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Results in Mathematics

New Solutions for (1+1)-Dimensional and (2+1)-Dimensional Kaup–Kupershmidt Equations

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Abstract. In this paper, using the exp-function method we obtain some new exact solutions for (1+1)-dimensional and (2+1)-dimensional Kaup–Kupershmidt (KK) equations. We show figures of some of the new solutions obtained here. We conclude that the exp-function method presents a wider applicability for handling nonlinear partial differential equations.

Mathematics Subject Classification (2010). 35D99, 65N99.

Keywords. (1+1)-dimensional and (2+1)-dimensional Kaup–Kupershmidt (KK) equations, Exact solutions, Exp-function method.

1. Introduction

In recent years, nonlinear evolution equations(NLEEs) have important role in several scientific and engineering fields [1–10, 21, 22]. Many effective and reliable methods are used in the literature to investigate exact solutions of NLEEs [1–10, 21, 22].

In this study, we consider the (2+1)-dimensional Kaup–Kupershmidt (KK) equation [11]

$$9u_t + u_{5x} + 15uu_{xxx} + \frac{75}{2}u_xu_{xx} + 45u^2u_x + 5\sigma u_{xxy} - 5\sigma\partial_x^{-1}u_{yy} + 15\sigma uu_y + 15\sigma u_x\partial_x^{-1}u_y = 0 \quad (1)$$

where $\sigma^2 = 1, \partial_x^{-1} = \int^d x$. If we take $u(x, y, t) = u(x, t)$, then Eq. (1) becomes the (1+1)-dimensional KK equation [11]

$$9u_t + u_{5x} + 15uu_{xxx} + \frac{75}{2}u_x u_{xx} + 45u^2 u_x = 0 \tag{2}$$

Some researchers investigated exact solutions of KK equations [11–14]. Recently, Ling and Qiang [11] used the symmetry method and obtained some new solutions of the (2+1)-dimensional KK equation.

We apply the exp-function method to obtain new solutions of the (1+1)-dimensional and (2+1)-dimensional KK equations. The exp-function method, firstly introduced by He and Wu in [15], shown to be effective and reliable for several nonlinear problems. It was successfully applied to NLEEs [16–21] and so on. The exp-function method is just a special case of the transformed rational function method [22], which generates various travelling wave solutions, and is generalized to a multiple exp-function method [23], which generates multiple wave solutions.

2. The Exp-Function Method

We consider the general nonlinear partial differential equation of the type

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xxx}, \ldots) = 0 \tag{3}$$

Using a transformation

$$\eta = kx + wt \tag{4}$$

where k and w are constants, we can rewrite Eq. (3) in the following nonlinear ODE

$$Q(u, u', u'', u''', u^{(iv)}, u^{(v)}, \ldots) = 0 \tag{5}$$

According to the exp-function method [15], we assume that the wave solutions can be expressed in the following form

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)} \tag{6}$$

where p, q, d and c are positive integers which are known to be further determined, a_n and b_m are unknown constants. We can rewrite Eq. (6) in the following equivalent form.

$$u(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d}\exp(-d\eta)}{b_p \exp(p\eta) + \ldots + b_{-q}\exp(-q\eta)} \tag{7}$$

This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems. To determine the value of c and p, we balance the linear term of highest order of Eq. (6) with the highest order nonlinear term. Similarly, to determine the value of d and q, we balance the linear term of lowest order of Eq. (5) with lowest order non linear term.

3. Exp-Function Method for (1+1)-Dimensional KK Equation

We consider the (1+1)-dimensional KK equation in the form [11]

$$9u_t + u_{5x} + 15uu_{xxx} + \frac{75}{2}u_x u_{xx} + 45u^2 u_x = 0 \tag{8}$$

Introducing a transformation as $\eta = kx + wt$, we can convert Eq. (8) into ordinary differential equation

$$9wu' + k^5 u^{(5)} + 15k^3 uu''' + \frac{75}{2}k^3 u'u'' + 45ku^2 u' = 0 \tag{9}$$

where the prime denotes the derivative with respect to η. The solution of Eq. (9) can be expressed in the form

$$u(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d}\exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q}\exp(-q\eta)}$$

To determine the value of c and p, we balance the linear term of highest order of Eq. (9) with the highest order nonlinear term

$$u^{(5)} = \frac{c_1 \exp[(31p + c)\eta] + \cdots}{c_2 \exp[(32p)\eta] + \cdots} \quad \text{and} \tag{10}$$

$$u^2 u' = \frac{c_3 \exp[(p + 3c)\eta] + \cdots}{c_4 \exp[(4p)\eta] + \cdots} = \frac{c_3 \exp[(29p + 3c)\eta] + \cdots}{c_4 \exp[(32p)\eta] + \cdots} \tag{11}$$

where c_i are determined coefficients only for simplicity; balancing the highest order of exp-function in (10) and (11), we have

$$31p + c = 29p + 3c \quad \text{which in turn gives} \tag{12}$$

$$p = c. \tag{13}$$

To determine the value of d and q, we balance the linear term of lowest order of Eq. (9) with the lowest order non-linear term

$$u^{(5)} = \frac{\cdots + d_1 \exp[(-d - 31q)\eta]}{\cdots + d_2 \exp[(-32q)\eta]} \quad \text{and} \tag{14}$$

$$uu''' = \frac{\cdots + d_3 \exp[(-2d - 7q)\eta]}{\cdots + d_4 \exp[(-9q)\eta]} = \frac{\cdots + d_3 \exp[(-2d - 30q)\eta]}{\ldots + d_4 \exp[(-32q)\eta]} \tag{15}$$

where d_i are determined coefficients only for simplicity. Now, balancing the lowest order of exp-function in (14) and (15), we have

$$-d - 31q = -2d - 30q \quad \text{which in turn gives} \tag{16}$$

$$q = d. \tag{17}$$

We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p = c = 1$ and $q = d = 1$, then the trial solution, Eq. (8) reduces to

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1}\exp(-\eta)} \tag{18}$$

Substituting Eq. (18) into Eq. (9) we have

$$\begin{aligned}&\frac{1}{A}[c_5\exp(5\eta)+c_4\exp(4\eta)+c_3\exp(3\eta)+c_2\exp(2\eta)+c_1\exp(\eta)\\&\quad+c_0+c_{-1}\exp(-\eta)+c_{-2}\exp(-2\eta)+c_{-3}\exp(-3\eta)\\&\quad+c_{-4}\exp(-4\eta)+c_{-5}\exp(-5\eta)]=0\end{aligned}\tag{19}$$

where $A=(b_1\exp(\eta)+b_0+b_{-1}\mathrm{exp}(-\eta))^6$.

Equating the coefficients of $\exp(\eta n)$ to be zero, we obtain

$$\begin{aligned}&\{c_{-5}=0,c_{-4}=0,c_{-3}=0,c_{-2}=0,c_{-1}=0,\\&\quad c_0=0,c_1=0,c_2=0,c_3=0,c_4=0,c_5=0\}\end{aligned}\tag{20}$$

Solution of (20) will yield

$$a_0=-\frac{2k^2b_0}{3},\quad a_{-1}=\frac{10k^2b_{-1}}{3},\quad a_1=0,\quad b_1=0,\quad w=-\frac{11k^5}{9}\tag{21}$$

We, therefore, obtain the following generalized solitary solution $u(x,t)$ of Eq. (8)

$$u(x,t)=\frac{\frac{10k^2b_{-1}}{3}e^{(-kx+\frac{11k^5}{9}t)}-\frac{2k^2b_0}{3}}{b_{-1}e^{(-kx+\frac{11k^5}{9}t)}+b_0}\tag{22}$$

where b_0,b_{-1} and k are real numbers. Also we can show Eq. (22) as

$$u(x,t)=\frac{\frac{10k^2b_{-1}}{3}\cosh\left(kx-\frac{11k^5}{9}t\right)-\frac{10k^2b_{-1}}{3}\sinh\left(kx-\frac{11k^5}{9}t\right)-\frac{2k^2b_0}{3}}{b_{-1}\cosh\left(kx-\frac{11k^5}{9}t\right)-b_{-1}\sinh\left(kx-\frac{11k^5}{9}t\right)+b_0}\tag{23}$$

Soliton solution of Eq. (8), when $b_0=k=1$ *and* $b_{-1}=-1$, is (Fig. 1)

$$u(x,t)=\frac{\frac{10}{3}\cosh\left(x-\frac{11t}{9}\right)-\frac{10}{3}\sinh\left(x-\frac{11t}{9}\right)+\frac{2}{3}}{\cosh\left(x-\frac{11t}{9}\right)-\sinh\left(x-\frac{11t}{9}\right)-1}\tag{24}$$

4. Exp-Function Method for (2+1)-Dimensional KK Equation

Now, we consider the (2+1)-dimensional KK equation in the form [11]

$$\begin{aligned}&9u_t+u_{5x}+15uu_{xxx}+\frac{75}{2}u_xu_{xx}+45u^2u_x+5\sigma u_{xxy}-5\sigma\partial_x^{-1}u_{yy}\\&\quad+15\sigma uu_y+15\sigma u_x\partial_x^{-1}u_y=0\end{aligned}\tag{25}$$

where $\sigma^2=1,\partial_x^{-1}=\int^d x$. Integrating the (2+1)-dimensional KK (25) respect to x, we get the equivalent form of Eq. (25) as follows

$$\begin{aligned}&9u_{xt}+u_{6x}+15u_xu_{xxxx}+\frac{75}{2}u_{xx}u_{xxx}+45u_x^2u_{xx}+5\sigma u_{xxxy}\\&\quad-5\sigma u_{yy}+15\sigma u_xu_{xy}+15\sigma u_{xx}u_y=0\end{aligned}\tag{26}$$

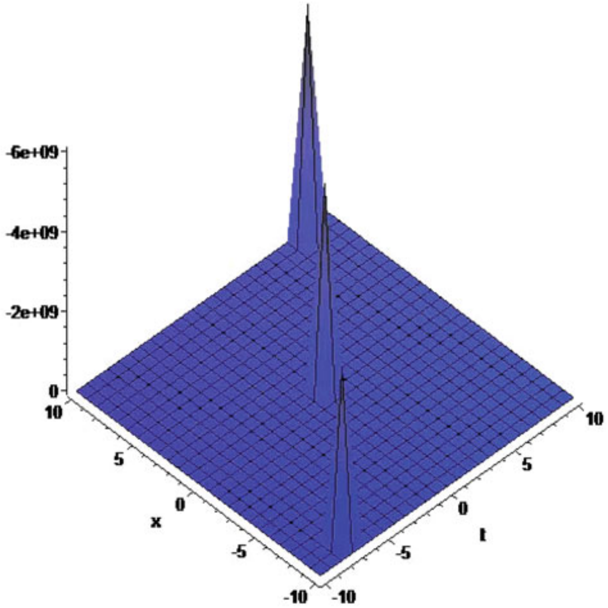

FIGURE 1. Soliton solution of Eq. (8)

Introducing a transformation as $\eta = kx + ly + wt$, we can convert Eq. (26) into ordinary differential equation

$$\begin{aligned} &9wku'' + k^6u^{(6)} + 15k^5u'u'''' + \frac{75}{2}k^5u''u''' + 45k^4u'^2u'' \\ &+5\sigma k^3lu'''' - 5\sigma l^2u'' + 30\sigma k^2lu'u'' = 0 \end{aligned} \tag{27}$$

where the prime denotes the derivative with respect to η. The solution of Eq. (27) can be expressed in the form

$$u(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d}\exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q}\exp(-q\eta)}$$

To determine the value of c and p, we balance the linear term of highest order of Eq. (27) with the highest order nonlinear term

$$u^{(6)} = \frac{c_1 \exp[(63p + c)\eta] + \cdots}{c_2 \exp[(64p)\eta] + \cdots} \quad \text{and} \tag{28}$$

$$u'^2u'' = \frac{c_3 \exp[(5p + 3c)\eta] + \cdots}{c_4 \exp[(8p)\eta] + \cdots} = \frac{c_3 \exp[(61p + 3c)\eta] + \cdots}{c_4 \exp[(64p)\eta] + \cdots} \tag{29}$$

where c_i are determined coefficients only for simplicity; balancing the highest order of exp-function in (28) and (29), we have

$$63p + c = 61p + 3c \quad \text{which in turn gives} \tag{30}$$
$$p = c. \tag{31}$$

To determine the value of d and q, we balance the linear term of lowest order of Eq. (27) with the lowest order non-linear term

$$u^{(6)} = \frac{\cdots + d_1 \exp[(-d - 63q)\eta]}{\cdots + d_2 \exp[(-64q)\eta]} \quad \text{and} \tag{32}$$
$$u'u'''' = \frac{\cdots + d_3 \exp[(-2d - 16q)\eta]}{\cdots + d_4 \exp[(-18q)\eta]} = \frac{\cdots + d_3 \exp[(-2d - 62q)\eta]}{\cdots + d_4 \exp[(-64q)\eta]} \tag{33}$$

where d_i are determined coefficients only for simplicity. Now, balancing the lowest order of exp-function in (32) and (33), we have

$$-d - 63q = -2d - 62q \quad \text{which in turn gives} \tag{34}$$
$$q = d. \tag{35}$$

We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p = c = 1$ and $q = d = 1$, then the trial solution of Eq. (26) reduces to

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)} \tag{36}$$

Substituting Eq. (36) into Eq. (27) we have

$$\begin{aligned} &\frac{1}{A}[c_6 \exp(6\eta) + c_5 \exp(5\eta) + c_4 \exp(4\eta) + c_3 \exp(3\eta) \\ &\quad + c_2 \exp(2\eta) + c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) \\ &\quad + c_{-2} \exp(-2\eta) + c_{-3} \exp(-3\eta) + c_{-4} \exp(-4\eta) + c_{-5} \exp(-5\eta) \\ &\quad + c_{-6} \exp(-6\eta)] = 0 \end{aligned} \tag{37}$$

where $A = (\exp(\eta) + b_0 + b_{-1} \exp(-\eta))^7$.

Equating the coefficients of $\exp(\eta n)$ to be zero, we obtain

$$\begin{aligned} &\{c_{-6} = 0, c_{-5} = 0, c_{-4} = 0, c_{-3} = 0, c_{-2} = 0, c_{-1} = 0, \\ &\quad c_0 = 0, c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0, c_5 = 0, c_6 = 0\}. \end{aligned} \tag{38}$$

Case 1. Solution of (38) will yield

$$a_0 = 0, a_1 = \frac{16kb_{-1} + a_{-1}}{b_{-1}}, b_0 = 0, l = -8k^3, w = \frac{464k^5}{9} \tag{39}$$

We obtain the following generalized solitary solution $u(x, y, t)$ of Eq. (25)

$$u(x, y, t) = \frac{a_{-1} e^{(-kx + 8k^3 y - \frac{464k^5}{9} t)} + \frac{16kb_{-1} + a_{-1}}{b_{-1}} e^{(kx - 8k^3 y + \frac{464k^5}{9} t)}}{b_{-1} e^{(-kx + 8k^3 y - \frac{464k^5}{9} t)} + e^{(kx - 8k^3 y + \frac{464k^5}{9} t)}} \tag{40}$$

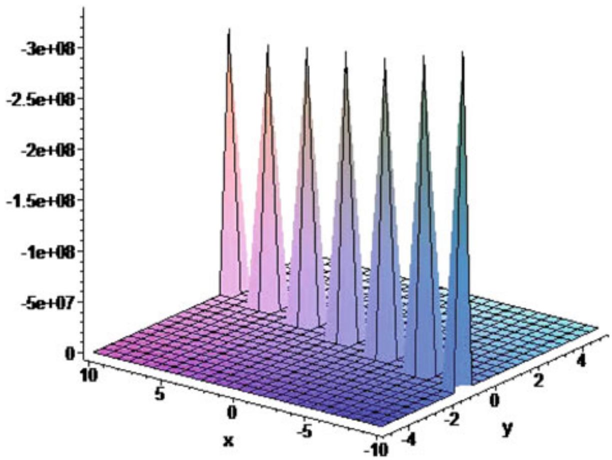

FIGURE 2. Soliton solution of Eq. (25) for $t = 0$. (Case 1)

where a_{-1}, b_{-1} and k are real numbers. We can also show Eq. (40) as

$$u\left(x,y,t\right)\frac{\left(\frac{16kb_{-1}+a_{-1}+a_{-1}b_{-1}}{b_{-1}}\right)\cosh\left(kx-8k^3y+\frac{464k^5}{9}t\right)+\left(\frac{16kb_{-1}+a_{-1}-a_{-1}b_{-1}}{b_{-1}}\right)\sinh\left(kx-8k^3y+\frac{464k^5}{9}t\right)}{\left(b_{-1}+1\right)\cosh\left(kx-8k^3y+\frac{464k^5}{9}t\right)+\left(-b_{-1}+1\right)\sinh\left(kx-8k^3y+\frac{464k^5}{9}t\right)} \tag{41}$$

Soliton solution of Eq. (25), when $a_{-1} = k = 1$ and $b_{-1} = -1$, is (Fig. 2)

$$u\left(x,y,t\right)=\frac{8\cosh\left(x-8y+\frac{464t}{9}\right)+7\sinh\left(x-8y+\frac{464t}{9}\right)}{\sinh\left(x-8y+\frac{464t}{9}\right)} \tag{42}$$

Case 2. Solution of (38) will yield

$$a_{-1}=-\frac{4}{27}kb_0^2+\frac{2}{9}a_0b_0,\quad a_1=\frac{kb_0+3a_0}{3b_0},\quad b_{-1}=\frac{2}{9}b_0^2,\quad l=-\frac{k^3}{4},\quad w=\frac{k^5}{16} \tag{43}$$

We obtain the following generalized solitary solution $u(x,y,t)$ of Eq. (25)

$$u\left(x,y,t\right)=\frac{\left(-\frac{4}{27}kb_0^2+\frac{2}{9}a_0b_0\right)e^{\left(-kx+\frac{k^3}{4}y-\frac{k^5}{16}t\right)}+a_0+\left(\frac{kb_0+3a_0}{3b_0}\right)e^{\left(kx-\frac{k^3}{4}y+\frac{k^5}{16}t\right)}}{\frac{2}{9}b_0^2e^{\left(-kx+\frac{k^3}{4}y-\frac{k^5}{16}t\right)}+b_0+e^{\left(kx-\frac{k^3}{4}y+\frac{k^5}{16}t\right)}} \tag{44}$$

where a_0, b_0 and k are real numbers.

Soliton solution of Eq. (25), when $a_0 = b_0 = -1$ and $k = 1$, is (Fig. 3)

$$u\left(x,y,t\right)=\frac{\frac{38}{27}\cosh\left(x-\frac{y}{4}+\frac{t}{16}\right)+\frac{34}{27}\sinh\left(x-\frac{y}{4}+\frac{t}{16}\right)-1}{\frac{11}{9}\cosh\left(x-\frac{y}{4}+\frac{t}{16}\right)+\frac{7}{9}\sinh\left(x-\frac{y}{4}+\frac{t}{16}\right)-1} \tag{45}$$

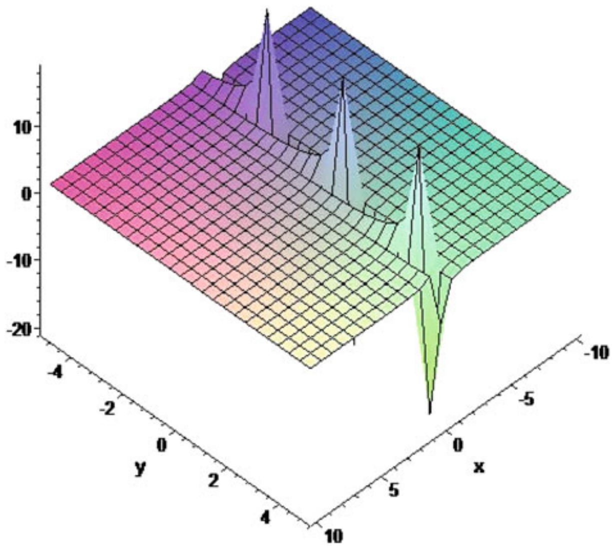

FIGURE 3. Soliton solution of Eq. (25) for $t = 0$. (Case 2)

Case 3. Solution of (38) will yield

$$a_{-1} = -kb_0^2 + \frac{1}{4}a_0 b_0, \quad a_1 = \frac{4kb_0 + a_0}{b_0}, \quad b_{-1} = \frac{1}{4}b_0^2, \quad l = -2k^3, \quad w = \frac{29k^5}{9} \tag{46}$$

We obtain the following generalized solitary solution $u(x, y, t)$ of Eq. (25)

$$u(x,y,t) = \frac{\left(-kb_0^2 + \frac{1}{4}a_0 b_0\right) e^{\left(-kx+2k^3y-\frac{29k^5}{9}t\right)} + a_0 + \left(\frac{4kb_0+a_0}{b_0}\right)e^{\left(kx-2k^3y+\frac{29k^5}{9}t\right)}}{\frac{1}{4}b_0^2 e^{\left(-kx+2k^3y-\frac{29k^5}{9}t\right)} + b_0 + e^{\left(kx-2k^3y+\frac{29k^5}{9}t\right)}} \tag{47}$$

where a_0, b_0 and k are real numbers.

Soliton solution of Eq. (25), when $a_0 = 1, b_0 = -1$ and $k = 1$, is (Fig. 4)

$$u(x,y,t) = \frac{\frac{7}{4}\cosh\left(x - 2y + \frac{29t}{9}\right) + \frac{17}{4}\sinh\left(x - 2y + \frac{29t}{9}\right) + 1}{\frac{5}{4}\cosh\left(x - 2y + \frac{29t}{9}\right) + \frac{3}{4}\sinh\left(x - 2y + \frac{29t}{9}\right) - 1} \tag{48}$$

Case 4. Solution of (38) will yield

$$a_{-1} = 0, a_1 = \frac{8kb_0 + a_0}{b_0}, b_{-1} = 0, l = -2k^3, \quad w = \frac{29k^5}{9} \tag{49}$$

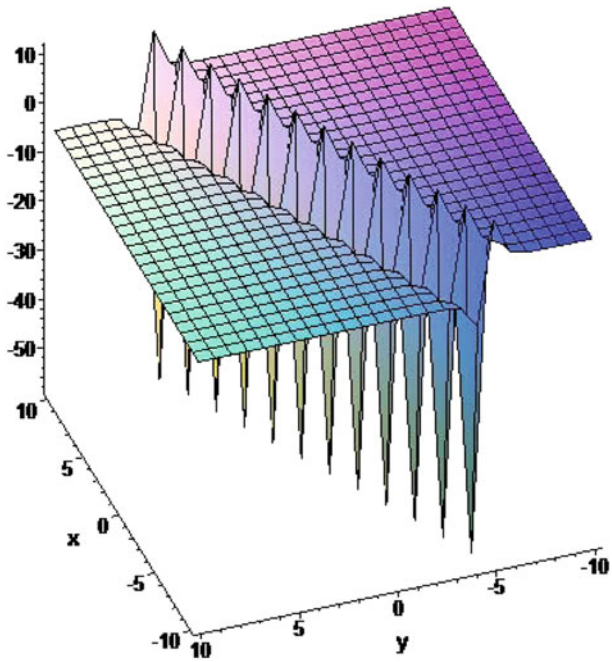

FIGURE 4. Soliton solution of Eq. (25) for $t = 0$. (Case 3)

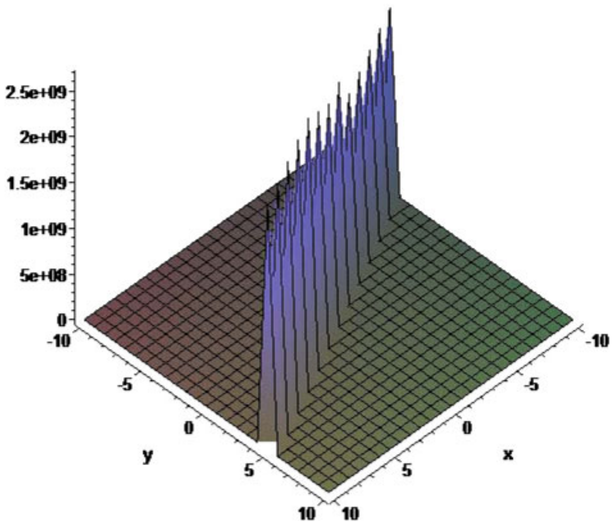

FIGURE 5. Soliton solution of Eq. (25) for $t = 0$. (Case 4)

We obtain the following generalized solitary solution $u(x, y, t)$ of Eq. (25)

$$u\left(x,y,t\right)=\frac{a_0+\left(\frac{8kb_0+a_0}{b_0}\right)e^{\left(kx-2k^3y+\frac{29k^5}{9}t\right)}}{b_0+e^{\left(kx-2k^3y+\frac{29k^5}{9}t\right)}} \tag{50}$$

where a_0, b_0 and k are real numbers.

Soliton solution of Eq. (25), when $a_0 = 1, b_0 = -1$ and $k = 1$, is (Fig. 5)

$$u\left(x,y,t\right)=\frac{7\cosh\left(x-2y+\frac{29t}{9}\right)+7\sinh\left(x-2y+\frac{29t}{9}\right)+1}{\cosh\left(x-2y+\frac{29t}{9}\right)+\sinh\left(x-2y+\frac{29t}{9}\right)-1} \tag{51}$$

5. Conclusion

In this paper, we applied the exp-function method to present soliton solitons of (1+1)-dimensional and (2+1)-dimensional Kaup–Kupershmidt (KK) equations. The solution procedure is very simple and efficient. These results also show that it is possible to construct directly exact solutions for NLEEs.

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