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Results in Mathematics

On Sharing Values of Meromorphic Functions and Their Differences

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Abstract. For a meromorphic function f in the complex plane, we prove that if f is a finite order transcendental entire function which has a finite Borel exceptional value a, if $f(z+\eta) \not\equiv f(z)$ for some $\eta \in \mathbb{C}$, and if $f(z+\eta) - f(z)$ and $f(z)$ share the value a CM, then

$$a = 0 \quad \text{and} \quad \frac{f(z+\eta) - f(z)}{f(z)} = A,$$

where A is a nonzero constant. We also consider problems on sharing values of meromorphic functions and their differences when their orders are not an integer or infinite.

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1. Introduction and Results

Let f and g be two nonconstant meromorphic functions, and let a be a finite value in the complex plane. We say that f and g share the value a CM (IM) provided that $f-a$ and $g-a$ have the same zeros counting multiplicities (ignoring multiplicities), that f and g share the value ∞ CM (IM) provided that f and g have the same poles counting multiplicities (ignoring multiplicities). Nevanlinna's four values theorem [13] says that if two nonconstant meromorphic functions f and g share four values CM, then $f \equiv g$ or f is a Möbius transformation of g. The condition "f and g share four values CM" has been weakened to "f and g share two values CM and two values IM" by Gundersen [5,6], as well as by Mues [12]. But whether the condition can be weakened to "f and g share three values IM and another value CM" is still an open question.

In a special case, we recall a well-known conjecture by Brück [1]:

Conjecture *Let f be a nonconstant entire function such that hyper order $\sigma_2(f) < \infty$ and $\sigma_2(f)$ is not a positive integer. If f and f' share the finite value a CM, then*

$$\frac{f'-a}{f-a} = c$$

where c is a nonzero constant.

We use the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (see [8,15]). In addition, we use the notation $\lambda(f)$ for the exponent of convergence of the sequence of zeros of a meromorphic function f, and $\sigma(f)$ to denote the order growth of f. Finally, $\sigma_2(f)$ denotes hyper-order (see [16]) of f which is defined by

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r}.$$

The conjecture has been verified in the special cases when $a = 0$ [1], or when f is of finite order [7], or when $\sigma_2(f) < \frac{1}{2}$ [2].

Recently, many authors [9–11] started to consider sharing values of meromorphic functions with their shifts.

Heittokangas et al. proved the following theorems.

Theorem A. (see [9]) *Let f be a meromorphic function with $\sigma(f) < 2$, and let $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share the values $a \in \mathbb{C}$ and ∞ CM, then*

$$\frac{f(z+c)-a}{f(z)-a} = \tau$$

for some constant τ.

In [9], Heittokangas et al. give the example $f(z) = e^{z^2} + 1$ which shows that $\sigma(f) < 2$ cannot be relaxed to $\sigma(f) \leq 2$.

Theorem B. (see [10]) *Let f be a meromorphic function of finite order, let $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share three distinct periodic functions $a_1,\ a_2,\ a_3 \in \widehat{S}(f)$ with period c CM (where $\widehat{S}(f) = S(f)\bigcup\{\infty\},\ S(f)$ denotes the family of all meromorphic functions which are small compared to f), then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

It is well known that $\Delta f(z) = f(z+\eta) - f(z)$ (where $\eta \in \mathbb{C}$ is a constant satisfying $f(z+\eta) - f(z) \not\equiv 0$) is regarded as the difference counterpart of f'. So, we consider the problem that $\Delta f(z)$ and $f(z)$ share one value a CM, and prove the following theorem.

Theorem 1.1. *Let f be a finite order transcendental entire function which has a finite Borel exceptional value a, and let $\eta \in \mathbb{C}$ be a constant such that $f(z+\eta) \not\equiv f(z)$. If $\Delta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share the value a CM, then*

$$a = 0 \quad \textit{and} \quad \frac{f(z+\eta) - f(z)}{f(z)} = A,$$

where A is a nonzero constant.

Remark 1.1. Theorem 1.1 shows that if f has a nonzero finite Borel exceptional value a, then for any $\eta \neq 0$, the value$a$ is not shared by $f(z+\eta) - f(z)$ and $f(z)$. For example, the function $f(z) = e^z + 1$ has the Borel exceptional value 1. Clearly, for any $\eta \neq 2k\pi i,\ k \in \mathbb{Z}$, the value 1 is not shared by $f(z+\eta) - f(z)$ $(= (e^\eta - 1)\, e^z)$ and $f(z)$.

Remark 1.2. In Theorem 1.1, the constant A is related to η. For example, the function $f(z) = e^z$ for $\eta \neq 2k\pi i,\ k \in \mathbb{Z}$, satisfies

$$\frac{f(z+\eta) - f(z)}{f(z)} = e^\eta - 1 = A.$$

The other aim of this paper is to consider that what can we say if the condition "f has a finite Borel exceptional value" is omitted? We obtain the following Theorems 1.2 and Corollary 1.3.

Theorem 1.2. *Let f be a transcendental meromorphic function such that its order of growth $\sigma(f)$ is not an integer or infinite, and let $\eta \in \mathbb{C}$ be a constant such that $f(z+\eta) \not\equiv f(z)$. If $\Delta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share three distinct values a, b, ∞ CM, then*

$$f(z+\eta) = 2f(z).$$

Corollary 1.3. *Let f be a transcendental entire function such that its order of growth $\sigma(f)$ is not an integer or infinite, and let $\eta \in \mathbb{C}$ be a constant such that $f(z+\eta) \not\equiv f(z)$. If $\Delta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share two distinct finite values a, b CM, then*

$$f(z+\eta) = 2f(z).$$

Example 1.1. Suppose that $f(z) = e^z D(z)$, where $D(z)$ is a periodic function with a period $\log 2$, and its order of growth $\sigma(D)$ is not an integer or infinite. Thus, $f(z + \log 2) - f(z)$ and $f(z)$ share values $1, 2$ CM, and satisfy $f(z + \log 2) = 2f(z)$. (By [14, Theorem 1, or 9, pp.354], we see that for any $\sigma \in [1, \infty)$ there exists a prime periodic entire function $D(z)$ of order $\sigma(D) = \sigma$).

This example shows existence of functions which satisfy the conditions of Theorem 1.2.

Example 1.2. The functions $f_1(z) = e^z$ and $f_2(z) = e^z e^{S(z)}$ where $S(z)$ is a periodic function with a period $\log 2$, such that $\sigma(f_1) = 1$ and $\sigma(f_2) = \infty$. For

f_j $(j=1,\ 2)$, we see that $f_j(z+\log 2)-f_j(z)$ and $f_j(z)$ share values $1, 2, \infty$ CM, and $f_j(z+\log 2)=2f_j(z)$.

Thus, we conjecture that in Theorem 1.2, the condition "order of growth $\sigma(f)$ is not an integer or infinite" can be omitted. But now we are unable to prove it.

2. Proof of Theorem 1.1

We need the following lemmas for the proof of Theorem 1.1.

Lemma 2.1. (see [3]) *Let f be a meromorphic function with a finite order σ, η be a nonzero constant. Let $\varepsilon>0$ be given, then exists a subset $E\subset(1,\infty)$ with finite logarithmic measure such that for all z satisfying $|z|=r\notin E\bigcup[0,1]$, we have*

$$\exp\{-r^{\sigma-1+\varepsilon}\}\leq\left|\frac{f(z+\eta)}{f(z)}\right|\leq\exp\{r^{\sigma-1+\varepsilon}\}.$$

Lemma 2.2. (see [4, p. 69–70 or 16, p. 79–80]) *Suppose that $n\geq 2$ and let $f_1,\cdots,f_n$ be meromorphic functions and $g_1,\ldots,g_n$ be entire functions such that*

(i) $\sum_{j=1}^{n}f_j\exp\{g_j\}\equiv 0$;
(ii) *when* $1\leq j<k\leq n$, g_j-g_k *is not constant;*
(iii) *when* $1\leq j\leq n,\quad 1\leq h<k\leq n$,

$$T(r,f_j)=o\{T(r,\exp\{g_h-g_k\})\}\ (r\to\infty,\ r\notin E),$$

where $E\subset(1,\infty)$ has finite linear measure or finite logarithmic measure.

Then $f_j\equiv 0,\quad j=1,\cdots,n.$

2.1. Proof of Theorem 1.1

Since f has the finite Borel exceptional value a, we see that f can be written as

$$f(z)=H(z)e^{h(z)}+a \tag{2.1}$$

where $H\not\equiv 0$ is an entire function, h is a polynomial, H and h satisfy

$$\lambda(H)=\sigma(H)=\lambda(f-a)<\sigma(f)=\deg h. \tag{2.2}$$

Since $\Delta f(z)$ and $f(z)$ share the value a CM, then

$$\frac{\Delta f(z)-a}{f(z)-a}=\frac{H(z+\eta)e^{h(z+\eta)}-H(z)e^{h(z)}-a}{H(z)e^{h(z)}}=e^{P(z)}, \tag{2.3}$$

where P is a polynomial. Set

$$h(z)=a_kz^k+a_{k-1}z^{k-1}+\cdots+a_0,\quad P(z)=b_sz^s+b_{s-1}z^{s-1}+\cdots+b_0 \tag{2.4}$$

where $k = \sigma(f), s = \deg P,\ a_k(\neq 0), a_{k-1}, \cdots, a_0, b_s(\neq 0), b_{s-1}, \cdots, b_0$ are constants. By (2.3), we see that

$$\deg P \leq \deg h.$$

First step We prove P is a constant, that is $s = 0$. To this end, we will deduce a contradiction for cases $1 \leq s < k$ and $s = k$ respectively.

Case I. Suppose that $1 \leq s < k$.

If $a \neq 0$, then by (2.3), we obtain

$$H(z+\eta)e^{h(z+\eta)-h(z)} - H(z) - H(z)e^{P(z)} = ae^{-h(z)}. \tag{2.5}$$

Since $\deg(h(z+\eta) - h(z)) = k - 1$, $\deg h = k$, $\deg P = s < k$ and $\sigma(H) < k$, we see that the order of growth of the left side of (2.5) is less than k, and the order of growth of the right side of (2.5) is equal to k. This is a contradiction.

If $a = 0$, then by (2.3), we obtain

$$\frac{H(z+\eta)}{H(z)}e^{h(z+\eta)-h(z)} - 1 = e^{P(z)}. \tag{2.6}$$

By (2.6), we see that $\frac{H(z+\eta)}{H(z)}$ is a nonzero entire function. Set $\sigma(H) = \sigma_1$. Then $\sigma_1 < \sigma(f) = k$. By Lemma 2.1, we see that for any given ε $(0 < 3\varepsilon < k - \sigma_1)$, there exists a set $E \subset (1, \infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \bigcup E$,

$$\exp\left\{-r^{\sigma_1 - 1+\varepsilon}\right\} \leq \left|\frac{H(z+\eta)}{H(z)}\right| \leq \exp\left\{r^{\sigma_1 - 1+\varepsilon}\right\}. \tag{2.7}$$

Since $\frac{H(z+\eta)}{H(z)}$ is an entire function, by (2.7) we see that

$$T\left(r, \frac{H(z+\eta)}{H(z)}\right) = m\left(r, \frac{H(z+\eta)}{H(z)}\right) \leq r^{\sigma_1 - 1+\varepsilon},$$

so that,

$$\sigma\left(\frac{H(z+\eta)}{H(z)}\right) \leq \sigma_1 - 1 + \varepsilon < k - 1. \tag{2.8}$$

Since $s < k$, we see that $\deg P \leq k - 1$. If $\deg P < k - 1$, then by (2.8) and $\deg(h(z+\eta) - h(z)) = k - 1$, we see that the order of growth of the left side of (2.6) is equal to $k - 1$, and the order of growth of the right side of (2.6) is equal to $\deg P$ which is less than $k - 1$. This is a contradiction.

If $\deg P = k-1$, then since $\frac{H(z+\eta)}{H(z)}$ is an entire function and $\deg(h(z+\eta) - h(z)) = k - 1$, by (2.8), we see that the entire function $\frac{H(z+\eta)}{H(z)}e^{h(z+\eta)-h(z)}$ has a Borel exceptional value 0, thus the value 1 must be not its Borel exceptional value. Hence, the left side of (2.6), $\frac{H(z+\eta)}{H(z)}e^{h(z+\eta)-h(z)} - 1$, has infinitely many zeros, but the right side of (2.6), e^P, has no zero. This is a contradiction.

Case II. Suppose that $1 \le s = k$. Thus, for P and h, there are three subcases: (1) $b_k = a_k$; (2) $b_k = -a_k$; (3) $b_k \ne a_k$ and $b_k \ne -a_k$.

Subcase (1) Suppose that $b_k = a_k$. First we suppose that $a \ne 0$. Thus, (2.3) is rewritten as

$$g_{11}(z)e^{P(z)} + g_{12}(z)e^{-h(z)} + g_{13}(z)e^{h_0(z)} = 0, \tag{2.9}$$

where $h_0(z) = 0$ and

$$\begin{cases} g_{11}(z) = -H(z) \\ g_{12}(z) = -a \\ g_{13}(z) = H(z+\eta)e^{h(z+\eta)-h(z)} - H(z). \end{cases}$$

Since $\deg(h(z+\eta) - h(z)) = k-1$ and $\sigma(H) < k$, we see that

$$\sigma(g_{1j}(z)) < k \ (j = 1, 2, 3).$$

On the other hand, by $b_k = a_k$, we see that

$$\deg(P - (-h)) = k, \quad \deg(P - h_0) = k, \quad \deg(-h - h_0) = k.$$

Since $e^{P-(-h)}$, e^{P-h_0} and e^{-h-h_0} are of regular growth, and $\sigma(g_{1j}) < k$ $(j = 1,\ 2,\ 3)$, we see that for $j = 1,\ 2,\ 3$

$$\begin{cases} T(r, g_{1j}) = o\left(T\left(r, e^{P-(-h)}\right)\right); \\ T(r, g_{1j}) = o\left(T\left(r, e^{P-h_0}\right)\right); \\ T(r, g_{1j}) = o\left(T\left(r, e^{-h-h_0}\right)\right). \end{cases} \tag{2.10}$$

Thus, applying Lemma 2.2 to (2.9), by (2.10), we obtain that

$$g_{1j}(z) \equiv 0 \quad (j = 1,\ 2,\ 3).$$

Clearly, this is a contradiction.

Now we suppose that $a = 0$. Thus, (2.3) is rewritten as

$$H(z)e^{P(z)} = H(z+\eta)e^{h(z(z+\eta)-h(z)} - H(z). \tag{2.11}$$

Since $H \not\equiv 0$, $\sigma(H) < k$, $\deg P = s = k$ and $\deg(h(z+\eta) - h(z)) = k-1$, we see that the order of growth of the left side of (2.11) is equal to k, and the order of growth of the right side of (2.11) is less than k. This is a contradiction.

Subcase (2) Suppose that $b_k = -a_k$. First we suppose that $a \ne 0$. Thus, (2.3) is rewritten as

$$\left[a + H(z)e^{P(z)+h(z)}\right] e^{-h(z)} = H(z+\eta)e^{h(z+\eta)-h(z)} - H(z). \tag{2.12}$$

We affirm that $a + H(z)e^{P(z)+h(z)} \not\equiv 0$. In fact, if $a + H(z)e^{P(z)+h(z)} \equiv 0$, then by (2.12), we obtain

$$H(z+\eta)e^{h(z+\eta)-h(z)} - H(z) \equiv 0,$$

that is

$$H(z+\eta)e^{h(z+\eta)} \equiv H(z)e^{h(z)},$$

this contradicts our condition $f(z+\eta) \not\equiv f(z)$. Hence $a + H(z)e^{P(z)+h(z)} \not\equiv 0$. Thus, since $\deg(P+h) \le k-1$, $\deg(-h) = k$, $\deg(h(z+\eta) - h(z)) = k-1$ and $\sigma(H) < k$, we see that the order of growth of the left side of (2.12) is equal to k, and the order of growth of the right side of (2.12) is less than k. This is a contradiction.

Now we suppose that $a = 0$. Thus, (2.3) is rewritten as (2.11). Using the same method as in the proof of Subcase (1), we get a contradiction.

Subcase (3) Suppose that $b_k \neq a_k$ and $b_k \neq -a_k$. First we suppose that $a \neq 0$. Thus, (2.3) is rewritten as (2.9). Since $b_k \neq a_k$ and $b_k \neq -a_k$, we see that

$$\deg(P - (-h)) = k, \quad \deg(P - h_0) = k, \quad \text{and} \quad \deg(-h - h_0) = k. \tag{2.13}$$

Using the same method as in Subcase (1), we deduce that

$$g_{1j} \equiv 0 \quad (j = 1,\ 2,\ 3).$$

Clearly, this is a contradiction.

Now we suppose that $a = 0$. Thus, (2.3) is rewritten as (2.11). Using the same method as in the proof of Subcase (1), we get a contradiction.

Thus, we have proved that P is only a constant, that is

$$\frac{f(z+\eta) - f(z) - a}{f(z) - a} = A \tag{2.14}$$

where A is a nonzero constant.

Second step We prove that $a = 0$. Suppose that $a \neq 0$. Thus, by (2.1) and (2.14), we deduce that

$$H(z+\eta)e^{h(z+\eta)-h(z)} - (1+A)H(z) = ae^{-h(z)} \tag{2.15}$$

Thus, since $\deg(h(z+\eta) - h(z)) = k-1$, $\sigma(H) < k$ and $\deg h = k$, we see that the order of growth of the left side of (2.15) is less than k, and the order of growth of the right side of (2.15) is equal to k. This is a contradiction.

Hence, $a = 0$ and Theorem 1.1 is proved.

3. Proof of Theorem 1.2

Since f is a finite order meromorphic function, and $f(z+\eta) - f(z)$ and $f(z)$ share the values a and ∞ CM, then

$$\frac{f(z+\eta) - f(z) - a}{f(z) - a} = e^{h_1(z)}, \tag{3.1}$$

where h_1 is a polynomial with $\deg h_1 \le \sigma(f)$. Since $\sigma(f)$ is not an integer, we see that

$$\deg h_1 < \sigma(f). \tag{3.2}$$

By (3.1), we obtain

$$f(z+\eta) - \left[1 + e^{h_1(z)}\right] f(z) = a - ae^{h_1(z)}. \tag{3.3}$$

Similarly, since $f(z+\eta) - f(z)$ and $f(z)$ share the values b and ∞ CM, we can obtain

$$f(z+\eta) - \left[1 + e^{h_2(z)}\right] f(z) = b - be^{h_2(z)}, \tag{3.4}$$

where h_2 is a polynomial, and satisfies

$$\deg h_2 < \sigma(f). \tag{3.5}$$

By (3.3) and (3.4), we obtain

$$\left[e^{h_2(z)} - e^{h_1(z)}\right] f(z) = a - b + be^{h_2(z)} - ae^{h_1(z)}. \tag{3.6}$$

We affirm $e^{h_2(z)} - e^{h_1(z)} \equiv 0$. In fact, if $e^{h_2(z)} - e^{h_1(z)} \not\equiv 0$, then by (3.2) and (3.5), we see that the order of growth of the left side of (3.6) is equal to $\sigma(f)$, but the order of growth of the right side of (3.6) is less than $\sigma(f)$. This is a contradiction.

Hence $e^{h_2(z)} \equiv e^{h_1(z)}$, so that by (3.6), we obtain

$$(a-b)\left[1 - e^{h_2(z)}\right] = 0. \tag{3.7}$$

By (3.7) and $a \neq b$, we obtain

$$e^{h_1(z)} = e^{h_2(z)} = 1. \tag{3.8}$$

Hence, by (3.1) and (3.8) we deduce that

$$f(z+\eta) \equiv 2f(z).$$

Thus, Theorem 1.2 is proved.

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