boundary condition are known a priori and for any $n(n\in\mathbf{N}_0)$,*

$$\lambda_n=\tilde{\lambda}_n,\ \mu_{l(n)}=\tilde{\mu}_{l(n)} \textit{ and } \frac{y'_{r(n)}(b,\lambda_{r(n)})}{y_{r(n)}(b,\lambda_{r(n)})}=\frac{\tilde{y}'_{r(n)}(b,\tilde{\lambda}_{r(n)})}{\tilde{y}_{r(n)}(b,\tilde{\lambda}_{r(n)})} \tag{4.5}$$

then

$$q(x)=\tilde{q}(x) \text{ a.e. on } [0,\pi]$$

and

$$R_{1k}(\lambda) = \tilde{R}_{1k}(\lambda)(k = 0, 1).$$

When $b \in (0, \frac{\pi}{2})$, Symmetrically, we have the following Theorem 4.3, which proof is therefore omitted.

Theorem 4.3. *Let $l(n)$ and $r(n)$ be subsequence of natural numbers satisfying (4.2) and (4.3), respectively, and $b \in (0, \frac{\pi}{2})$ be such that $\sigma_1 > 1 - \frac{2b}{\pi}, \sigma_2 > \frac{2b}{\pi}$. Suppose that $\{\lambda_n\}_0^\infty$ and $\{\tilde{\lambda}_n\}_0^\infty$ be spectrum of both Sturm–Liouville problem (1.1)–(1.3) and Sturm–Liouville problem (3.1), (1.2) and (3.2), respectively. If coefficient functions $R_{0k}(\lambda)(k = 0, 1)$ of the boundary condition are known a priori and for any $n(n \in \mathbf{N}_0)$,*

$$\lambda_n = \tilde{\lambda}_n, \qquad \mu_{l(n)} = \tilde{\mu}_{l(n)} \text{ and } \frac{y'_{r(n)}(b, \lambda_{r(n)})}{y_{r(n)}(b, \lambda_{r(n)})} = \frac{\tilde{y}'_{r(n)}(b, \tilde{\lambda}_{r(n)})}{\tilde{y}_{r(n)}(b, \tilde{\lambda}_{r(n)})} \tag{4.6}$$

then

$$q(x) = \tilde{q}(x) \text{ a.e. on } [0, \pi]$$

and

$$R_{1k}(\lambda) = \tilde{R}_{1k}(\lambda)(k = 0, 1).$$

Proof of Theorem 4.1. We give the proof of Theorem 4.1 by two steps. □

Step 1: By multiplying (3.1) by y_1 and (1.1) by y_2 and subtracting and integrating from $\frac{\pi}{2}$ to π, we have

$$\begin{aligned} &\int_{\frac{\pi}{2}}^{\pi} Q(x) y_1(x, \lambda) y_2(x, \lambda) dx \\ &\quad = [y_1(x, \lambda) {y_2}'(x, \lambda) - y_2(x, \lambda) {y_1}'(x, \lambda)]_{\frac{\pi}{2}}^{\pi} \\ &\quad = F(\pi, \lambda) - F\left(\frac{\pi}{2}, \lambda\right), \end{aligned} \tag{4.7}$$

where $Q(x) = \tilde{q}(x) - q(x)$.

From the assumptions of Theorem 4.1, we get

$$F\left(\frac{\pi}{2}, \lambda_n\right) = 0. \tag{4.8}$$

Similar to the proof of Theorem 3.1, this yields

$$F\left(\frac{\pi}{2}, \lambda\right) = 0, \qquad \forall \lambda \in \mathbf{C}. \tag{4.9}$$

Step 2: Consider the following supplementary problem

$$\begin{aligned} L_1 \tilde{y} = -\tilde{y}'' + q_1(x)\tilde{y} = \lambda \tilde{y}, \\ q_1(x) = q(\pi - x), \ x \in [0, \pi], \end{aligned} \tag{4.10}$$

with the boundary conditions

$$R_{10}(\lambda)\tilde{y}(0, \lambda) - R_{11}(\lambda)\tilde{y}'(0, \lambda) = 0, \tag{4.11}$$

$$R_{00}(\lambda)\tilde{y}(\pi,\lambda) - R_{01}(\lambda)\tilde{y}'(\pi,\lambda) = 0, \tag{4.12}$$

and

$$\begin{aligned} \tilde{L}_1\tilde{y} &= -\tilde{y}'' + \tilde{q}_1(x)\tilde{y} = \lambda\tilde{y}, \\ \tilde{q}_1(x) &= \tilde{q}(\pi - x),\ x \in [0,\pi], \end{aligned} \tag{4.13}$$

with the boundary conditions

$$\tilde{R}_{10}(\lambda)\tilde{y}(0,\lambda) - \tilde{R}_{11}(\lambda)\tilde{y}'(0,\lambda) = 0, \tag{4.14}$$

$$\tilde{R}_{00}(\lambda)\tilde{y}(\pi,\lambda) - \tilde{R}_{01}(\lambda)\tilde{y}'(\pi,\lambda) = 0, \tag{4.15}$$

where $\tilde{y}(x,\lambda) = y(\pi - x, \lambda)$.

By virtue of Step 1, this yields

$$\begin{aligned} &\int_{\frac{\pi}{2}}^{\pi} Q_1(x)\tilde{y}_1(x,\lambda)\tilde{y}_2(x,\lambda)dx \\ &\quad = [\tilde{y}_1(\pi,\lambda)\tilde{y}_2'(\pi,\lambda) - \tilde{y}_2(\pi,\lambda)\tilde{y}_1'(\pi,\lambda)]_{\frac{\pi}{2}}^{\pi} \\ &\quad = \tilde{F}(\pi,\lambda) - \tilde{F}\left(\frac{\pi}{2},\lambda\right), \end{aligned} \tag{4.16}$$

where $Q_1(x) = \tilde{q}_1(x) - q_1(x)$, $\tilde{F}(x,\lambda) = \tilde{y}_1(x,\lambda)\tilde{y}_2'(x,\lambda) - \tilde{y}_1'(x,\lambda)\tilde{y}_2(x,\lambda)$, $\tilde{y}_1$ is the solution of the Sturm–Liouville problem (4.10)–(4.12) and $\tilde{y}_2$ is the solution of the Sturm–Liouville problem (4.13)–(4.15).

In virtue of (4.9) and $\tilde{y}_k(x,\lambda) = y_k(\pi - x,\lambda)(k = 0,1)$, this yields

$$\tilde{F}\left(\frac{\pi}{2},\lambda\right) = -F\left(\frac{\pi}{2},\lambda\right) = 0, \quad \forall\lambda \in \mathbf{C}. \tag{4.17}$$

Note that $\tilde{F}(\pi,\lambda_n) = -F(0,\lambda_n) = 0$, from (4.16) and repeating the Step 1 for the supplementary problem, we obtain

$$F(0,\lambda) = -\tilde{F}(\pi,\lambda) = 0, \quad \forall\lambda \in \mathbf{C}. \tag{4.18}$$

By virtue of (4.18), this yields

$$M(\lambda) = \tilde{M}(\lambda), \quad \forall\lambda \in \mathbf{C}. \tag{4.19}$$

From Lemma 2.1, together with (4.19), we have

$$q(x) = \tilde{q}(x) \ a.e. \ on \ [0,\pi] \tag{4.20}$$

and

$$R_{1k}(\lambda) = \tilde{R}_{1k}(\lambda)(k = 0,1). \tag{4.21}$$

The proof of Theorem 4.1 is now completed.

Next, we show that Theorem 4.2 holds.

Proof of Theorem 4.2. By multiplying (3.1) by y_1 and (1.1) by y_2, and subtracting and integrating from b to π, we obtain

$$G_b(\lambda) := [y_1(\pi,\lambda)y_2'(\pi,\lambda) - y_2(\pi,\lambda)y_1'(\pi,\lambda)] - \int_b^\pi Qy_1y_2dx$$
$$= [y_1(b,\lambda)y_2'(b,\lambda) - y_2(b,\lambda)y_1'(b,\lambda)], \tag{4.22}$$

where $Q(x) = \tilde{q}(x) - q(x)$. □

From the assumption

$$\lambda_{m(n)} = \tilde{\lambda}_{m(n)} \textit{ and } \frac{y'_{m(n)}(b,\lambda_{m(n)})}{y_{m(n)}(b,\lambda_{m(n)})} = \frac{\tilde{y'}_{m(n)}(b,\tilde{\lambda}_{m(n)})}{\tilde{y}_{m(n)}(b,\tilde{\lambda}_{m(n)})},$$

we get

$$G_b(\lambda_{m(n)}) = 0, \quad n \in \mathbf{N_0}. \tag{4.23}$$

Next, we will prove $G_b(\lambda) = 0, \quad \forall \lambda \in \mathbf{C}$.

Clearly, the entire function $G_b(\lambda)$ is a function of exponential type $\leq 2(\pi - b)$ and for sufficiently large r, we have

$$|G_b(\lambda)| \leq Mr^{r_{01}+\tilde{r}_{01}+\frac{1}{2}}e^{2(\pi-b)r|\sin\theta|}, \tag{4.24}$$

where M is a positive constant, $\lambda = re^{i\theta}$.

Define the indicator of function $G_b(\lambda)$ by

$$h(\theta) = \limsup_{\lambda\to+\infty} \frac{\ln|G_b(re^{i\theta})|}{r}. \tag{4.25}$$

Since $|Im\lambda| = r|\sin\theta|$, $\theta = arg\lambda$, from (4.24) and (4.25), we get

$$h(\theta) \leq 2(\pi - b)|\sin\theta|. \tag{4.26}$$

Let $n(r)$ be the number of zeros of $G_b(\lambda)$ in the disk $|\lambda| \leq r$. From the assumption of Theorem 4.2 and the asymptotic form (2.10) of the eigenvalues λ_n, we obtain

$$n(r) \geq 2 \sum_{|\frac{n}{\sigma_2}[1-\frac{r_{01}+r_{11}}{n}+O(\frac{1}{n^2})]|<r} 1 \geq 2\sigma_2 r[1 + o(1)], r \to \infty, \tag{4.27}$$

where $[x]$ is the integer part of x.

For the case $\sigma_2 > 2 - \frac{2b}{\pi}$,

$$\liminf_{n\to\infty} \frac{n(r)}{r} \geq 2\sigma_2 > \frac{4(\pi - b)}{\pi} \geq \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta. \tag{4.28}$$

According to the theorem 3([[29], theorem 3, p.273]), for any entire function $G_b(\lambda)$ of exponential type, not identically zero, we have

$$\liminf_{n\to\infty} \frac{n(r)}{r} \leq \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta. \tag{4.29}$$

The inequalities (4.28) and (4.29) imply that

$$G_b(\lambda) = 0, \quad \forall \lambda \in \mathbf{C}. \tag{4.30}$$

Hence

$$F(b,\lambda)=y_1(b,\lambda)y_2'(b,\lambda)-y_2(b,\lambda)y_1'(b,\lambda)]=0,\qquad \forall\lambda\in\mathbf{C}. \tag{4.31}$$

Define the entire function $G(\lambda)$ by

$$\begin{aligned} G(\lambda):&=\int_0^b Q(x)y_1(\pi,\lambda)y_2(\pi,\lambda)dx\\ &=[y_1(x,\lambda)y_2'(x,\lambda)-y_2(x,\lambda)y_1'(x,\lambda)]_0^b. \end{aligned} \tag{4.32}$$

By virtue of (4.31) and (4.32), this yields

$$G(\lambda)=-[y_1(0,\lambda)y_2'(0,\lambda)-y_2(0,\lambda)y_1'(0,\lambda)]. \tag{4.33}$$

From (4.33), we obtain

$$G(\lambda_n)=0,\qquad n\in\mathbf{N_0} \tag{4.34}$$

and

$$G(\mu_{l(n)})=0,\qquad n\in\mathbf{N_0}, \tag{4.35}$$

where λ_n and $\mu_{l(n)}$ satisfy (2.10).

Let us count the number of the λ_n and $\mu_{l(n)}$ located inside the disc of radius r. It is easy to see that there are $1+2r[1+o(1)]$ of λ_n and $1+2r\sigma_1[1+o(1)]$ of $\mu_{l(n)}$ located inside the disc of radius r (sufficiently large r), respectively. Therefore

$$n(r)=2+2r[1+\sigma_1+o(1)]. \tag{4.36}$$

From (4.36), similar to the proof of $G_b(\lambda)=0(\forall\lambda\in\mathbf{C})$, we have

$$G(\lambda)=0,\qquad \forall\lambda\in\mathbf{C}. \tag{4.37}$$

In virtue of (4.37) and (4.33), this yields

$$F(0,\lambda)=0,\qquad \forall\lambda\in\mathbf{C}. \tag{4.38}$$

From (4.38), we obtain

$$M(\lambda)=\tilde{M}(\lambda). \tag{4.39}$$

By virtue of (4.39), together with Lemma 2.1, this yields

$$q(x)=\tilde{q}(x)\ a.e.\ on\ [0,\pi] \tag{4.40}$$

and

$$R_{1k}(\lambda)=\tilde{R}_{1k}(\lambda)(k=0,1). \tag{4.41}$$

By now this completes the proof of Theorem 4.2.

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