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Results in Mathematics

On a Generalization of Szász–Mirakjan–Kantorovich Operators

Francesco Altomare, Mirella Cappelletti Montano
and Vita Leonessa

Abstract. In this paper we introduce and study a sequence of positive linear operators acting on suitable spaces of measurable functions on $[0, +\infty[$, including $L^p([0, +\infty[)$ spaces, $1 \leq p < +\infty$, as well as continuous function spaces with polynomial weights. These operators generalize the Szász–Mirakjan–Kantorovich operators and they allow to approximate (or to reconstruct) suitable measurable functions by knowing their mean values on a sequence of subintervals of $[0, +\infty[$ that do not constitute a subdivision of it. We also give some estimates of the rates of convergence by means of suitable moduli of smoothness.

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positive approximation process, weighted space, modulus of smoothness.

1. Introduction

In the 1940s G. M. Mirakjan [16], J. Favard [13] and O. Szàsz [19] independently studied a sequence $(S_n)_{n\geq 1}$ of positive linear operators, that nowadays are called Szász–Mirakjan operators. These operators are defined by

$$S_n(f)(x) := \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \qquad (n \geq 1, x \geq 0)$$

for all functions $f : [0, +\infty[\longrightarrow \mathbf{R}$ for which the series at the right-hand side is absolutely convergent. The space of such functions includes, in particular, the space $\mathscr{S}([0, +\infty[)$ of all functions $f : [0, +\infty[\longrightarrow \mathbf{R}$ such that $|f(x)| \leq M \exp(\alpha x)$ $(x \geq 0)$, for some $M \geq 0$ and $\alpha \in \mathbf{R}$.

Later on, in order to furnish an approximation process for spaces of locally integrable functions on unbounded intervals, Butzer [8] introduced and studied an integral modification of the operators S_n; they are defined by setting, for every $n \geq 1, f \in \mathscr{T}([0,+\infty[)$ and $x \geq 0$,

$$K_n(f)(x) := n \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt,$$

where $\mathscr{T}([0,+\infty[)$ is the space of all Borel measurable locally integrable functions $f : [0,+\infty[\longrightarrow \mathbf{R}$ such that the antiderivative $F(x) := \int_0^x f(t)\, dt (x \geq 0)$ belongs to $\mathscr{S}([0,+\infty[)$.

The operators $(K_n)_{n\geq 1}$ were named in [20] as to Szàsz–Mirakjan–Kantorovich operators by analogy with the Kantorovich operators that constitute a similar integral modification of Bernstein operators (see, e.g. [2, pp. 333–335]).

In more recent years, Szász–Mirakjan–Kantorovich operators and their modifications have been object of investigation by several mathematicians. For example, some saturation results were discussed by Totik in [21] (see also [20]) and further properties were studied in [11, Chapter 9]. More recent results may be found in [12].

In this paper we deal with a further generalization of such operators, that extends to the unbounded setting an idea first developed in [4], where the authors introduced and studied a generalization of Kantorovich operators. Namely, we will focus our attention on a sequence $(C_n)_{n\geq 1}$ of positive linear operators defined by

$$C_n(f)(x) := \frac{n}{b_n - a_n} \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \int_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} f(t)\, dt \qquad (n \geq 1,\ x \geq 0)$$

for every $f \in \mathscr{T}([0,+\infty[)$, where $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences of the real numbers satisfying $0 \leq a_n < b_n \leq 1$ for every $n \geq 1$.

Of course, if $a_n = 0$ and $b_n = 1$ for all $n \geq 1$, then the C_n's turn into the Szàsz–Mirakian–Kantorovich operators. A possible interest in the study of C_n's rests on the fact that, by means of them, it is possible to reconstruct a continuous or an integrable function by knowing its mean values on subintervals of $[0,+\infty[$ which do not necessarily constitute a subdivision of $[0,+\infty[$.

We investigate the approximation properties of the sequence $(C_n)_{n\geq 1}$ on several continuous and weighted continuous function spaces as well as on Lebesgue spaces and we also establish some estimates of the rate of convergence by means of suitable moduli of smoothness.

The paper is organized as follows. After some preliminaries, in Sect. 2 we present our operators and their main properties.

Subsequently, in Sect. 3, we discuss their behavior on some continuous function spaces and on weighted continuous function spaces with polynomial weights by proving that they are an approximation process with respect to the uniform norm and to particular weighted norms. Moreover, we prove that our operators are an approximation process also on $L^p([0,+\infty[)(1 \le p < +\infty)$.

In the last section, we estimate the rate of convergence; in particular, with a similarity technique, we prove that, up an isometric isomorphism, the study of such questions is equivalent to the study of the rate of convergence of suitable approximation processes acting on $[0,1]$. Similar arguments can be found in [7].

2. Generalizing Szász–Mirakjan–Kantorovich Operators

Throughout this paper we shall denote by $\mathscr{C}([0,+\infty[)$ the space of all continuous real valued functions on $[0,+\infty[$. We shall also denote by $\mathscr{C}_b([0,+\infty[)$ the subspace of all functions in $\mathscr{C}([0,+\infty[)$ that are bounded.

The space $\mathscr{C}_b([0,+\infty[)$, endowed with the sup-norm $\|\cdot\|_\infty$ and the natural pointwise ordering, is a Banach lattice. The space of all continuous functions that converge at infinity will be denoted by $\mathscr{C}_*([0,+\infty[)$; clearly, $\mathscr{C}_*([0,+\infty[)$ is a Banach sublattice of $\mathscr{C}_b([0,+\infty[)$.

Further, $\mathscr{C}_0([0,+\infty[)$ stands for the subspace of $\mathscr{C}_*([0,+\infty[)$ consisting of all continuous real valued functions on $[0,+\infty[$ that vanish at infinity.

Moreover, for $m \ge 1$ we set $w_m(x) := (1+x^m)^{-1}$ $(x \ge 0)$ and

$$E_m := \left\{ f \in \mathscr{C}([0,+\infty[) \mid \sup_{x\ge 0} w_m(x)|f(x)| \in \mathbf{R} \right\};$$

E_m is a Banach lattice, provided that it is endowed with the pointwise ordering and the weighted norm

$$\|f\|_m := \sup_{x\ge 0} w_m(x)|f(x)| \qquad (f \in E_m).$$

Further, we shall consider the spaces

$$E_m^* := \left\{ f \in E_m \mid \lim_{x\to+\infty} w_m(x)f(x) \in \mathbf{R} \right\}$$

and

$$E_m^0 := \left\{ f \in E_m^* \mid \lim_{x\to+\infty} w_m(x)f(x) = 0 \right\},$$

that turn out to be Banach sublattices of E_m.

Note that, by Stone–Weierstrass theorem, $\mathscr{C}_0([0,+\infty[)$ is dense in each $E_m^0, m \ge 1$.

As usual, if $1 \le p < +\infty$, we shall denote by $L^p([0,+\infty[)$ the space of all (equivalence classes of) Borel measurable functions on $[0,+\infty[$ such that $\|f\|_p := \left(\int_0^{+\infty} |f(t)|^p \, dt\right)^{\frac{1}{p}} < +\infty$. Moreover, $L^\infty([0,+\infty[)$ stands for the space of all (equivalence classes of) Borel measurable functions on $[0,+\infty[$ that are λ_1-a.e. bounded, λ_1 being the Borel-Lebesgue measure on $[0,+\infty[$.

Consider the sequence $(S_n)_{n\geq 1}$ of Szász–Mirakjan operators defined e.g., on the space $\mathscr{S}([0,+\infty[)$ of all functions $f:[0,+\infty[\longrightarrow \mathbf{R}$ such that $|f(x)|\leq M\exp(\alpha x)$ $(x\geq 0)$, for some $M\geq 0$ and $\alpha\in\mathbf{R}$, by setting

$$S_n(f)(x) := \sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \qquad (n\geq 1, x\geq 0). \tag{2.1}$$

An integral modification of them was introduced by Butzer [8] in order to furnish a positive approximation process for locally integrable functions on $[0,+\infty[$. These operators are defined on the space $\mathscr{T}([0,+\infty[)$ of all Borel measurable locally integrable functions $f:[0,+\infty[\longrightarrow\mathbf{R}$ such that the antiderivative $F(x):=\int_0^x f(t)\,dt (x\geq 0)$ belongs to $\mathscr{S}([0,+\infty[)$, by setting

$$K_n(f)(x) := n\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt \qquad (n\geq 1, x\geq 0).$$

Note that $\mathscr{T}([0,+\infty[)$ contains $\mathscr{S}([0,+\infty[)\cap\mathscr{C}([0,+\infty[)$ (and hence $E_m, m\geq 0$) as well as $L^p([0,+\infty[)$ spaces, $1\leq p\leq+\infty$.

The operators $K_n, n\geq 1$, were named in [20] as to Szász–Mirakjan–Kantorovich operators. Thus, they can be used to approximate functions in $\mathscr{T}([0,+\infty[)$ by having information about their mean values on the intervals $\left[\frac{k}{n},\frac{k+1}{n}\right], n\geq 1, k\geq 1$.

In this paper, in the spirit of a similar idea developed in [4] for compact intervals (see also [3]), we introduce a generalization of the operators K_n by involving the mean values of the approximating functions on possibly smaller subintervals of $\left[\frac{k}{n},\frac{k+1}{n}\right], n\geq 1, k\geq 1$.

More precisely, consider two sequences of real numbers $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that $0\leq a_n<b_n\leq 1$ for every $n\geq 1$ and, for every $f\in\mathscr{T}([0,+\infty[), x\geq 0$ and $n\geq 1$, set

$$C_n(f)(x) := \frac{n}{b_n-a_n}\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\int_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} f(t)\,dt. \tag{2.2}$$

Thus, $C_n(f)$ is defined throughout the mean values of f on the sets $\left[\frac{k+a_n}{n},\frac{k+b_n}{n}\right]$ $(k\geq 0)$ that do not cover the whole $[0,+\infty[$.

Of course, if $a_n=0$ and $b_n=1$ for every $n\geq 1$, then the $C_n's$ turn into the operators K_n.

For a given $f\in\mathscr{T}([0,+\infty[)$, considering the antiderivative $F(x)=\int_0^x f(t)\,dt (x\geq 0)$, we can also write

$$\begin{aligned} C_n(f)(x) &= \frac{n}{b_n-a_n}\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\left[F\left(\frac{k+b_n}{n}\right)-F\left(\frac{k+a_n}{n}\right)\right] \\ &= \frac{n}{b_n-a_n} S_n(\sigma_n(F))(x), \end{aligned} \tag{2.3}$$

where S_n is given by (2.1) and the mapping σ_n is defined as

$$\sigma_n(F)(x) := F\left(x + \frac{b_n}{n}\right) - F\left(x + \frac{a_n}{n}\right) \quad (x \geq 0). \tag{2.4}$$

It will be also useful to represent $C_n(f)$ as

$$C_n(f)(x) := \int_0^{+\infty} f\, d\mu_{n,x} \qquad (n \geq 1, x \geq 0), \tag{2.5}$$

where

$$\mu_{n,x} := \frac{n}{b_n - a_n} e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \mu_{n,k}$$

and each $\mu_{n,k}$ designates the Borel measure on $[0, +\infty[$ having density the characteristic function of $\left[\frac{k+a_n}{n}, \frac{k+b_n}{n}\right]$ with respect to the Borel-Lebesgue measure λ_1 on $[0, +\infty[$.

From now on, for every $m \geq 0$ the symbol e_m will denote the function defined by setting $e_m(x) = x^m (x \geq 0)$; in particular $e_0 = \mathbf{1}$, where $\mathbf{1}$ denotes the constant function on $[0, +\infty[$ of constant value 1. Finally, for a fixed $x \geq 0$, we shall set $\psi_x(y) := y - x \ (y \geq 0)$.

It will be useful to recall the behavior of the Szász–Mirakjan operators (2.1) on the above mentioned functions; more precisely (see [6, Lemma 3]), for every $n \geq 1$ and $m \geq 0$,

$$S_n(e_m) = \sum_{j=0}^{m} a_{m,j} n^{j-m} e_j, \tag{2.6}$$

for some positive coefficients $a_{m,j}$ satisfying the following properties:

(i) $a_{j,j} = 1$ for every $j = 0, \ldots, m$ and $a_{j,0} = 0$ for every $j \geq 1$;
(ii) $a_{j,1} = 1$ for every $j = 1, \ldots, m$;
(iii) $a_{j,j-1} = j(j-1)/2$ for every $j = 1, \ldots, m$;
(iv) $a_{j+2,j+1} - 2a_{j+1,j} + a_{j,j-1} = 1$ for every $j = 1, \ldots, m-2$.

Hence, for every $m \geq 1, S_n(e_m)$ is a polynomial of degree m with no constant terms.

In particular,

$$S_n(\mathbf{1}) = \mathbf{1}, \qquad S_n(e_1) = e_1 \quad \text{and} \quad S_n(e_2) = e_2 + \frac{1}{n} e_1. \tag{2.7}$$

Moreover, for every $x \geq 0$,

$$S_n(\psi_x)(x) = 0 \quad \text{and} \quad S_n(\psi_x^2)(x) = \frac{x}{n}. \tag{2.8}$$

Finally, for a given $\lambda > 0$, if we set

$$f_\lambda(x) := e^{-\lambda x} \quad (x \geq 0), \tag{2.9}$$

we get

$$S_n(f_\lambda)(x) = \exp\left(nx\left(e^{-\frac{\lambda}{n}} - 1\right)\right) \tag{2.10}$$

(see, e.g., [2, pp. 339–340]).

For the sake of brevity we omit the details of the proof of the next result that can be achieved by direct calculations on account also of the formula

$$C_n(f) = K_n(f_n), \tag{2.11}$$

where

$$f_n(x) = \int_0^1 f(x + ((b_n - a_n)y + a_n)/n)dy \tag{2.12}$$

for every $f \in \mathscr{T}([0, +\infty[)$, $x \geq 0$ and $n \geq 1$.

Proposition 2.1. *For every $n \geq 1$ and $m \geq 0$,*

$$\begin{aligned} C_n(e_m) &= \frac{1}{(m+1)n^m} \sum_{k=0}^{m} \binom{m+1}{k} \sum_{p=0}^{m-k} b_n^p a_n^{m-k-p} \sum_{j=0}^{k} a_{k,j} n^j e_j \\ &= e_m + \frac{1}{n} F_{m-1}, \end{aligned} \tag{2.13}$$

where the coefficients $a_{k,j}$ are the same as in (2.6) and F_{m-1} is a positive polynomial of degree $m-1$. In particular, $e_m \leq C_n(e_m)$ for every $m \geq 0$.

Further, denoting by $\mathbb{P}_m$ the space of (the restrictions to $[0, +\infty[$ of) all polynomials of degree no grater than $m, m \geq 1$, then

$$C_n(\mathbb{P}_m) \subset \mathbb{P}_m$$

for every $n, m \geq 1$.

Moreover, for every $m \geq 1, n \geq 1$ and $x \geq 0$,

$$w_m(x) C_n(e_m)(x) \leq w_m(x) e_m(x) + \frac{d_m}{n}, \tag{2.14}$$

where

$$d_m := \max_{x \geq 0} w_m(x) \left\{ \sum_{j=0}^{m-1} a_{m,j} x^j + \sum_{k=0}^{m-1} \binom{m}{k} \sum_{j=0}^{k} a_{k,j} x^j \right\} \tag{2.15}$$

and $w_m(x) = (1 + x^m)^{-1} (x \geq 0)$.

Hence, for every $m \geq 0$,

$$\lim_{n \to \infty} ||C_n(e_m) - e_m||_m = 0. \tag{2.16}$$

Finally, for every $n, m \geq 1$ and $x \geq 0$,

$$C_n(\psi_x^m) = \sum_{h=0}^{m} \binom{m}{h} \frac{(-1)^{m-h} x^{m-h}}{(h+1)n^h} \sum_{k=0}^{h} \binom{h+1}{k} \sum_{p=0}^{h-k} b_n^p a_n^{h-k-p} \sum_{j=0}^{k} a_{k,j} n^j e_j. \tag{2.17}$$

In particular,

$$C_n(\mathbf{1}) = \mathbf{1},\ C_n(e_1) = e_1 + \frac{a_n + b_n}{2n}\mathbf{1}, \tag{2.18}$$

$$C_n(e_2) = e_2 + \frac{b_n + a_n + 1}{n}e_1 + \frac{a_n^2 + a_n b_n + b_n^2}{3n^2}\mathbf{1} \tag{2.19}$$

and, for every $x \geq 0$,

$$C_n(\psi_x)(x) = \frac{a_n + b_n}{2n} \quad \text{and} \quad C_n(\psi_x^2)(x) = \frac{x}{n} + \frac{a_n^2 + a_n b_n + b_n^2}{3n^2}. \tag{2.20}$$

Another useful result is stated below.

Proposition 2.2. *For every* $\lambda > 0$, *let* f_λ *be as in* (2.9). *Then,*

$$C_n(f_\lambda) = \frac{n}{\lambda(b_n - a_n)}\left(e^{-\frac{\lambda a_n}{n}} - e^{-\frac{\lambda b_n}{n}}\right)S_n(f_\lambda) \qquad (n \geq 1), \tag{2.21}$$

where $S_n(f_\lambda)$ *is evaluated in* (2.10) (*see also* (2.1)).

Moreover, for every $n \geq 1$ *and* $\lambda > 0$,

$$C_n(f_\lambda) \leq S_n(f_\lambda) \leq S_1(f_\lambda). \tag{2.22}$$

Proof. Formula (2.21) follows after a straightforward computation. As regards (2.22), the first inequality is an easy consequence of the fact that

$$\frac{n}{\lambda(b_n - a_n)}\left(e^{-\frac{\lambda a_n}{n}} - e^{-\frac{\lambda b_n}{n}}\right) \leq \frac{n}{\lambda(b_n - a_n)}\left(1 - e^{-\left(\frac{\lambda b_n}{n} - \frac{\lambda a_n}{n}\right)}\right) \leq 1;$$

here we have used the well-known inequality $1 - e^{-x} \leq x$ $(x \geq 0)$.

On the other hand, by the monotonicity of $(S_n(f))_{n\geq 1}$ on convex functions (see [9, p. 247]), we get the second inequality in (2.22). □

3. Approximation Properties

In this section we deal with some approximation properties of the sequence $(C_n)_{n\geq 1}$ in several spaces of both continuous and integrable functions.

We begin with the following result.

Theorem 3.1. *Consider the operators* $C_n, n \geq 1$, *defined by* (2.2). *Then, for* $n \geq 1$ *and* $m \geq 1$ *fixed,*

- *(a)* C_n *is a positive continuous linear operator from* $\mathscr{C}_b([0, +\infty[)$ *into itself and* $||C_n||_{\mathscr{C}_b([0,+\infty[)} = 1$;
- *(b)* $C_n(\mathscr{C}_0([0, +\infty[) \subset \mathscr{C}_0([0, +\infty[)$;
- *(c)* C_n *is a positive continuous linear operator from* E_m *into itself and* $||C_n||_{E_m} \leq 1 + d_m/n$, d_m *being defined by* (2.15); *in particular,*

$$\sup_{n\geq 1} ||C_n||_{E_m} \leq 1 + d_m; \tag{3.1}$$

- *(d)* $C_n(E_m^0) \subset E_m^0$.

Proof. Statement (a) can be easily verified, since $C_n(\mathbf{1}) = \mathbf{1}$. To prove statement (b), fix $f \in \mathscr{C}_0([0,+\infty[)$ and $\epsilon > 0$; then, there exists $x_1 \geq 0$ such that $|f(x)| \leq \epsilon/2$ for any $x \geq x_1$.

Moreover, consider $x_2 > x_1$ such that, for every $x \geq x_2$,

$$\frac{(nx)^h e^{-nx}}{h!} \leq \frac{\epsilon}{2||f||_\infty (n[x_1]+1)},$$

for any $h = 0, \ldots, n[x_1]$, where $[x_1]$ denotes the integer part of x_1.

Now, for $x \geq x_2$,

$$\begin{aligned} |C_n(f)(x)| &\leq \frac{n}{b_n - a_n} \sum_{k=0}^{n[x_1]} e^{-nx} \frac{(nx)^k}{k!} \int_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} |f(t)|\, dt \\ &\quad + \frac{n}{b_n - a_n} \sum_{k=n[x_1]+1}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \int_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} |f(t)|\, dt \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} = \epsilon. \end{aligned}$$

Statement (c) is a consequence of (2.14) because, for every $f \in E_m$ and $x \geq 0$,

$$w_m(x)|C_n(f)(x)| \leq ||f||_m w_m(x) C_n(\mathbf{1} + e_m)(x) \leq ||f||_m \left(1 + \frac{d_m}{n}\right).$$

To prove statement (d), consider the subspace D generated by the family $(f_\lambda)_{\lambda>0}$ defined by (2.9), that by Stone–Weierstrass theorem is dense in $\mathscr{C}_0([0,+\infty[)$ and hence in E_m^0. Since

$$C_n(D) \subset \mathscr{C}_0([0,+\infty[) \subset E_m^0,$$

we get inclusion (d). □

Remark 3.2. *We point out that, since $C_n(\mathbf{1}) = \mathbf{1}$, from Theorem 3.1, (b), it also follows that $C_n(\mathscr{C}_*([0,+\infty[)) \subset \mathscr{C}_*([0,+\infty[)$ for all $n \geq 1$.*

Moreover, (2.13) implies that $C_n(\mathbf{1} + e_m) = \mathbf{1} + C_n(e_m) \in E_m^*$, i.e., $C_n(e_m) \in E_m^*$. Then, again by Theorem 3.1, (d), we also get that $C_n(E_m^*) \subset E_m^*$.

In order to state the next approximation results, we first notice that if $\lambda > 0, n \geq 1$ and $0 \leq a_n < b_n \leq 1$, then

$$0 \leq 1 - \frac{n}{\lambda(b_n - a_n)} \left(e^{-\frac{\lambda a_n}{n}} - e^{-\frac{\lambda b_n}{n}}\right) \leq \frac{\lambda}{n}. \tag{3.2}$$

In fact, by using the inequalities $1-e^{-x} \le x, 1-e^{-x} \ge x - x^2/2$ $(x \ge 0)$, we get

$$\begin{aligned}0 &\le 1 - \frac{n}{\lambda(b_n - a_n)}\left(e^{-\frac{\lambda a_n}{n}} - e^{-\frac{\lambda b_n}{n}}\right) \\ &= 1 - \frac{n}{\lambda(b_n - a_n)} e^{-\frac{\lambda a_n}{n}}\left(1 - e^{-\frac{\lambda(b_n - a_n)}{n}}\right) \\ &\le 1 - \frac{n}{\lambda(b_n - a_n)} e^{-\frac{\lambda a_n}{n}}\left(\lambda\frac{b_n - a_n}{n} - \lambda^2\frac{(b_n - a_n)^2}{2n^2}\right) \\ &= 1 - e^{-\frac{\lambda a_n}{n}} + \lambda\frac{b_n - a_n}{2n} \le \frac{\lambda}{2n}(a_n + b_n) \le \frac{\lambda}{n}.\end{aligned}$$

As far as bounded continuous functions are concerned, the following approximation result holds.

Theorem 3.3. *If* $f \in \mathscr{C}_*([0, +\infty[)$, *then* $\lim_{n\to+\infty} C_n(f) = f$ *uniformly on* $[0, +\infty[$.

Moreover, if $f \in \mathscr{C}_b([0, +\infty[)$, *then* $\lim_{n\to+\infty} C_n(f) = f$ *uniformly on compacts subsets of* $[0, +\infty[$.

Proof. It suffices to show the first part of the statement for $f \in \mathscr{C}_0([0, +\infty[)$ or, in fact, for each function $f_\lambda, \lambda > 0$, defined by (2.9), since the subspace generated by them is dense in $\mathscr{C}_0([0, +\infty[)$ and the sequence $(C_n)_{n\ge1}$ is equibounded on $\mathscr{C}_0([0, +\infty[)$. Now, by using (2.21) and (3.2), for every $x \ge 0$ and $n \ge 1$, we get

$$\begin{aligned}&|C_n(f_\lambda)(x) - f_\lambda(x)| \\ &\quad\le \left|\frac{n}{\lambda(b_n - a_n)}\left(e^{-\frac{\lambda a_n}{n}} - e^{-\frac{\lambda b_n}{n}}\right) - 1\right| S_n(f_\lambda)(x) + |S_n(f_\lambda)(x) - f_\lambda(x)| \\ &\quad\le \left(1 - \frac{n}{\lambda(b_n - a_n)}\left(e^{-\frac{\lambda a_n}{n}} - e^{-\frac{\lambda b_n}{n}}\right)\right) + ||S_n(f_\lambda) - f_\lambda||_\infty \\ &\quad\le \frac{\lambda}{n} + ||S_n(f_\lambda) - f_\lambda||_\infty.\end{aligned}$$

Since the sequence $(S_n)_{n\ge1}$ of Szász–Mirakjan operators (see (2.1)) is an approximation process on $\mathscr{C}_0([0, +\infty[)$ (see [2, Sect. 5.3.9]), the result is obviously achieved.

In order to prove the final statement, we notice that, from (2.18) and (2.19), it follows that $\lim_{n\to+\infty} C_n(h) = h$ uniformly on compact subsets of $[0 + \infty[$ for every $h \in \{\mathbf{1}, e_1, e_2\}$. Since $\{\mathbf{1}, e_1, e_2\} \subset E_2^*$, the result follows from [1, Theorem 3.5]. □

The approximation properties of the operators C_n on the weighted function spaces E_m^0, E_m^* and E_m are shown below.

Theorem 3.4. *For a given $m \geq 1$, if $f \in E_m^*$ (and, in particular, if $f \in E_m^0$), then*

$$\lim_{n\to+\infty} w_m(C_n(f)-f)=0 \qquad \textit{uniformly on } [0,+\infty[\tag{3.3}$$

i.e., $\lim_{n\to+\infty} C_n(f)=f$ with respect to $\|\cdot\|_m$.

If $f \in E_m$, then

$$\lim_{n\to+\infty} w_m(C_n(f)-f)=0 \tag{3.4}$$

uniformly on compact subsets of $[0,+\infty[$.

Proof. Consider again the functions $f_\lambda, \lambda > 0$, defined by (2.9). Then $\lim_{n\to+\infty} C_n(f_\lambda)=f_\lambda$ with respect to $\|\cdot\|_\infty$ and hence with respect to $\|\cdot\|_m$. Since the sequence $(C_n)_{n\geq1}$ is equibounded on E_m^0 (see (3.1)) and the linear subspace generated by $(f_\lambda)_{\lambda>0}$ is dense in E_m^0, (3.3) is certainly true for $f \in E_m^0$. On the other hand, if $f \in E_m^*$, then $f = g + \alpha_m(\mathbf{1}+e_m)$, where $\alpha_m := \lim_{x\to+\infty} w_m(x)f(x) \in \mathbf{R}$ and $g = f-\alpha_m(\mathbf{1}+e_m) \in E_m^0$. Therefore (3.3) follows for f too on account of (2.16).

The preceding result and the inclusion $E_m \subset E_{m+1}^0$ imply formula (3.4) because, if J is a compact subset of $[0,+\infty[$, then

$$w_m(x)|C_n(f)(x)-f(x)| \leq M\|C_n(f)-f\|_{m+1}$$

for every $x \in J$, where $M := \sup\limits_{x\in J} \frac{w_m(x)}{w_{m+1}(x)}$. □

Finally, we show that, in some particular cases, the C_n's furnish an approximation process in $L^p([0,+\infty[)$ spaces $(1 \leq p < +\infty)$.

Theorem 3.5. *Let $(C_n)_{n\geq1}$ be the sequence of operators defined by (2.2) and fix $1 \leq p < +\infty$. Then, $C_n(L^p([0,+\infty[)) \subset L^p([0,+\infty[)$ and*

$$\|C_n\|_{L^p,L^p} \leq (b_n-a_n)^{-\frac{1}{p}}.$$

for every $n \geq 1$.

Moreover, if there exists $M>0$ such that $\frac{1}{b_n-a_n} \leq M$ for every $n \geq 1$, then, for every $f \in L^p([0,+\infty[)$,

$$\lim_{n\to+\infty} C_n(f)=f \quad \textit{in } \; L^p([0,+\infty[).$$

Proof. Consider $n \geq 1$ and $f \in L^p([0,+\infty[)$. A twofold application of Jensen's inequality (see, e.g., [5, Theorem 3.9]) yields

$$
\begin{aligned}
|C_n(f)(x)|^p &\leq \sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\left|\frac{n}{b_n-a_n}\int_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} f(t)\,dt\right|^p \\
&\leq \sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\frac{n}{b_n-a_n}\int_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} |f(t)|^p\,dt
\end{aligned}
$$

for every $x \geq 0$.

By integrating with respect to $x \geq 0$ and by using the identities

$$
\int_0^{+\infty} e^{-nx}x^k\,dx = \frac{k!}{n^{k+1}} \quad (k \geq 0),
$$

we get

$$
\begin{aligned}
\int_0^{+\infty} |C_n(f)(x)|^p\,dx &\leq \sum_{k=0}^{\infty}\left(\frac{n}{b_n-a_n}\int_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} |f(t)|^p\,dt\right)\frac{n^k}{k!}\int_0^{+\infty} e^{-nx}x^k\,dx \\
&= \frac{1}{n}\sum_{k=0}^{\infty}\left(\frac{n}{b_n-a_n}\int_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} |f(t)|^p\,dt\right) \leq \frac{1}{b_n-a_n}\|f\|_p^p,
\end{aligned}
$$

which shows the first part of the theorem.

Assume now that there exists $M > 0$ such that $\frac{1}{b_n-a_n} \leq M$ for every $n \geq 1$; then, the sequence $(C_n)_{n\geq 1}$ is equibounded in $L^p([0,+\infty[)$.

Moreover, by [2, Proposition 4.2.5, (2)] (see also [1, Corollary 8.9]), the subset $\{f_\lambda \mid \lambda > 0\}$ is a Korovkin set in $L^p([0,+\infty[)$ (see (2.9)) (in fact, any subset $\{f_{\lambda_1}, f_{\lambda_2}, f_{\lambda_3}\}$ with $0 < \lambda_1 < \lambda_2 < \lambda_3$ is a Korovkin subset in $L^p([0,+\infty[)$). So, in order to prove the final claim, it is sufficient to ascertain that $C_n(f_\lambda) \to f_\lambda$ in $L^p([0,+\infty[)$ for every $\lambda > 0$. By Theorem 3.3 we already know that $C_n(f_\lambda) \to f_\lambda$ uniformly and, hence, pointwise on $[0,+\infty[$. On the other hand, by (2.10) and (2.22) we get

$$
0 \leq |C_n(f_\lambda)|^p \leq |S_1(f_\lambda)|^p \in L^1([0,+\infty[).
$$

Then, by the dominated convergence theorem,

$$
\lim_{n\to+\infty} C_n(f_\lambda) = f_\lambda \quad \text{in} \;\; L^p([0,+\infty[).
$$

□

4. Estimating the Rate of Convergence

We pass now to present several estimates of the rate of convergence of $(C_n(f))_{n\geq 1}$ to f by means of suitable moduli of smoothness.

For the convenience of the reader we split up the discussion of the several types of convergence into the following three subsections.

4.1. Pointwise and Uniform Estimates for the Rate of Convergence

In what follows, we state some estimates of the pointwise and uniform rate of convergence involving the usual moduli of smoothness of the first and second order $\omega(f,\delta)$ and $\omega_2(f,\delta)$ (for the definitions of the above-mentioned moduli of smoothness we refer, e.g., to [2, Sect. 5.1] or [10, Chapter 2, § 7]).

By using some results in [17], we start to provide for some estimates for the rate of pointwise convergence.

Theorem 4.1. *Consider* $f \in \mathscr{C}_b([0,+\infty[), n \geq 1$ *and* $x \geq 0$. *Then*

$$|C_n(f)(x) - f(x)| \leq \frac{a_n + b_n}{2\sqrt{n}}\omega\left(f, \frac{1}{\sqrt{n}}\right) + \left[1 + \frac{1}{2}\left(x + \frac{a_n^2 + a_n b_n + b_n^2}{3n}\right)\right]\omega_2\left(f, \frac{1}{\sqrt{n}}\right). \tag{4.1}$$

Proof. Since (2.5) holds, by means of [17, Theorem 2.2.1] and (2.20), we have that, for every $\delta > 0$,

$$\begin{aligned} &|C_n(f)(x) - f(x)| \\ &\quad\leq |C_n(\mathbf{1})(x) - 1||f(x)| + \frac{1}{\delta}|C_n(\psi_x)(x)|\omega(f,\delta) \\ &\qquad + \left[C_n(\mathbf{1})(x) + \frac{1}{2\delta^2}C_n(\psi_x^2)(x)\right]\omega_2(f,\delta) \\ &\quad= \frac{1}{\delta}\frac{a_n + b_n}{2n}\omega(f,\delta) + \left[1 + \frac{1}{2\delta^2}\left(\frac{x}{n} + \frac{a_n^2 + a_n b_n + b_n^2}{3n^2}\right)\right]\omega_2(f,\delta). \end{aligned}$$

If $\delta = n^{-1/2}$, then we get (4.1). □

It is possible to present other estimates of the rate of convergence of the C_n's in $\mathscr{C}_b([0,+\infty[)$. In fact from (2.3) it follows that, for every $f \in \mathscr{C}_b([0,+\infty[)$ and $x \geq 0$,

$$\begin{aligned} &|C_n(f)(x) - f(x)| \\ &\quad\leq \frac{n}{b_n - a_n}|S_n(\sigma_n(F))(x) - \sigma_n(F)(x)| + \left|\frac{n}{b_n - a_n}\sigma_n(F)(x) - f(x)\right|, \end{aligned}$$

where σ_n is given by (2.4) and F is the antiderivative of f.

Therefore we can obtain some quantitative estimates for the rate of pointwise convergence of $(C_n(f))_{n\geq 1}$ by means of similar ones held by Szàsz–Mirakjan operators and by using the following lemma (see also [4, Theorem 3.3]).

Lemma 4.2. *Let $0 \leq a_n < b_n \leq 1$ $(n \geq 1)$, $f \in \mathscr{C}_b([0,+\infty[)$ and $F(x) = \int_0^x f(t)\,dt$ $(x \geq 0)$. Then, for every $x \geq 0$ and $n \geq 1$,*

$$\left|\frac{n}{b_n - a_n}\sigma_n(F)(x) - f(x)\right| \leq \omega\left(f, \frac{b_n - a_n}{n}\right). \tag{4.2}$$

Moreover, for every $\delta > 0$,

$$\omega(\sigma_n(F), \delta) \leq \frac{b_n - a_n}{n}\omega\left(f, \delta + \frac{b_n - a_n}{n}\right). \tag{4.3}$$

Proof. Fix $x \geq 0$ and $n \geq 1$; by applying Lagrange's theorem to the function F and the interval $\left[x + \frac{a_n}{n}, x + \frac{b_n}{n}\right]$, there exists $\zeta_{n,x} \in \left[x + \frac{a_n}{n}, x + \frac{b_n}{n}\right]$ such that

$$\frac{n}{b_n - a_n}\sigma_n(F)(x) = f(\zeta_{n,x}).$$

Then

$$\left|\frac{n}{b_n - a_n}\sigma_n(F)(x) - f(x)\right| = |f(\zeta_{n,x}) - f(x)| \leq \omega(f, |\zeta_{n,x} - x|) \leq \omega\left(f, \frac{b_n - a_n}{n}\right).$$

Now fix $\delta > 0$ and $x, y \geq 0$ such that $|x-y| < \delta$; then, again by Lagrange's theorem,

$$|\sigma_n(F)(x) - \sigma_n(F)(y)| = \frac{b_n - a_n}{n}|f(\zeta_{n,x}) - f(\eta_{n,y})| \leq \frac{b_n - a_n}{n}\omega(f, |\zeta_{n,x} - \eta_{n,y}|),$$

where $\zeta_{n,x}$ is defined as above and $\eta_{n,y}$ is a suitable element of the interval $\left[y + \frac{a_n}{n}, y + \frac{b_n}{n}\right]$, and hence the claim, since

$$|\zeta_{n,x} - \eta_{n,y}| \leq |x - y| + \frac{b_n - a_n}{n} \leq \delta + \frac{b_n - a_n}{n}.$$

□

We are now in a position to state the following result.

Theorem 4.3. *Consider $f \in \mathscr{C}_b([0,+\infty[), n \geq 1$ and $x \geq 0$. Then*

$$|C_n(f)(x) - f(x)| \leq \left(2 + \sqrt{x}\right)\omega\left(f, \frac{\sqrt{n} + b_n - a_n}{n}\right). \tag{4.4}$$

Furthermore, if f is differentiable on $[0,+\infty[$ and $f' \in \mathscr{C}_b([0,+\infty[)$, then

$$|C_n(f)(x) - f(x)| \leq \sqrt{\frac{x}{n}}(1 + \sqrt{x})\omega\left(f', \frac{\sqrt{n} + b_n - a_n}{n}\right) + \|f'\|_\infty\frac{b_n - a_n}{n}. \tag{4.5}$$

Proof. From [2, Theorem 5.2.4], it follows that, for every $\delta > 0$,

$$|S_n(f)(x) - f(x)| \le \left(1 + \frac{1}{\delta}\sqrt{\frac{x}{n}}\right)\omega(f,\delta). \tag{1}$$

From this and from (4.2) and (4.3), we obtain

$$\begin{aligned}
&|C_n(f)(x) - f(x)| \\
&\quad \le \frac{n}{b_n - a_n}|S_n(\sigma_n(F))(x) - \sigma_n(F)(x)| + \left|\frac{n}{b_n - a_n}\sigma_n(F)(x) - f(x)\right| \\
&\quad \le \left(1 + \frac{1}{\delta}\sqrt{\frac{x}{n}}\right)\frac{n}{b_n - a_n}\omega(\sigma_n(F),\delta) + \omega\left(f, \frac{b_n - a_n}{n}\right) \\
&\quad \le \left(1 + \frac{1}{\delta}\sqrt{\frac{x}{n}}\right)\omega\left(f, \delta + \frac{b_n - a_n}{n}\right) + \omega\left(f, \frac{b_n - a_n}{n}\right) \\
&\quad \le \left(2 + \frac{1}{\delta}\sqrt{\frac{x}{n}}\right)\omega\left(f, \delta + \frac{b_n - a_n}{n}\right).
\end{aligned}$$

Setting $\delta = n^{-1/2}$, we get (4.4).

As for (4.5), assume that f is differentiable on $[0, +\infty[$ and $f' \in \mathscr{C}_b([0, +\infty[)$; then

$$\omega\left(f, \frac{b_n - a_n}{n}\right) \le \|f'\|_\infty \frac{b_n - a_n}{n}. \tag{2}$$

Moreover, from [2, Theorem 5.2.4], we obtain, for $\delta > 0$,

$$|S_n(f)(x) - f(x)| \le \sqrt{\frac{x}{n}}\left(1 + \frac{1}{\delta}\sqrt{\frac{x}{n}}\right)\omega(f',\delta). \tag{3}$$

We notice that, along with $f, \sigma_n(F)$ is differentiable and its derivative is continuous and bounded. Moreover, for every $x \ge 0$ and $n \ge 1$,

$$\sigma_n(F)'(x) = f\left(x + \frac{b_n}{n}\right) - f\left(x + \frac{a_n}{n}\right).$$

Pick now $x, y \ge 0$, such that $|x - y| < \delta$; then, arguing as in the proof of Lemma 4.2, by applying Lagrange's theorem, there exist $\zeta_{n,x} \in \left[x + \frac{a_n}{n}, x + \frac{b_n}{n}\right]$ and $\eta_{n,y} \in \left[y + \frac{a_n}{n}, y + \frac{b_n}{n}\right]$ such that

$$\begin{aligned}
&|\sigma_n(F)'(x) - \sigma_n(F)'(y)| \\
&= \left|f\left(x + \frac{b_n}{n}\right) - f\left(x + \frac{a_n}{n}\right) - f\left(y + \frac{b_n}{n}\right) + f\left(y + \frac{a_n}{n}\right)\right| \\
&= \frac{b_n - a_n}{n}|f'(\zeta_{n,x}) - f'(\eta_{n,y})| \le \frac{b_n - a_n}{n}\omega\left(f', \delta + \frac{b_n - a_n}{n}\right).
\end{aligned}$$

Hence,

$$\omega(\sigma_n(F)', \delta) \le \frac{b_n - a_n}{n}\omega\left(f', \delta + \frac{b_n - a_n}{n}\right) \tag{4}$$

and by (2), (3) and (4),

$$
\begin{aligned}
&|C_n(f)(x) - f(x)| \\
&\leq \frac{n}{b_n - a_n} |S_n(\sigma_n(F))(x) - \sigma_n(F)(x)| + \left| \frac{n}{b_n - a_n} \sigma_n(F)(x) - f(x) \right| \\
&\leq \frac{n}{b_n - a_n} \sqrt{\frac{x}{n}} \left(1 + \frac{1}{\delta}\sqrt{\frac{x}{n}}\right) \omega(\sigma_n(F)', \delta) + \omega\left(f, \frac{b_n - a_n}{n}\right) \\
&\leq \sqrt{\frac{x}{n}} \left(1 + \frac{1}{\delta}\sqrt{\frac{x}{n}}\right) \omega\left(f', \delta + \frac{b_n - a_n}{n}\right) + \|f'\|_\infty \frac{b_n - a_n}{n}.
\end{aligned}
$$

In particular, for $\delta = n^{-1/2}$, we get (4.5). □

We pass now to state some uniform estimates of the rate of convergence. To this end, some preliminaries are needed.

Lemma 4.4. *Let $(C_n)_{n\geq 1}$ be the sequence of operators defined by (2.2). Then*

$$
\left| C_n(f_\lambda)(-\log x) - x^\lambda \right| \leq \frac{5\lambda}{4n} \tag{4.6}
$$

for every $\lambda > 0, n \geq 1$ and $0 < x \leq 1$, where f_λ is given by (2.9).

Proof. We note that, for every $0 < x \leq 1$, there exists $s > 0$ such that $x = e^{-s}$. Therefore, using [15, Lemma 3.1],

$$
x^{n(1-e^{-\frac{\lambda}{n}})} - x^\lambda \leq \frac{\lambda}{2ne}.
$$

Then, on account of (2.21), (2.10) and (3.2) for every $n \geq 1, \lambda > 0$ and $0 < x \leq 1$ we get

$$
\begin{aligned}
\left| C_n(f_\lambda)(-\log x) - x^\lambda \right| &= \left| \frac{n}{\lambda(b_n - a_n)} \left(e^{-\frac{\lambda a_n}{n}} - e^{-\frac{\lambda b_n}{n}}\right) x^{n(1-e^{-\frac{\lambda}{n}})} - x^\lambda \right| \\
&\leq \frac{n}{\lambda(b_n - a_n)} \left(e^{-\frac{\lambda a_n}{n}} - e^{-\frac{\lambda b_n}{n}}\right) \left(x^{n(1-e^{-\frac{\lambda}{n}})} - x^\lambda\right) \\
&\quad + x^\lambda \left(1 - \frac{n}{\lambda(b_n - a_n)} \left(e^{-\frac{\lambda a_n}{n}} - e^{-\frac{\lambda b_n}{n}}\right)\right) \\
&\leq \frac{n}{\lambda(b_n - a_n)} e^{-\frac{\lambda a_n}{n}} \left(1 - e^{-\frac{\lambda(b_n - a_n)}{n}}\right) \frac{\lambda}{2ne} + \frac{\lambda}{n} \leq \frac{5\lambda}{4n}.
\end{aligned}
$$

□

In order to present some uniform estimates, we shall also use a similarity technique. In other words, given an approximation process $(L_n)_{n\geq 1}$ on some Banach space X, if $\Phi : X \to Y$ is an isometric isomorphism between X and another Banach space Y, then it is possible to construct an approximation process on Y by setting

$$
L_n^* := \Phi \circ L_n \circ \Phi^{-1} \qquad (n \geq 1).
$$

In this case, we say that $(L_n)_{n\geq 1}$ and $(L_n^*)_{n\geq 1}$ are similar or isomorphic. Clearly, for every $u \in X$,

$$\|L_n(u) - u\|_X = \|L_n^*(\Phi(u)) - \Phi(u)\|_Y, \tag{4.7}$$

which transfers the problem of estimating the rate of convergence for $(L_n)_{n\geq 1}$ in X to the (possibly easier to handle) sequence $(L_n^*)_{n\geq 1}$ in Y.

As a first example of application of this technique, consider the isometric isomorphism $\Phi : \mathscr{C}_*([0,+\infty[) \to \mathscr{C}([0,1])$ defined by setting

$$\Phi(f)(t) = \begin{cases} f(-\log t) & \text{if } 0 < t \leq 1, \\ \lim\limits_{x\to+\infty} f(x) & \text{if } t = 0, \end{cases} \quad \text{for every } f \in \mathscr{C}_*([0,+\infty[).$$

We observe that $\Phi^{-1} : \mathscr{C}([0,1]) \to \mathscr{C}_*([0,+\infty[)$ is defined as $\Phi^{-1}(g)(t) := g(e^{-t})$ for every $g \in \mathscr{C}([0,1])$ and $t \geq 0$.

Moreover, for every $n \geq 1$ and $g \in \mathscr{C}([0,1])$, set

$$C_n^*(g) := \Phi(C_n(\Phi^{-1}(g))). \tag{4.8}$$

A simple computation shows that $C_n^*(\mathbf{1}) = \mathbf{1}$,

$$C_n^*(\psi_x)(t) = \begin{cases} C_n(f_1 - x\mathbf{1})(-\log t) & \text{if } 0 < t \leq 1, \\ 0 & \text{if } t = 0 \end{cases} \tag{4.9}$$

and

$$C_n^*(\psi_x^2)(t) = \begin{cases} C_n(f_2 - 2xf_1 + x^2\mathbf{1})(-\log t) & \text{if } 0 < t \leq 1, \\ 0 & \text{if } t = 0, \end{cases} \tag{4.10}$$

$(x \in [0,1])$, where $f_\lambda, \lambda = 1,2$, is defined by (2.9).

Theorem 4.5. *Let $(C_n)_{n\geq 1}$ be the sequence of operators defined by* (2.2) *and acting on $\mathscr{C}_*([0,+\infty[)$. Then, for every $n \geq 1$ and $f \in \mathscr{C}_*([0,+\infty[)$,*

$$\|C_n(f) - f\|_\infty \leq \frac{5}{4\sqrt{n}}\omega\left(\Phi(f), \frac{1}{\sqrt{n}}\right) + \frac{7}{2}\omega_2\left(\Phi(f), \frac{1}{\sqrt{n}}\right).$$

Proof. Thanks to the general equality (4.7) we pass to establish a uniform estimate for $\|C_n^*(\Phi(f)) - \Phi(f)\|_\infty$. To this end we apply [17, Theorem 2.2.1] (see also [14, Theorem 10]) from which, for every $n \geq 1, f \in \mathscr{C}_*([0,+\infty[), 0 \leq x \leq 1$ and $\delta > 0$, we get

$$\begin{aligned}|C_n^*(\Phi(f))(x) - \Phi(f)(x)| &\leq |C_n^*(\mathbf{1})(x) - 1||\Phi(f)(x)| \\ &+\frac{1}{\delta}|C_n^*(\psi_x)(x)|\omega(\Phi(f),\delta) + \left(C_n^*(\mathbf{1})(x) + \frac{1}{2\delta^2}C_n^*(\psi_x^2)(x)\right)\omega_2(\Phi(f),\delta).\end{aligned}$$

From (4.9) and from estimate (4.6) it follows that

$$|C_n^*(\psi_x)(x)| \leq \frac{5}{4n}$$

and, by (4.10),

$$C_n^*(\psi_x^2)(x) = C_n(f_2)(-\log x) - x^2 - 2x(C_n(f_1)(-\log x) - x) \leq \frac{5}{2n} + \frac{5}{2n} = \frac{5}{n}.$$

Setting $\delta = n^{-1/2}$ the claim is proved. □

4.2. Weighted Uniform Estimates of the Rate of Convergence

Now we present some estimates of the rate of convergence in (3.4). To this end we have to introduce some preliminaries.

For every $n \geq 1$ and $x \geq 0$, consider the Borel measure on $[0, +\infty[$

$$\nu_{n,x} := w_m(x)\frac{n}{b_n - a_n}e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\nu_{n,k}$$

where each $\nu_{n,k}$ designates the Borel measure on $[0, +\infty[$ having density the product of w_m^{-1} and the characteristic function of $\left[\frac{k+a_n}{n}, \frac{k+b_n}{n}\right]$ with respect to the Borel-Lebesgue measure on $[0, +\infty[$. Clearly, $C_b([0, +\infty[) \subset L^1(\mu_{n,x}, [0, +\infty[)$.

Moreover, for every $g \in L^1(\mu_{n,x}, [0, +\infty[)$, set

$$V_n^*(g)(x) := \int_0^{+\infty} g\, d\nu_{n,x} \in \mathbf{R}.$$

Now, consider the isometric isomorphism $\Theta_m : E_m \to \mathscr{C}_b([0, +\infty[)$ defined by setting $\Theta_m(f) = w_m f$ for every $f \in E_m$; we observe that $\Theta_m^{-1} : \mathscr{C}_b([0, +\infty[) \to E_m$ is defined as $\Theta_m^{-1}(g) := w_m^{-1}g$ for every $g \in \mathscr{C}_b([0, +\infty[)$. Finally, for every $n \geq 1$, define the positive linear operator $L_n^* : \mathscr{C}_b([0, +\infty[) \to \mathscr{C}_b([0, +\infty[)$ by setting, for every $g \in \mathscr{C}_b([0, +\infty[)$,

$$L_n^*(g) := \Theta_m(C_n(\Theta_m^{-1}(g))).$$

It is easy to prove that, for every $n \geq 1, x \geq 0$ and $g \in \mathscr{C}_b([0, +\infty[)$,

$$L_n^*(g)(x) = V_n^*(g)(x) = \int_0^{+\infty} g\, d\nu_{n,x}. \tag{4.11}$$

Moreover, for every $x \geq 0$ and $f \in E_m$,

$$\begin{aligned} w_m(x)|C_n(f)(x) - f(x)| &= |\Theta_m(C_n(f))(x) - \Theta_m(f)(x)| \\ &= |L_n^*(\Theta_m(f))(x) - \Theta_m(f)(x)|, \end{aligned}$$

so that, in order to study the rate of convergence in (3.4), it is enough establish the relevant result for the sequence $(L_n^*(\Theta_m(f))(x))_{n\geq 1}(x \geq 0)$.

Theorem 4.6. *Consider the sequence $(C_n)_{n\geq 1}$ of operators defined by (2.2) and acting on $E_m (m \geq 1)$. Then, for every $f \in E_m, n \geq 1$ and $x \geq 0$,*

$$\begin{aligned} w_m(x)|C_n(f)(x) - f(x)| &\leq \frac{d_m}{n}w_m(x)f(x) \\ &+ \frac{K_m}{\sqrt{n}}\omega\left(w_m f, \frac{1}{\sqrt{n}}\right) + \left[1 + \frac{d_m}{n} + \frac{x + K_m'}{2}\right]\omega_2\left(w_m f, \frac{1}{\sqrt{n}}\right), \end{aligned} \tag{4.12}$$

where d_m is defined by (2.15) and K_m and K'_m are suitable positive constants depending on m, only.

Proof. Since (4.11) holds, by means of [17, Theorem 2.2.1], for every $x \geq 0, f \in E_m, n \geq 1$ and $\delta > 0$, we obtain

$$\begin{aligned}
&|L_n^*(\Theta_m(f))(x) - \Theta_m(f)(x)| \\
&\quad \leq |L_n^*(\mathbf{1})(x) - 1||\Theta_m(f)(x)| \\
&\qquad + \frac{1}{\delta}|L_n^*(\psi_x)(x)|\omega(\Theta_m(f), \delta) + \left[L_n^*(\mathbf{1})(x) + \frac{1}{2\delta^2}L_n^*(\psi_x^2)(x)\right]\omega_2(\Theta_m(f), \delta).
\end{aligned}$$

Since, by (2.14),

$$L_n^*(\mathbf{1})(x) = w_m(x)C_n(\mathbf{1} + e_m)(x) \leq w_m(x) + w_m(x)e_m(x) + \frac{d_m}{n} = 1 + \frac{d_m}{n},$$

where d_m is defined by (2.15), we get

$$|L_n^*(\mathbf{1})(x) - 1| \leq \frac{d_m}{n}. \tag{1}$$

Moreover, from (2.13) and (2.20) it follows that

$$\begin{aligned}
L_n^*(\psi_x)(x) &= w_m(x)C_n((\mathbf{1} + e_m)\psi_x)(x) \\
&= w_m(x)\left(C_n(\psi_x)(x) + C_n(e_{m+1})(x) - xC_n(e_m)(x)\right) \\
&= w_m(x)\left(\frac{a_n + b_n}{2n} + \frac{1}{(m+2)n^{m+1}}\sum_{k=0}^{m+1}\binom{m+2}{k}\sum_{p=0}^{m+1-k} b_n^p a_n^{m+1-k-p}\right. \\
&\qquad \left.\times \sum_{j=0}^{k} a_{k,j}n^j x^j - \frac{1}{(m+1)n^m}\sum_{k=0}^{m}\binom{m+1}{k}\sum_{p=0}^{m-k} b_n^p a_n^{m-k-p}\sum_{j=0}^{k} a_{k,j}n^j x^{j+1}\right) \\
&= \frac{1}{n}w_m(x)\left(\frac{a_n + b_n}{2} + \frac{1}{n^m}\sum_{j=0}^{m} a_{m+1,j}n^j x^j + \frac{1}{(m+2)n^m}\sum_{k=0}^{m}\binom{m+2}{k}\right. \\
&\qquad \times \sum_{p=0}^{m+1-k} b_n^p a_n^{m+1-k-p}\sum_{j=0}^{k} a_{k,j}n^j x^j - \frac{1}{n^{m-1}}\sum_{j=0}^{m-1} a_{m,j}n^j x^{j+1} \\
&\qquad \left. - \frac{1}{(m+1)n^{m-1}}\sum_{k=0}^{m-1}\binom{m+1}{k}\sum_{p=0}^{m-k} b_n^p a_n^{m-k-p}\sum_{j=0}^{k} a_{k,j}n^j x^{j+1}\right);
\end{aligned}$$

therefore, there exists $K_m > 0$ such that, for every $x \geq 0$,

$$|L_n^*(\Psi_x)(x)| \leq \frac{K_m}{n}. \tag{2}$$

Finally, we prove that for every $m \geq 1$ there exists a constant $K'_m > 0$ such that

$$L_n^*(\psi_x^2)(x) = w_m(x)C_n(\psi_x^2(1+e_m))(x) \le \frac{(x+K_m')}{n} \tag{3}$$

for all $n \ge 1$ and $x \ge 0$.

In fact, since $\psi_x^2 e_m = e_{m+2} - 2xe_{m+1} + x^2 e_m$, taking (2.6) and (2.13) into account, we get

$$\begin{aligned}
C_n(\psi_x^2 e_m)(x) &= \frac{1}{(m+3)n^{m+2}} \sum_{k=0}^{m+2} \binom{m+3}{k} \sum_{p=0}^{m+2-k} b_n^p a_n^{m+2-k-p} \sum_{j=0}^{k} a_{k,j} n^j x^j \\
&\quad - \frac{2}{(m+2)n^{m+1}} \sum_{k=0}^{m+1} \binom{m+2}{k} \sum_{p=0}^{m+1-k} b_n^p a_n^{m+1-k-p} \sum_{j=0}^{k} a_{k,j} n^j x^{j+1} \\
&\quad + \frac{1}{(m+1)n^m} \sum_{k=0}^{m} \binom{m+1}{k} \sum_{p=0}^{m-k} b_n^p a_n^{m-k-p} \sum_{j=0}^{k} a_{k,j} n^j x^{j+2} \\
&= \frac{1}{n^{m+2}} \sum_{j=0}^{m+2} a_{m+2,j} n^j x^j + \frac{1}{(m+3)n^{m+2}} \sum_{k=0}^{m+1} \binom{m+3}{k} \sum_{p=0}^{m+2-k} b_n^p a_n^{m+2-k-p} \\
&\quad \times \sum_{j=0}^{k} a_{k,j} n^j x^j - \frac{2}{n^{m+1}} \sum_{j=0}^{m+1} a_{m+1,j} n^j x^{j+1} - \frac{2}{(m+2)n^{m+1}} \sum_{k=0}^{m} \binom{m+2}{k} \\
&\quad \times \sum_{p=0}^{m+1-k} b_n^p a_n^{m+1-k-p} \sum_{j=0}^{k} a_{k,j} n^j x^{j+1} + \frac{1}{n^m} \sum_{j=0}^{m} a_{m,j} n^j x^{j+2} + \frac{1}{(m+1)n^m} \\
&\quad \times \sum_{k=0}^{m-1} \binom{m+1}{k} \sum_{p=0}^{m-k} b_n^p a_n^{m-k-p} \sum_{j=0}^{k} a_{k,j} n^j x^{j+2} = \frac{1}{n} \\
&\quad \times \left(\left(\frac{(m+1)(m+2)}{2} + \frac{1}{m+3} \binom{m+3}{m+1} \sum_{p=0}^{1} b_n^p a_n^{1-p} - m(m+1) \right.\right. \\
&\quad - \frac{2}{m+2} \binom{m+2}{m} \sum_{p=0}^{1} b_n^p a_n^{1-p} + \frac{(m-1)m}{2} + \frac{1}{m+1} \binom{m+1}{m-1} \\
&\quad \left.\left. \times \sum_{p=0}^{1} b_n^p a_n^{1-p} \right) x^{m+1} + F_m(x) \right) = \frac{1}{n} \left(\left(1 + \frac{m+2}{2}(a_n+b_n) - (m+1)(a_n+b_n) \right.\right. \\
&\quad \left.\left. + \frac{m}{2}(a_n+b_n) \right) x^{m+1} + F_m(x) \right) = \frac{1}{n}(x^{m+1} + F_m(x)),
\end{aligned}$$

where $F_m(x)$ is a polynomial of degree m.

Hence, from (2.20) it follows that

$$\begin{aligned}
&w_m(x)C_n(\psi_x^2(1+e_m))(x) \\
&\quad = w_m(x)C_n(\psi_x^2)(x) + w_m(x)C_n(\psi_x^2 e_m)(x) \\
&\quad = \frac{1}{n}\left(w_m(x)\left(x + \frac{a_n^2 + a_n b_n + b_n^2}{3n} \right) + w_m(x)(x^{m+1} + F_m(x)) \right) \\
&\quad \le \frac{1}{n}(x + K_m'),
\end{aligned}$$

for a suitable $K'_m > 0$ and, taking (1), (2) and (3) into account, we get (4.12) when $\delta = n^{-1/2}$. □

In order to estimate the rate of the convergence with respect to the weighted norm $\|\cdot\|_m$, we introduce the isometric isomorphism $\Phi_m : E^*_m \to \mathscr{C}([0,1])$ defined by

$$\Phi_m(f)(t) = \begin{cases} (w_m f)\,(-\log t) & \text{if } 0 < t \le 1, \\ \lim\limits_{x\to+\infty}(w_m f)(x) & \text{if } t = 0, \end{cases} \quad \text{for every } f \in E^*_m. \quad (4.13)$$

We observe that $\Phi_m^{-1} : \mathscr{C}([0,1]) \to E^*_m$ is defined as $\Phi_m^{-1}(g)(t) := w_m^{-1}(t)g(e^{-t})$ for every $g \in \mathscr{C}([0,1])$ and $t \ge 0$.

Moreover, for every $n \ge 1$, we consider the similar positive linear operator $W^*_n : \mathscr{C}([0,1]) \to \mathscr{C}([0,1])$ defined by setting, for every $g \in \mathscr{C}([0,1])$,

$$W^*_n(g) := \Phi_m(C_n(\Phi_m^{-1}(g))). \quad (4.14)$$

Theorem 4.7. *Let $(C_n)_{n\ge1}$ be the sequence of operators defined by (2.2) and acting on E^*_m. Then, for every $n \ge 1$ and $f \in E^*_m$,*

$$\|C_n(f) - f\|_m \le \frac{H_{1,m}}{n}||\Phi_m(f)||_\infty + H_{2,m}\omega\left(\Phi_m(f), \frac{1}{\sqrt{n}}\right) + \left(1 + \frac{d_m + H_{3,m}\sqrt{n}}{n}\right)\omega_2\left(\Phi_m(f), \frac{1}{\sqrt{n}}\right),$$

where d_m is defined by (2.15) and $H_{1,m}, H_{2,m}, H_{3,m}$ are suitable positive constants which depend on m, only.

Proof. Thanks to (4.7) we establish a uniform estimate for $\|W^*_n(\Phi_m(f)) - \Phi_m(f)\|_\infty$. To this end by [17, Theorem 2.2.1] (see also [14, Theorem 10]), for every $n \ge 1, f \in E^*_m, 0 \le x \le 1$ and $\delta > 0$, we get

$$|W^*_n(\Phi_m(f))(x) - \Phi_m(f)(x)| \le |W^*_n(\mathbf{1})(x) - 1||\Phi_m(f)(x)| + \frac{1}{\delta}|W^*_n(\psi_x)(x)|\omega(\Phi_m(f),\delta) + \left(W^*_n(\mathbf{1})(x) + \frac{1}{2\delta^2}W^*_n(\psi_x^2)(x)\right)\omega_2(\Phi_m(f),\delta).$$

From (4.14), (4.13) and Proposition 2.1 it follows that

$$W^*_n(\mathbf{1})(x) = \begin{cases} (w_m C_n(\mathbf{1} + e_m))(-\log x) & \text{if } 0 < x \le 1, \\ 1 & \text{if } x = 0, \end{cases}$$

$$W^*_n(\psi_x)(x) = \begin{cases} (w_m C_n((1 + e_m)(f_1 - x\mathbf{1})))(-\log x) & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0 \end{cases}$$

and

$$W^*_n(\psi_x^2)(x) = \begin{cases} (w_m C_n((1 + e_m)(f_2 - 2xf_1 + x^2\mathbf{1})))(-\log x) & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0, \end{cases}$$

where $f_\lambda, \lambda = 1, 2$, is defined by (2.9).

Now, fix $0 < x \leq 1$; hence, thanks to (2.13),

$$|w_m(-\log x)C_n(1+e_m)(-\log x) - 1| = \frac{1}{n} w_m(-\log x)F_{m-1}(-\log x),$$

where F_{m-1} is a positive polynomial of degree $m-1$ (see Proposition 2.1); hence, there exists $H_{1,m} > 0$ such that

$$||W_n^*(\mathbf{1}) - \mathbf{1}||_\infty \leq \frac{H_{1,m}}{n}.$$

Moreover, for every $0 < x \leq 1$, by using (4.10) and the Cauchy–Schwartz inequality, we get

$$\begin{aligned}|W_n^*(\psi_x)(x)| &\leq w_m(-\log x)C_n(|(1+e_m)(f_1 - x)|)(-\log x)\\ &\leq w_m(-\log x)\sqrt{C_n((1+e_m)^2)(-\log x)}\sqrt{C_n((f_1-x)^2)(-\log x)}\\ &= w_m(-\log x)\sqrt{C_n((1+e_m)^2)(-\log x)}\sqrt{C_n^*(\psi_x^2)(x)}.\end{aligned}$$

Then, arguing as in the proof of Theorem 4.5 and since, by (2.13), $L_m := \sup_{0<x\leq 1} w_m(-\log x)\sqrt{C_n((1+e_m)^2)(-\log x)} \in \mathbf{R}$, there exists $H_{2,m} > 0$ such that

$$|W_n^*(\psi_x)(x)| \leq \frac{H_{2,m}}{\sqrt{n}}.$$

Finally, for every $0 < x \leq 1$, we get

$$\begin{aligned}W_n^*(\psi_x^2)(x) &= w_m(-\log x)C_n((1+e_m)(f_2 - 2xf_1 + x^2))(-\log x)\\ &\leq w_m(-\log x)\sqrt{C_n((1+e_m)^2)(-\log x)}\sqrt{C_n(f_2-2xf_1+x^2)^2)(-\log x)}\\ &\leq L_m\sqrt{C_n^*(\psi_x^4)(x)},\end{aligned}$$

where, again, C_n^* is defined by (4.8).

Because of Lemma 4.4, there exists $K_3 > 0$ such that

$$\begin{aligned}C_n^*(\psi_x^4)(x) &= C_n(f_4)(-\log x) - x^4 - 4x(C_n(f_3)(-\log x) - x^3)\\ &\quad +6x^2(C_n(f_2)(-\log x) - x^2) - 4x^3(C_n(f_1)(-\log x) - x) \leq \frac{K_3}{n},\end{aligned}$$

so that

$$W_n^*(\psi_x^2)(x) \leq \frac{H_{3,m}}{\sqrt{n}},$$

for a suitable constant $H_{3,m} > 0$ depending on m, only.

Then,

$$\begin{aligned}&|W_n^*(\Phi_m(f))(x) - \Phi_m(f)(x)| \leq \frac{H_{1,m}}{n}|\Phi_m(f)(x)|\\ &\quad +\frac{1}{\delta}\frac{H_{2,m}}{\sqrt{n}}\omega(\Phi_m(f),\delta) + \left(1 + \frac{d_m}{n} + \frac{1}{\delta^2}\frac{H_{3,m}}{\sqrt{n}}\right)\omega_2(\Phi_m(f),\delta),\end{aligned}$$

In particular, for $\delta := n^{-1/2}$ we get the required assertion. □

4.3. L^p-Estimates of the Rate of Convergence

Finally, we deal with the rate of convergence in Theorem 3.5 by means of the second-order integral modulus of smoothness $\omega_2(f,\delta)_p$ in $L^p([0,1])$ (for the definition see, e.g., [10, Chapter 2, § 7]) by using again a similarity technique.

From now on we assume that the sequence $(1/(b_n - a_n))_{n\geq 1}$ is bounded.

First of all, consider the isometric isomorphism $\Phi_p : L^p([0,+\infty[) \to L^p([0,1])$ defined by setting, for every $f \in L^p([0,+\infty[)$,

$$\Phi_p(f)(t) = \begin{cases} t^{-\frac{1}{p}} f(-\log t) & \text{if } 0 < t \leq 1, \\ S_1(f)(0) & \text{if } t = 0, \end{cases}$$

S_1 being the first Szász–Mirakjan operator (see (2.1)). We note that its inverse $\Phi_p^{-1} : L^p([0,1]) \to L^p([0,+\infty[)$ is defined as $\Phi_p^{-1}(g)(t) := e^{-\frac{t}{p}} g(e^{-t})$ for every $g \in L^p([0,1])$ and $t \geq 0$.

Moreover, for every $n \geq 1$, define the similar positive linear operator $P_n^* : L^p([0,1]) \to L^p([0,1])$ as follows

$$P_n^*(g) := \Phi_p(C_n(\Phi_p^{-1}(g))) \qquad (g \in L^p([0,1])). \tag{4.15}$$

Before stating the main result, some preliminary lemmas are needed.

Lemma 4.8. *For every $n \geq 1, \lambda > 0$ and $x \in]0,1]$ we have*

$$0 < x^{n(1-e^{-\frac{\lambda}{n}})-\lambda} - 1 \leq \frac{\lambda^2}{2n} x^{-\frac{\lambda^2}{2}} \log \frac{1}{x}. \tag{4.16}$$

Moreover, the function $g_\lambda(x) = x^{-\frac{\lambda^2}{2}} \log \frac{1}{x}$ $(x \in]0,1])$ belongs to $L^p([0,1])$ provided that

$$\lambda^2 p < 2. \tag{4.17}$$

Proof. Indeed, by means of the classical inequalities $1 - e^{-x} \geq x - x^2/2$ and $e^x - 1 \leq xe^x (x \geq 0)$, we get

$$0 \leq x^{n(1-e^{-\frac{\lambda}{n}})-\lambda} - 1 \leq e^{\frac{\lambda^2}{2n}\log\frac{1}{x}} - 1 \leq \frac{\lambda^2}{2n} x^{-\frac{\lambda^2}{2n}} \log \frac{1}{x} \leq \frac{\lambda^2}{2n} x^{-\frac{\lambda^2}{2}} \log \frac{1}{x}.$$

□

Lemma 4.9. *For every $p \in [1+\infty[, x \in]0,1], k \geq 1$ and $n \geq (\frac{1}{p}+k)^2/k$,*

$$x^{n\left(1-e^{-\frac{1}{n}\left(\frac{1}{p}+k\right)}\right)-\frac{1}{p}} - x^{n(1-e^{-\frac{1}{np}})-\frac{1}{p}+k} \leq \frac{1}{n}\left(\frac{1}{p}+k\right)^2 \log \frac{1}{x}.$$

Proof. First of all, notice that, for every $k, n \geq 1$ and $p \in [1,+\infty[$

$$n\left(1 - e^{-\frac{1}{n}\left(\frac{1}{p}+k\right)}\right) - \frac{1}{p} \leq n\left(1 - e^{-\frac{1}{np}}\right) - \frac{1}{p} + k.$$

Moreover, for $n \geq \dfrac{(\frac{1}{p}+k)^2}{k}$, we have that

$$n\left(1 - e^{-\frac{1}{n}\left(\frac{1}{p}+k\right)}\right) - \frac{1}{p} \geq 0,$$

since

$$n\left(1-e^{-\frac{1}{n}\left(\frac{1}{p}+k\right)}\right)-\frac{1}{p}\geq\frac{k}{n}-\frac{1}{n^2}\left(\frac{1}{p}+k\right)^2\geq 0.$$

Finally,

$$n\left(1-e^{-\frac{1}{np}}\right)-\frac{1}{p}+k\geq 0,$$

because

$$n\left(1-e^{-\frac{1}{np}}\right)-\frac{1}{p}+k\geq-\frac{1}{np^2}+k\geq 0.$$

Summing up, for every $x\in]0,1], k\geq 1$ and $n\geq\dfrac{(\frac{1}{p}+k)^2}{k}$, we get

$$\begin{aligned}
&x^{n\left(1-e^{-\frac{1}{n}\left(\frac{1}{p}+k\right)}\right)-\frac{1}{p}}-x^{n\left(1-e^{-\frac{1}{np}}\right)-\frac{1}{p}+k}\\
&=e^{-n\left(1-e^{-\frac{1}{n}\left(\frac{1}{p}+k\right)}-\frac{1}{p}\right)\log\frac{1}{x}}-e^{-\left(n\left(1-e^{-\frac{1}{np}}\right)-\frac{1}{p}+k\right)\log\frac{1}{x}}\\
&\leq\log\frac{1}{x}\left(n\left(1-e^{-\frac{1}{np}}\right)-\frac{1}{p}+k-n\left(1-e^{-\frac{1}{n}\left(\frac{1}{p}+k\right)}\right)+\frac{1}{p}\right)\\
&\leq\log\frac{1}{x}\left(\frac{1}{p}+k-n\left(\frac{1}{n}\left(\frac{1}{p}+k\right)-\frac{1}{n^2}\left(\frac{1}{p}+k\right)^2\right)\right)\\
&=\frac{1}{n}\left(\frac{1}{p}+k\right)^2\log\frac{1}{x},
\end{aligned}$$

where, for the last inequality, we have used again the inequalities $1-e^{-x}\leq x$ and $1-e^{-x}\geq x-x^2/2$ $(x\geq 0)$. □

We are now in a position to get the desired L^p-estimate.

Theorem 4.10. *Let* $(C_n)_{n\geq 1}$ *be the sequence of positive linear operators defined by* (2.2) *and acting on* $L^p([0,+\infty[), p\in[1,+\infty[$*. Then, for every* $f\in L^p([0,+\infty[)$ *and* $n\geq(\frac{1}{p}+2)^2/2$*,*

$$\|C_n(f)-f\|_p\leq K_p\left(n^{-\frac{2p}{2p+1}}\|\Phi_p(f)\|_p+\omega_2\left(\Phi_p(f),n^{-\frac{p}{2p+1}}\right)_p\right),$$

where K_p *is a positive constant that depends on* p*, only.*

Proof. By applying to the sequence $(P_n^*)_{n\geq 1}$ a result due to Swetits and Wood (see [18, Theorem 1]), setting

$$\mu_{n,p}:=\left(\max\left\{\|P_n^*(\mathbf{1})-\mathbf{1}\|_p,\ \|\alpha_n\|_p,\ \|\beta_n\|_p^{\frac{2p}{2p+1}}\right\}\right)^{1/2},$$

where $\alpha_n(x) := P_n^*(\psi_x)(x)$ and $\beta_n(x) := P_n^*(\psi_x^2)(x)(0 \leq x \leq 1)$, it is enough to show that

$$\lim_{n\to+\infty} \mu_{n,p} = 0,$$

in order to obtain that, for every $f \in L^p([0,+\infty[)$,

$$\|C_n(f)-f\|_p = \|P_n^*(\Phi_p(f)) - \Phi_p(f)\|_p \leq K_p\left(\mu_{n,p}^2\|\Phi_p(f)\|_p + \omega_2(\Phi_p(f),\mu_{n,p})_p\right).$$

We start by evaluating P_n^* on $\mathbf{1}, \psi_x$ and ψ_x^2. Taking (4.15), (2.7) and (2.8) into account, we get

$$P_n^*(\mathbf{1})(t) = \begin{cases} t^{-\frac{1}{p}} C_n\left(f_{\frac{1}{p}}\right)(-\log t) & \text{if } 0 < t \leq 1, \\ 1 & \text{if } t = 0, \end{cases}$$
$$P_n^*(\psi_x)(t) = \begin{cases} t^{-\frac{1}{p}} C_n\left(f_{\frac{1}{p}+1} - x f_{\frac{1}{p}}\right)(-\log t) & \text{if } 0 < t \leq 1, \\ 0 & \text{if } t = 0 \end{cases}$$

and

$$P_n^*(\psi_x^2)(t) = \begin{cases} t^{-\frac{1}{p}} C_n\left(f_{\frac{1}{p}+2} - 2x f_{\frac{1}{p}+1} + x^2 f_{\frac{1}{p}}\right)(-\log t) & \text{if } 0 < t \leq 1, \\ 0 & \text{if } t = 0, \end{cases}$$

where $f_\lambda, \lambda = \frac{1}{p}, \frac{1}{p}+1, \frac{1}{p}+2$, is defined by (2.9).

Thanks to (4.16), with $\lambda = \frac{1}{p}$, and (3.2) we get

$$\begin{aligned} |P_n^*(\mathbf{1}) - \mathbf{1}| &= \left|\frac{np}{b_n - a_n}\left(e^{-\frac{a_n}{np}} - e^{-\frac{b_n}{np}}\right) x^{n(1-e^{-\frac{1}{np}})-\frac{1}{p}} - 1\right| \\ &\leq \frac{np}{b_n - a_n}\left(e^{-\frac{a_n}{np}} - e^{-\frac{b_n}{np}}\right)\left(x^{n(1-e^{-\frac{1}{np}})-\frac{1}{p}} - 1\right) \\ &\quad +1 - \frac{np}{b_n - a_n}\left(e^{-\frac{a_n}{np}} - e^{-\frac{b_n}{np}}\right) \leq \frac{1}{2np^2} x^{-\frac{1}{2p^2}} \log\frac{1}{x} + \frac{1}{np}; \end{aligned}$$

hence

$$|P_n^*(\mathbf{1}) - \mathbf{1}|^p \leq 2^{p-1}\left(\left(\frac{1}{2np^2}\right)^p x^{-\frac{1}{2p}} \log^p\frac{1}{x} + \left(\frac{1}{np}\right)^p\right)$$

so that

$$\|P_n^*(\mathbf{1}) - \mathbf{1}\|_p \leq \frac{A_p}{n} \tag{1}$$

(for a suitable positive constant A_p), since $\lambda = \frac{1}{p}$ satisfies (4.17).

In order to estimate $\|\alpha_n\|_p$, fix $0 < x \leq 1$ and $n \geq p^{-2}$; taking Lemma 4.9 with $k = 1$ and (3.2) into account, we have

$$
\begin{aligned}
|\alpha_n(x)| &= \Bigg| \frac{n}{\left(\frac{1}{p}+1\right)(b_n-a_n)} \left(e^{-\left(\frac{1}{p}+1\right)\frac{a_n}{n}} - e^{-\left(\frac{1}{p}+1\right)\frac{b_n}{n}}\right) x^{n\left(1-e^{-\frac{1}{n}\left(\frac{1}{p}+1\right)}\right)-\frac{1}{p}} \\
&\quad - \frac{np}{b_n-a_n}\left(e^{-\frac{a_n}{pn}} - e^{-\frac{b_n}{pn}}\right) x^{n\left(1-e^{-\frac{1}{np}}\right)-\frac{1}{p}+1} \Bigg| \\
&\le \left(1 - \frac{n}{\left(\frac{1}{p}+1\right)(b_n-a_n)} \left(e^{-\left(\frac{1}{p}+1\right)\frac{a_n}{n}} - e^{-\left(\frac{1}{p}+1\right)\frac{b_n}{n}}\right)\right) x^{n\left(1-e^{-\frac{1}{n}\left(\frac{1}{p}+1\right)}\right)-\frac{1}{p}} \\
&\quad + x^{n\left(1-e^{-\frac{1}{n}\left(\frac{1}{p}+1\right)}\right)-\frac{1}{p}} - x^{n\left(1-e^{-\frac{1}{np}}\right)-\frac{1}{p}+1} \\
&\quad + \left(1 - \frac{pn}{b_n-a_n}\left(e^{-\frac{a_n}{pn}} - e^{-\frac{b_n}{np}}\right)\right) x^{n\left(1-e^{-\frac{1}{np}}\right)-\frac{1}{p}+1} \\
&\le \frac{1}{n}\left(\frac{1}{p}+1\right) + \frac{1}{n}\left(\frac{1}{p}+1\right)^2 \log\frac{1}{x} + \frac{1}{np}.
\end{aligned}
$$

Then,

$$
|\alpha_n(x)|^p \le 2^{2(p-1)} \frac{1}{n^p} \left(\left(\frac{1}{p}+1\right)^p + \left(\frac{1}{p}+1\right)^{2p} \log^p \frac{1}{x} + \left(\frac{1}{p}\right)^p \right)
$$

and hence ther exists $B_p > 0$ such that

$$
\|\alpha_n\|_p \le \frac{B_p}{n}. \tag{2}
$$

In order to evaluate $P_n^*(\psi_x^2)(x)$, by means of Lemma 4.9, for $k = 2$, and (3.2), we have that, for every $x \in]0,1]$ and $n \ge (\frac{1}{p}+2)^2/2$,

$$
\begin{aligned}
|P_n^*(\psi_x^2)(x)| &= x^{-\frac{1}{p}} \Big| C_n\left(f_{\frac{1}{p}+2}\right)(-\log x) \\
&\quad - x^2 C_n\left(f_{\frac{1}{p}}\right)(-\log x) - 2x\left(C_n\left(f_{\frac{1}{p}+1}\right)(-\log x) - xC_n\left(f_{\frac{1}{p}}\right)(-\log x)\right)\Big| \\
&\le \Bigg| \frac{n}{\left(\frac{1}{p}+2\right)(b_n-a_n)} \left(e^{-\left(\frac{1}{p}+2\right)\frac{a_n}{n}} - e^{-\left(\frac{1}{p}+2\right)\frac{b_n}{n}}\right) x^{n\left(1-e^{-\frac{1}{n}\left(\frac{1}{p}+2\right)}\right)-\frac{1}{p}} \\
&\quad - \frac{np}{b_n-a_n}\left(e^{-\frac{a_n}{pn}} - e^{-\frac{b_n}{pn}}\right) x^{n\left(1-e^{-\frac{1}{np}}\right)-\frac{1}{p}+2} \Bigg| + 2|P_n^*(\psi_x)(x)| \\
&\le \left(1 - \frac{n}{\left(\frac{1}{p}+2\right)(b_n-a_n)} \left(e^{-\left(\frac{1}{p}+2\right)\frac{a_n}{n}} - e^{-\left(\frac{1}{p}+2\right)\frac{b_n}{n}}\right)\right) x^{n\left(1-e^{-\frac{1}{n}\left(\frac{1}{p}+2\right)}\right)-\frac{1}{p}} \\
&\quad + x^{n\left(1-e^{-\frac{1}{n}\left(\frac{1}{p}+2\right)}\right)-\frac{1}{p}} - x^{n\left(1-e^{-\frac{1}{np}}\right)-\frac{1}{p}+2} \\
&\quad + \left(1 - \frac{pn}{b_n-a_n}\left(e^{-\frac{a_n}{pn}} - e^{-\frac{b_n}{np}}\right)\right) x^{n\left(1-e^{-\frac{1}{np}}\right)-\frac{1}{p}+2} + 2|P_n^*(\psi_x)(x)|
\end{aligned}
$$

$$\leq \frac{1}{n}\left(\frac{1}{p}+2\right)+\frac{1}{n}\left(\frac{1}{p}+2\right)^2\log\frac{1}{x}+\frac{1}{np}+2|P_n^*(\psi_x)(x)|.$$

Hence,

$$\|\beta_n\|_p^{\frac{2p}{2p+1}} \leq C_p\, n^{-\frac{2p}{2p+1}}, \tag{3}$$

for a suitable $C_p > 0$.

Collecting (1)–(3), there exists a constant M_p that only depends on p, such that

$$\mu_{n,p} \leq M_p\, n^{-\frac{p}{2p+1}},$$

for any $n \geq (\frac{1}{p}+2)^2/2$. Then $\mu_{n,p} \to 0$ as $n \to +\infty$ and the claim easily follows. □

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Francesco Altomare and Mirella Cappelletti Montano
Dipartimento di Matematica
Università degli Studi di Bari
Campus Universitario, Via E. Orabona, 4,
70125, Bari, Italy
e-mail: altomare@dm.uniba.it;
montano@dm.uniba.it

Vita Leonessa
Dipartimento di Matematica e Informatica
Università degli Studi della Basilicata
Viale dell'Ateneo Lucano no. 10, Macchia Romana,
85100, Potenza, Italy
e-mail: vita.leonessa@unibas.it

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