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Results in Mathematics

Complete Linear Weingarten Spacelike Hypersurfaces Immersed in a Locally Symmetric Lorentz Space

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Abstract. In this paper, we establish a characterization theorem concerning the complete linear Weingarten spacelike hypersurfaces immersed in a locally symmetric Lorentz space, whose sectional curvature is supposed to obey certain appropriated conditions. Under a suitable restriction on the length of the second fundamental form, we prove that a such spacelike hypersurface must be either totally umbilical or an isoparametric hypersurface with two distinct principal curvatures one of which is simple.

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1. Introduction

Let L_1^{n+1} be an $(n+1)$-dimensional Lorentz space, that is, a semi-Riemannian manifold of index 1. When the Lorentz space L_1^{n+1} has constant sectional curvature c, it is called a Lorentz space form, denoted by $L_1^{n+1}(c)$. In particular, for $n \geq 2$, the de Sitter space $\mathbb{S}_1^{n+1}$ is the standard simply connected Lorentz space form of positive constant sectional curvature 1.

We recall that a hypersurface M^n immersed in a Lorentz space L_1^{n+1} is said to be *spacelike* if the metric on M^n induced from that of the ambient space L_1^{n+1} is positive definite.

The interest in the study of spacelike hypersurfaces immersed in a Lorentz space is motivated by their nice Bernstein-type properties. As for the case of the de Sitter space, Goddard [10] conjectured that every complete spacelike hypersurface with constant mean curvature H in $\mathbb{S}_1^{n+1}$ should be totally umbilical.

Although the conjecture turned out to be false in its original statement, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under appropriate additional hypotheses. For instance, in [1] Akutagawa showed that Goddard's conjecture is true when $0 \leq H^2 \leq 1$ in the case $n = 2$, and when $0 \leq H^2 < 4(n-1)/n^2$ in the case $n \geq 3$. Later, Montiel [15] solved Goddard's problem in the compact case proving that the only closed spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature are the totally umbilical hypersurfaces.

Another Goddard-like problem is to study hypersurfaces immersed in a Lorentz space with constant scalar curvature. An interesting result of Cheng and Ishikawa [7] states that the totally umbilical round spheres are the only compact spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant normalized scalar curvature $R < 1$. Many other authors, such as Brasil et al. [3], Camargo et al. [4], Caminha [5], Hu et al. [11] and Li [12] have also worked on related problems.

It is natural to study the geometry of spacelike hypersurfaces immersed in more general Lorentz spaces since they have important meaning in the relativity theory and are of substantial interest from geometric and mathematical cosmology points of view. In this setting, for constants c_1 and c_2, Choi et al. [9,18] introduced the class of $(n+1)$-dimensional Lorentz spaces L_1^{n+1} which satisfy the following two conditions (here, K denotes the sectional curvature of L_1^{n+1}):

$$K(u, v) = -\frac{c_1}{n}, \tag{1.1}$$

for any spacelike vector u and timelike vector v; and

$$K(u, v) \geq c_2, \tag{1.2}$$

for any spacelike vectors u and v.

We observe that Lorentz space forms $L_1^{n+1}(c)$ satisfy conditions (1.1) and (1.2) for $-\frac{c_1}{n} = c_2 = c$. Moreover, there are several examples of Lorentz spaces which are not Lorentz space forms and satisfy (1.1) and (1.2). For instance, semi-Riemannian product manifolds $\mathbb{H}_1^k(-c_1/n) \times N^{n+1-k}(c_2)$, where $c_1 > 0$, and $\mathbb{R}_1^k \times \mathbb{S}^{n+1-k}$. In particular, $\mathbb{R}_1^1 \times \mathbb{S}^n$ is a so-called *Einstein Static Universe*. Also the so-called *Robertsonfb́Walker spacetime* $N(c, f) = I \times_f N^3(c)$ is another general example of Lorentz space, where I denotes an open interval of $\mathbb{R}_1^1$, f is a positive smooth function defined on the interval I and $N^3(c)$ is a 3-dimensional Riemannian manifold of constant curvature c. $N(c, f)$ also satisfies conditions (1.1) and (1.2) for an appropriate choice of the function f (for more details, see [9] and [18]).

Here, our purpose is to study the rigidity of complete *linear Weingarten* spacelike hypersurfaces, that is, complete spacelike hypersurfaces whose mean curvature H and normalized scalar curvature R satisfy

$$R = aH + b,$$

for some $a, b \in \mathbb{R}$. In this setting, as a suitable application of the Hopf's strong maximum principle and under an appropriated restriction on the squared norm S of the second fundamental form, we are able to establish a characterization theorem concerning to such spacelike hypersurfaces immersed in a locally symmetric Lorentz space L_1^{n+1}, which is supposed to obey conditions (1.1) and (1.2). We recall that a Lorentz space L_1^{n+1} is said *locally symmetric* if all the covariant derivative components $\bar{R}_{ABCD;E}$ of the curvature tensor of L_1^{n+1} vanish identically.

In order to state our result, we will need some basic facts. Denote by $\bar{R}_{CD}$ the components of the Ricci tensor of L_1^{n+1} satisfying conditions (1.1) and (1.2), then the scalar curvature $\bar{R}$ of L_1^{n+1} is given by

$$\bar{R} = \sum_{A=1}^{n+1} \varepsilon_A \bar{R}_{AA} = \sum_{i,j=1}^{n} \bar{R}_{ijji} - 2\sum_{i=1}^{n} \bar{R}_{(n+1)ii(n+1)} = \sum_{i,j=1}^{n} \bar{R}_{ijji} + 2c_1.$$

Moreover, it is well known that the scalar curvature of a locally symmetric Lorentz space is constant. Consequently, $\sum_{i,j=1}^{n} \bar{R}_{ijji}$ is a constant naturally attached to a locally symmetric Lorentz space satisfying conditions (1.1) and (1.2).

Now, we are in position to present our result.

Theorem 1.1. *Let L_1^{n+1} be a locally symmetric Lorentz space satisfying conditions (1.1) and (1.2), with $c = \frac{c_1}{n} + 2c_2 > 0$. Let M^n be a complete linear Weingarten spacelike hypersurface immersed in L_1^{n+1}, such that $R = aH + b$ with $b < \frac{1}{n(n-1)} \sum_{i,j} \bar{R}_{ijji}$. If H can attain the maximum on M^n and $S \leq 2\sqrt{n-1}\, c$, then M^n is either totally umbilical or an isoparametric hypersurface with two distinct principal curvatures one of which is simple.*

We note that the previous theorem can be regarded as an extension of rigidity results of the current literature concerning to spacelike hypersurfaces with either constant mean curvature or constant scalar curvature in locally symmetric Lorentz spaces. In this sense, we refer the readers to the works of Ok Baek et al. [16], Liu and Sun [14], and Zhang and Wu [19]. Moreover, we point out that Li et al. [13] have obtained rigidity theorems related to linear Weingarten hypersurfaces immersed in the unit Euclidean sphere.

2. Preliminaries

'From now on, we will consider complete spacelike hypersurfaces M^n immersed in a Lorentz space L_1^{n+1}. We choose a local field of semi-Riemannian orthonormal frame $\{e_A\}_{1\leq A\leq n+1}$ in L_1^{n+1}, with dual coframe $\{\omega_A\}_{1\leq A\leq n+1}$, such that, at each point of $M^n, e_1, \ldots, e_n$ are tangent to M^n and e_{n+1} is normal to M^n. We will use the following convention for the indices:

$$1 \leq A, B, C, \ldots \leq n+1, \ 1 \leq i, j, k, \ldots \leq n.$$

In this setting, denoting by $\{\omega_{AB}\}$ the connection forms of L_1^{n+1}, we have that the structure equations of L_1^{n+1} are given by:

$$d\omega_A = -\sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \quad \varepsilon_i = 1, \varepsilon_{n+1} = -1, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} \varepsilon_C \varepsilon_D \bar{R}_{ABCD} \omega_C \wedge \omega_D. \tag{2.2}$$

Here, $\bar{R}_{ABCD}, \bar{R}_{CD}$ and $\bar{R}$ denote respectively the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of the Lorentz space L_1^{n+1}. In this setting, we have

$$\bar{R}_{CD} = \sum_B \varepsilon_B \bar{R}_{BCDB}, \quad \bar{R} = \sum_A \varepsilon_A \bar{R}_{AA}.$$

Moreover, the components $\bar{R}_{ABCD,E}$ of the covariant derivative of the Riemannian curvature tensor of L_1^{n+1} are defined by

$$\sum_E \varepsilon_E \bar{R}_{ABCD,E} \omega_E = d\bar{R}_{ABCD} - \sum_E \varepsilon_E (\bar{R}_{EBCD}\omega_{EA} + \bar{R}_{AECD}\omega_{EB} + \bar{R}_{ABED}\omega_{EC} + \bar{R}_{ABCE}\omega_{ED}).$$

Next, we restrict all the tensors to the spacelike hypersurface M^n in L_1^{n+1}. First of all, $\omega_{n+1} = 0$ on M^n, so $\sum_i \omega_{(n+1)i} \wedge \omega_i = d\omega_{n+1} = 0$. Consequently, by *Cartan's Lemma* [6], there are h_{ij} such that

$$\omega_{(n+1)i} = \sum_j h_{ij}\omega_j \quad \text{and} \quad h_{ij} = h_{ji}. \tag{2.3}$$

This gives the second fundamental form of $M^n, h = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1}$, and its square length $S = \sum_{i,j} h_{ij}^2$. Furthermore, the mean curvature H of M^n is defined by $H = \frac{1}{n}\sum_i h_{ii}$.

The connection forms $\{\omega_{ij}\}$ of M^n are characterized by the structure equations of M^n:

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.4}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.5}$$

where R_{ijkl} are the components of the curvature tensor of M^n.

Using the structure equations we obtain the Gauss equation

$$R_{ijkl} = \bar{R}_{ijkl} - (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{2.6}$$

The components R_{ij} of the Ricci tensor and the scalar curvature R of M^n are given, respectively, by

$$R_{ij} = \sum_k \bar{R}_{kijk} - nHh_{ij} + \sum_k h_{ik}h_{kj} \tag{2.7}$$

and

$$n(n-1)R = \sum_{j,k} \bar{R}_{kjjk} - n^2H^2 + S. \tag{2.8}$$

The first covariant derivatives h_{ijk} of h_{ij} satisfy

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{ik}\omega_{kj} - \sum_k h_{jk}\omega_{ki}. \tag{2.9}$$

Then, by exterior differentiation of (2.3), we obtain the *Codazzi equation*

$$h_{ijk} - h_{ikj} = \bar{R}_{(n+1)ijk}. \tag{2.10}$$

Similarly, the second covariant derivatives h_{ijkl} of h_{ij} are given by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l h_{ljk}\omega_{li} - \sum_l h_{ilk}\omega_{lj} - \sum_l h_{ijl}\omega_{lk}. \tag{2.11}$$

By exterior differentiation of (2.9), we can get the following *Ricci formula*

$$h_{ijkl} - h_{ijlk} = -\sum_m h_{im}R_{mjkl} - \sum_m h_{jm}R_{mikl}. \tag{2.12}$$

Restricting the covariant derivative $\bar{R}_{ABCD;E}$ of $\bar{R}_{ABCD}$ on M^n, then $\bar{R}_{(n+1)ijk;l}$ is given by

$$\begin{aligned}\bar{R}_{(n+1)ijk;l} &= \bar{R}_{(n+1)ijkl} + \bar{R}_{(n+1)i(n+1)k}h_{jl}\\ &\quad+\bar{R}_{(n+1)ij(n+1)}h_{kl} + \sum_m \bar{R}_{mijk}h_{ml},\end{aligned} \tag{2.13}$$

where $\bar{R}_{(n+1)ijkl}$ denotes the covariant derivative of $\bar{R}_{(n+1)ijk}$ as a tensor on M^n so that

$$\begin{aligned}\sum_l \bar{R}_{(n+1)ijkl}\omega_l &= d\bar{R}_{(n+1)ijk} - \sum_l \bar{R}_{(n+1)ljk}\omega_{li}\\ &\quad - \sum_l \bar{R}_{(n+1)ilk}\omega_{lj} - \sum_l \bar{R}_{(n+1)ijl}\omega_{lk}.\end{aligned}$$

The Laplacian Δh_{ij} of h_{ij} is defined by $\Delta h_{ij} = \sum_k h_{ijkk}$. From (2.10), (2.12) and (2.13), after a straightforward computation we obtain

$$\begin{aligned}\Delta h_{ij} &= (nH)_{ij} - nH\sum_l h_{il}h_{lj} + Sh_{ij}\\ &\quad+\sum_k(\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j})\\ &\quad-\sum_k(h_{kk}\bar{R}_{(n+1)ij(n+1)} + h_{ij}\bar{R}_{(n+1)k(n+1)k})\\ &\quad-\sum_{k,l}(2h_{kl}\bar{R}_{lijk} + h_{jl}\bar{R}_{lkik} + h_{il}\bar{R}_{lkjk}).\end{aligned} \tag{2.14}$$

Since $\Delta S = 2\left(\sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\Delta h_{ij}\right)$, from (2.14) we get

$$\begin{aligned}\frac{1}{2}\Delta S &= S^2 + \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j}(nH)_{ij}h_{ij}\\ &\quad + \sum_{i,j,k}(\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j})h_{ij}\\ &\quad - \left(\sum_{i,j} nHh_{ij}\bar{R}_{(n+1)ij(n+1)} + S\sum_k \bar{R}_{(n+1)k(n+1)k}\right)\\ &\quad -2\sum_{i,j,k,l}(h_{kl}h_{ij}\bar{R}_{lijk} + h_{il}h_{ij}\bar{R}_{lkjk}) - nH\sum_{i,j,l} h_{il}h_{lj}h_{ij}. \qquad (2.15)\end{aligned}$$

Now, let $\phi = \sum_{i,j}\phi_{ij}\omega_i\omega_j$ be a symmetric tensor on M^n defined by

$$\phi_{ij} = nH\delta_{ij} - h_{ij}.$$

Following Cheng–Yau [8], we introduce a operator $\square$ associated to ϕ acting on any smooth function f by

$$\square f = \sum_{i,j}\phi_{ij}f_{ij} = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}. \qquad (2.16)$$

Setting $f = nH$ in (2.16) and taking a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda\delta_{ij}$, from Eq. (2.8) we obtain the following

$$\begin{aligned}\square(nH) &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i \lambda_i(nH)_{ii}\\ &= \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_i \lambda_i(nH)_{ii}\\ &\quad + \frac{1}{2}\Delta\left(\sum_{i,j}\bar{R}_{ijji} - n(n-1)R\right). \qquad (2.17)\end{aligned}$$

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we will need three lemmas. The first one is a classic algebraic lemma due to Okumura in [17], and completed with the equality case proved in [2] by Alencar and do Carmo.

Lemma 3.1. *Let $\mu_1, \ldots \mu_n$ be real numbers such that $\sum_i\mu_i = 0$ and $\sum_i\mu_i^2 = \beta^2$, where β is constant and $\beta \geq 0$. Then*

$$-\frac{(n-2)}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{(n-2)}{\sqrt{n(n-1)}}\beta^3, \qquad (3.1)$$

and equality holds if, and only if, either at least $(n-1)$ *of the numbers* μ_i *are equal to* $\beta/\sqrt{(n-1)n}$ *or at least* $(n-1)$ *of the numbers* μ_i *are equal to* $-\beta/\sqrt{(n-1)n}$.

Now, we present our second auxiliary lemma. Following the steps of the proof of Lemma 2.1 of [13], we get

Lemma 3.2. *Let* M^n *be a linear Weingarten spacelike hypersurface immersed in a locally symmetric Lorentz space* L_1^{n+1}*, such that* $R = aH + b$*. Suppose that*

$$(n-1)^2a^2 + 4\sum_{i,j}\bar{R}_{ijji} - 4n(n-1)b \geq 0. \tag{3.2}$$

Then,

$$\sum_{i,j,k} h_{ijk}^2 \geq n^2|\nabla H|^2. \tag{3.3}$$

Moreover, if the inequality (3.2) *is strict and the equality holds in* (3.3) *on* M^n*, then* H *is constant on* M^n.

Proof. Since we are supposing that $R = aH + b$, from Eq. (2.8) we get

$$2\sum_{i,j} h_{ij}h_{ijk} = \left(2n^2H + n(n-1)a\right)(H)_k.$$

Thus,

$$4\sum_k\left(\sum_{i,j} h_{ij}h_{ijk}\right)^2 = \left(2n^2H + n(n-1)a\right)^2|\nabla H|^2.$$

Consequently, using Cauchy–Schwartz inequality, we obtain that

$$\begin{aligned} 4S\sum_{i,j,k} h_{ijk}^2 &= 4\left(\sum_{i,j} h_{ij}^2\right)\left(\sum_{i,j,k} h_{ijk}^2\right) \\ &\geq 4\sum_k\left(\sum_{i,j} h_{ij}h_{ijk}\right)^2 \\ &= \left(2n^2H + n(n-1)a\right)^2|\nabla H|^2. \end{aligned} \tag{3.4}$$

On the other hand, since $R = aH + b$, using again Eq. (2.8) we easily verify that

$$\begin{aligned} \left(2n^2H + n(n-1)a\right)^2 = 4n^2\sum_{i,j}\bar{R}_{ijji} - 4n^3(n-1)b \\ + n^2(n-1)^2a^2 + 4n^2S. \end{aligned} \tag{3.5}$$

Consequently, from (3.2), (3.4) and (3.5), we get

$$S\sum_{i,j,k} h_{ijk}^2 \geq n^2 S|\nabla H|^2.$$

Therefore, we obtain either $S=0$ and $\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2$ or $\sum_{i,j,k} h_{ijk}^2 \geq n^2|\nabla H|^2$. Moreover, if the inequality (3.2) is strict, from (3.5) we get that

$$\left(2n^2H + n(n-1)a\right)^2 > 4n^2 S.$$

Consequently, if $\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2$ holds on M^n, from (3.4) we conclude that $\nabla H = 0$ on M^n and, hence, H is constant on M^n. □

Now, we consider the Cheng–Yau's modified operator

$$L = \Box + \frac{n-1}{2}a\Delta. \tag{3.6}$$

Related to such operator, we have the following sufficient criteria of ellipticity.

Lemma 3.3. *Let M^n be a linear Weingarten spacelike hypersurface immersed in a locally symmetric Lorentz space L_1^{n+1}, such that $R = aH + b$ with $b < \frac{1}{n(n-1)}\sum_{i,j}\bar{R}_{ijji}$. Then, L is elliptic.*

Proof. From Eq. (2.8), since $R = aH + b$ with $b < \frac{1}{n(n-1)}\sum_{i,j}\bar{R}_{ijji}$, we easily see that H can not vanish on M^n and, by choosing the appropriate Gauss mapping, we may assume that $H > 0$ on M^n.

Let us consider the case that $a = 0$. Since $R = b < \frac{1}{n(n-1)}\sum_{i,j}\bar{R}_{ijji}$, from Eq. (2.8) if we choose a (local) orthonormal frame $\{e_1,\ldots,e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$, we have that $\sum_{i<j}\lambda_i\lambda_j > 0$. Consequently,

$$n^2H^2 = \sum_i \lambda_i^2 + 2\sum_{i<j}\lambda_i\lambda_j > \lambda_i^2$$

for every $i = 1,\ldots,n$ and, hence, we have that $nH - \lambda_i > 0$ for every i. Therefore, in this case, we conclude that L is elliptic.

Now, suppose that $a \neq 0$. From Eq. (2.8) we get that

$$a = \frac{1}{n(n-1)H}\left(S - n^2H^2 + \sum_{i,j}\bar{R}_{ijji} - n(n-1)b\right).$$

Consequently, for every $i = 1,\ldots,n$, with a straightforward algebraic computation we verify that

$$\begin{aligned} nH - \lambda_i + \frac{n-1}{2}a &= nH - \lambda_i + \frac{1}{2nH}\left(S - n^2H^2 + \sum_{i,j}\bar{R}_{ijji} - n(n-1)b\right) \\ &= \frac{1}{2nH}\left(\sum_{j\neq i}\lambda_j^2 + (\sum_{j\neq i}\lambda_j)^2 + \sum_{i,j}\bar{R}_{ijji} - n(n-1)b\right). \end{aligned}$$

Therefore, since $b < \frac{1}{n(n-1)}\sum_{i,j}\bar{R}_{ijji}$, we also conclude in this case that L is elliptic. $\square$

Proof of Theorem 1.1. Initially, we observe that the local symmetry of L_1^{n+1} implies that

$$\sum_{i,j,k}(\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j})h_{ij} = 0.$$

Consequently, if we choose a (local) orthonormal frame $\{e_1,\ldots,e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$, taking into account Eqs. (2.15) and (2.17) we get from (3.6) that

$$\begin{aligned} L(nH) = &\sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + S^2 - nH\sum_i \lambda_i^3 \\ &-2\sum_{i,k}(\lambda_i\lambda_k\bar{R}_{kiik} + \lambda_i^2\bar{R}_{ikik}) \\ &-\left(\sum_i nH\lambda_i\bar{R}_{(n+1)ii(n+1)} + S\sum_k \bar{R}_{(n+1)k(n+1)k}\right). \end{aligned} \tag{3.7}$$

Thus, from Lemma 3.2, we have

$$\begin{aligned} L(nH) \geq\ & S^2 - nH\sum_i \lambda_i^3 - 2\sum_{i,k}(\lambda_i\lambda_k\bar{R}_{kiik} + \lambda_i^2\bar{R}_{ikik}) \\ &-\left(\sum_i nH\lambda_i\bar{R}_{(n+1)ii(n+1)} + S\sum_k \bar{R}_{(n+1)k(n+1)k}\right). \end{aligned} \tag{3.8}$$

Now, set $\Phi_{ij} = h_{ij} - H\delta_{ij}$. We will consider the following symmetric tensor

$$\Phi = \sum_{i,j}\Phi_{ij}\omega_i\omega_j.$$

Let $|\Phi|^2 = \sum_{i,j}\Phi_{ij}^2$ be the square of the length of Φ. It is easy to check that Φ is traceless and

$$|\Phi|^2 = S - nH^2.$$

If we take a (local) frame field $e_1,\ldots,e_n$ at $p \in M^n$, such that

$$h_{ij} = \lambda_i\delta_{ij} \quad \text{and} \quad \Phi_{ij} = \mu_i\delta_{ij},$$

it is straightforward to check that

$$\sum_i \mu_i = 0,\ \sum_i \mu_i^2 = |\Phi|^2 \quad \text{and} \quad \sum_i \mu_i^3 = \sum_i \lambda_i^3 - 3H|\Phi|^2 - nH^3.$$

Consequently, by applying Lemma 3.1 to the real numbers $\mu_1,\ldots,\mu_n$, we get

$$\begin{aligned} S^2 - nH\sum_i \lambda_i^3 &= (|\Phi|^2 + nH^2)^2 - n^2H^4 \\ &\quad -3nH^2|\Phi|^2 - nH\sum_i \mu_i^3 \\ &\geq |\Phi|^4 - nH^2|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3. \end{aligned} \tag{3.9}$$

Using curvature conditions (1.1) and (1.2), we get

$$-\left(\sum_{i,j} nH\lambda_i \bar{R}_{(n+1)ii(n+1)} + S\sum_k \bar{R}_{(n+1)k(n+1)k}\right) = c_1(S - nH^2) \tag{3.10}$$

and

$$\begin{aligned} -2\sum_{i,j,k,l}(\lambda_i\lambda_k\bar{R}_{kiik} + \lambda_i^2\bar{R}_{ikik}) &\geq c_2\sum_{i,k}(\lambda_i - \lambda_k)^2 \\ &= 2nc_2(S - nH^2). \end{aligned} \tag{3.11}$$

Hence, setting $c = \frac{c_1}{n} + 2c_2$, from (3.8), (3.9), (3.10) and (3.11) we obtain that

$$L(nH) \geq |\Phi|^2\left(nc + S - 2nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|\right). \tag{3.12}$$

On the other hand, with a straightforward computation we verify that

$$\begin{aligned} S - 2nH^2 &= \frac{1}{2\sqrt{n-1}}\left((\sqrt{n-1}+1)|\Phi| - (\sqrt{n-1}-1)\sqrt{n}H\right)^2 \\ &\quad + \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - \frac{n}{2\sqrt{n-1}}S. \end{aligned}$$

Thus, since we are supposing that $S \leq 2\sqrt{n-1}\,c$, from (3.12) we get

$$L(nH) \geq |\Phi|^2\left(nc - \frac{n}{2\sqrt{n-1}}S\right) \geq 0. \tag{3.13}$$

Since Lemma 3.3 guarantees that L is elliptic and as we are supposing that H attains its maximum on M^n, from (3.13) we conclude that H is constant on M^n. Thus, taking into account Eq. (3.7), we get

$$\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2 = 0,$$

and it follows that λ_i is constant for every $i = 1,\ldots,n$. Moreover, from (3.13) we have

$$|\Phi|^2\left(nc - \frac{n}{2\sqrt{n-1}}S\right) = 0. \tag{3.14}$$

If $S < 2\sqrt{n-1}\,c$, then $|\Phi|^2 = 0$ and M^n is totally umbilical. If $S = 2\sqrt{n-1}\,c$, since all the inequalities that we have obtained are, in fact, equalities, we easily verify that

$$|\Phi| = \frac{(\sqrt{n-1}-1)\sqrt{n}}{\sqrt{n-1}+1}H. \tag{3.15}$$

Thus, in the case that $n = 2$, from (3.15) we obtain that $|\Phi|^2 = 0$. Hence, M^2 is totally umbilical. Finally, when $n \geq 3$, since the equality holds in (3.1) of Lemma 3.1, we conclude that M^n is either totally umbilical or an isoparametric hypersurface with two distinct principal curvatures one of which is simple. □

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