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Results in Mathematics

Almost Periodic Solutions to a Stochastic Differential Equation in Hilbert Spaces

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Abstract. In this paper, we prove the existence and uniqueness of quadratic mean almost periodic mild solutions for a class of stochastic differential equations in a real separable Hilbert space. The main technique is based upon an appropriate composition theorem combined with the Banach contraction mapping principle and an analytic semigroup of linear operators.

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1. Introduction

Stochastic differential equation has attracted great interest due to its applications in characterizing many problems in physics, biology, mechanics and so on. Qualitative properties such as existence, uniqueness, controllability and stability for various stochastic differential systems have been extensively studied by many researchers, see for instance [9, 15, 19, 21–23, 27–32, 34] and the references therein. On the other hand, the existence of almost periodic solutions for deterministic differential equations have been considerably investigated in lots of publications because of its significance and applications in physics, mechanics and mathematical biology, see for example [1, 2, 13, 14, 16, 18, 25, 35] and the references therein.

Recently, the concept of quadratic mean almost periodicity was introduced by Bezandry and Diagana [4]. In [4], such a concept was subsequently applied to proving the existence and uniqueness of a quadratic mean almost periodic solution to the following stochastic differential equations

$$dx(t) = Ax(t)dt + F\left(t, x(t)\right)dt + G\left(t, x(t)\right)dw(t), \quad t \in \mathbb{R},$$

where $A : D(A) \subset L^2(\Omega; \mathbb{H}) \to L^2(\Omega; \mathbb{H})$ for $t \in \mathbb{R}$ is a densely defined closed linear operators, $F : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \to L^2(\Omega; \mathbb{H})$ and $G : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \to L^2(\Omega; L_2^0)$ are jointly continuous satisfying some additional conditions, and $w(t)$ is a Wiener process.

Bezandry and Diagana [5] have studied the existence and uniqueness of a quadratic mean almost periodic solution to a non-autonomous semi-linear stochastic differential equations such as

$$dx(t) = A(t)x(t)dt + F(t, x(t))\,dt + G(t, x(t))dw(t), \quad t \in \mathbb{R},$$

where $A(t)$ for $t \in \mathbb{R}$ is a family of densely defined closed linear operators satisfying the so-called Acquistapace-Terreni condition in [3], $F : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \to L^2(\Omega; \mathbb{H})$ and $G : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \to L^2(\Omega; L_2^0)$ are jointly continuous satisfying some additional conditions, and $w(t)$ is a Wiener process. And Bezandry in [6] has considered the existence of quadratic mean almost periodic solutions to a semi-linear functional stochastic integro-differential equations in the form

$$\begin{aligned} x'(t) = Ax(t) &+ \int_{-\infty}^{t} C(t-u)G(u, x(u))dw(u) \\ &+ \int_{-\infty}^{t} B(t-u)F_2(u, x(u))du + F_1(t, x(t)), \end{aligned}$$

where $t \in \mathbb{R}, A : D(A) \subset L^2(\Omega; \mathbb{H}) \to L^2(\Omega; \mathbb{H})$ is a densely defined closed (possibly unbounded) linear operator; B and C are convolution-type kernels in $L^1(0, \infty)$ and $L^2(0, \infty)$, respectively, satisfying Assumptions 3.2 in [21]; $F_1, F_2 : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \to L^2(\Omega; \mathbb{H})$ and $G : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \to L^2(\Omega; L_2^0)$ are jointly continuous functions. For more results on this topic, we refer the reader to the papers [7,8,12,33] and the references therein.

Motivated by the above mentioned works [4–6], the main purpose of this paper is to deal with the existence and uniqueness of quadratic mean almost periodic solutions to a class of neutral stochastic functional differential equations in the abstract form

$$d[x(t) - g(t, x(t))] = Ax(t)dt + G(t, x(t))dw(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $A : D(A) \subset L^2(\Omega; \mathbb{H}) \to L^2(\Omega; \mathbb{H})$ is the infinitesimal generator of an analytic semigroup of linear operators $\{T(t)\}_{t \geq 0}$ on $L^2(\Omega; \mathbb{H}), g : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \to L^2(\Omega; \mathbb{H}_\alpha)$ and $G : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \to L^2(\Omega; L_2^0)$ are jointly continuous functions, $w(t)$ is a Brownian motion. The main technique is based upon an appropriate composition theorem combined with the Banach contraction mapping principle and an analytic semigroup of linear operators. The obtained result can be seen as a contribution to this emerging field.

The rest of this paper is organized as follows: In Sect. 2 we recall some basic definitions, lemmas and preliminary facts which will be need in the sequel. Our main result and its proofs are arranged in Sect. 3. In the last section, an example is given to illustrate our main result.

2. Preliminaries

This section is mainly concerned with some notations, definitions, lemmas and preliminary facts which are used in what follows. For more details on this section, we refer the reader to [4,5,10,11]

Throughout the paper, $(\mathbb{H}, \|\cdot\|)$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}})$ denote two real Hilbert spaces. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. We let $L_2(\mathbb{K}, \mathbb{H})$ denote the space of all Hilbert–Schmidt operators $\Phi : \mathbb{K} \to \mathbb{H}$, equipped with the Hilbert–Schmidt norm $\|\cdot\|_2$.

For a symmetric nonnegative operator $Q \in L_2(\mathbb{K}, \mathbb{H})$ with finite trace we suppose that $\{w(t) : t \in \mathbb{R}\}$ is a Q-Wiener process defined on $(\Omega, \mathscr{F}, P)$ and with values in $\mathbb{K}$. So, actually, w can be obtained as follows: let $w_i(t), t \in \mathbb{R}$, $i = 1, 2$, be independent $\mathbb{K}$-valued Q-Wiener processes, then

$$w(t) = \begin{cases} w_1(t) & \text{if } t \geq 0, \\ w_2(-t) & \text{if } t \leq 0 \end{cases}$$

is Q-Wiener process with $\mathbb{R}$ as time parameter. We then let $\mathscr{F}_t = \sigma\{w(s) : s \leq t\}$ is the σ-algebra generated by w.

The collection of all strongly measurable, square integrable, $\mathbb{H}$-valued random variable, denoted by $L^2(\Omega; \mathbb{H})$, is a Banach space equipped with norm $\|x\|_{L^2(\Omega;\mathbb{H})} = (E\|X\|^2)^{\frac{1}{2}}$, where the expectation E is defined $E[x] = \int_\Omega x(w)dP(w)$.

Let $\mathbb{K}_0 = Q^{\frac{1}{2}}\mathbb{K}$ and $L_2^0 = L_2(\mathbb{K}_0, \mathbb{H})$ with respect to the norm

$$\|\Phi\|^2_{L_2^0} = \|\Phi Q^{\frac{1}{2}}\|_2^2 = Tr(\Phi Q \Phi^*).$$

Let $0 \in \rho(A)$ where $\rho(A)$ is the resolvent of A. Then for $0 < \alpha \leq 1$, it is possible to define the fractional power $(-A)^\alpha$, as a closed linear operator on its domain $D((-A)^\alpha)$. Furthermore, the subspace $D((-A)^\alpha)$ is dense in $L^2(\Omega; \mathbb{H})$ and the expression

$$\|x\|_\alpha = \|(-A)^\alpha x\|_{L^2(\Omega;\mathbb{H})}, \ x \in D((-A)^\alpha),$$

defines a norm on $D((-A)^\alpha)$. Hereafter we denote by $L^2(\Omega; \mathbb{H}_\alpha)$ the Banach space $D((-A)^\alpha)$ with norm $\|x\|_\alpha$.

The following properties hold by [26].

Lemma 2.1. *Let $0 < \gamma \le \mu \le 1$. Then the following properties hold:*

(i) $L^2(\Omega;\mathbb{H}_\mu)$ *is a Banach space and* $L^2(\Omega;\mathbb{H}_\mu) \hookrightarrow L^2(\Omega;\mathbb{H}_\gamma)$ *is continuous.*
(ii) *The function* $s \to (-A)^\mu T(s)$ *is continuous in the uniform operator topology on* $(0,\infty)$ *and there exists* $M_\mu > 0$ *such that* $\|(-A)^\mu T(t)\| \le M_\mu e^{-\delta t} t^{-\mu}$ *for each* $t > 0$.
(iii) *For each* $x \in D((-A)^\mu)$ *and* $t \ge 0, (-A)^\mu T(t)x = T(t)(-A)^\mu x$.
(iv) $(-A)^{-\mu}$ *is a bounded linear operator in* $L^2(\Omega;\mathbb{H})$ *with* $D((-A)^\mu) = Im((-A)^{-\mu})$.

In the following results and definitions, we let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}}), (\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ and $(\mathbb{Z}, \|\cdot\|_{\mathbb{Z}})$ be Banach spaces and let $L^2(\Omega;\mathbb{X})$, $L^2(\Omega;\mathbb{Y})$ and $L^2(\Omega;\mathbb{Z})$ be their corresponding L^2-spaces, respectively.

Definition 2.1. [4] A stochastic process $x : \mathbb{R} \to L^2(\Omega;\mathbb{X})$ is said to be continuous whenever

$$\lim_{t\to s} E\|x(t) - x(s)\|_{\mathbb{X}}^2 = 0.$$

Definition 2.2. [4] A continuous stochastic process $x : \mathbb{R} \to L^2(\Omega;\mathbb{X})$ is said to be quadratic mean almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t\in\mathbb{R}} E\|x(t+\tau) - x(t)\|_{\mathbb{X}}^2 < \varepsilon.$$

The collection of all stochastic processes $x : \mathbb{R} \to L^2(\Omega;\mathbb{X})$ which are quadratic mean almost periodic is then denoted by $AP\left(\mathbb{R}; L^2(\Omega;\mathbb{X})\right)$.

Lemma 2.2. [4] *If x belongs to $AP\left(\mathbb{R}; L^2(\Omega;\mathbb{X})\right)$, then the following hold true:*

(i) *the mapping* $t \to E\|x(t)\|_{\mathbb{X}}^2$ *is uniformly continuous,*
(ii) *there exists a constant* $N > 0$, *such that* $E\|x(t)\|_{\mathbb{X}}^2 \le N$, *for each* $t \in \mathbb{R}$,
(iii) x *is stochastically bounded.*

Let $C(\mathbb{R}, L^2(\Omega;\mathbb{H}))$ denote the space of all continuous stochastic processes $x : \mathbb{R} \to L^2(\Omega;\mathbb{H})$. The notation $CUB(\mathbb{R}; L^2(\Omega;\mathbb{X}))$ stands for the collection of all stochastic processes $x : \mathbb{R} \to L^2(\Omega;\mathbb{X})$, which are continuous and uniformly bounded. It is known from [4] that $CUB(\mathbb{R}; L^2(\Omega;\mathbb{X}))$ is a Banach space endowed with the norm:

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} (E\|x(t)\|_{\mathbb{X}}^2)^{\frac{1}{2}}.$$

Lemma 2.3. [4] $AP(\mathbb{R}; L^2(\Omega;\mathbb{X})) \subset CUB(\mathbb{R}; L^2(\Omega;\mathbb{X}))$ *is a closed subspace.*

Lemma 2.4. [4] $(AP(\mathbb{R}; L^2(\Omega;\mathbb{X})), \|\cdot\|_{AP(\mathbb{R};L^2(\Omega;\mathbb{X}))})$ *is a Banach space endowed with the norm:*

$$\|x\|_{AP(\mathbb{R};L^2(\Omega;\mathbb{X}))} = \sup_{t\in\mathbb{R}} (E\|x(t)\|_{\mathbb{X}}^2)^{\frac{1}{2}}.$$

Definition 2.3. [4] A function $F : \mathbb{R} \times L^2(\Omega; \mathbb{Y}) \to L^2(\Omega; \mathbb{Z}), (t, y) \to F(t, y)$, which is jointly continuous, is said to be quadratic mean almost periodic in $t \in \mathbb{R}$ uniformly in $y \in \mathbb{B}$ where $\mathbb{B} \subset L^2(\Omega; \mathbb{Y})$ is compact if for any $\varepsilon > 0$, there exists $l(\varepsilon, \mathbb{B}) > 0$ such that any interval of length $l(\varepsilon, \mathbb{B})$ contains at least a number τ for which

$$\sup_{t\in\mathbb{R}} E\|F(t+\tau, y) - F(t, y)\|_{\mathbb{Z}}^2 < \varepsilon$$

for each stochastic process $y : \mathbb{R} \to \mathbb{B}$.

Lemma 2.5. [4] *Let $F : \mathbb{R}\times L^2(\Omega; \mathbb{Y}) \to L^2(\Omega; \mathbb{Z}), (t, y) \to F(t, y)$ be a quadratic mean almost periodic process in $t \in \mathbb{R}$ uniformly in $y \in \mathbb{B}$, where $\mathbb{B} \subset L^2(\Omega; \mathbb{Y})$ is compact. Suppose that F is Lipschitz in the following sense:*

$$E\|F(t, x) - F(t, y)\|_{\mathbb{Z}}^2 \leq \tilde{M} E\|x - y\|_{\mathbb{Y}}^2$$

for all $x, y \in L^2(\Omega; \mathbb{Y})$ and for each $t \in \mathbb{R}$, where $\tilde{M} > 0$. Then for any quadratic mean almost periodic process $\Psi : \mathbb{R} \to L^2(\Omega; \mathbb{Y})$, the stochastic process $t \to F(t, \Psi(t))$ is quadratic mean almost periodic.

Definition 2.4. A $\mathscr{F}_t$-progressively process $\{x(t)\}_{t\in\mathbb{R}}$ is called a mild solution of the problem (1.1) on $\mathbb{R}$ if the function $s \to AT(t-s)g(s, x(s))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$, and $x(t)$ satisfies

$$x(t) = T(t-a)[x(a) - g(a, x(a))] + g(t, x(t)) + \int_a^t AT(t-s)g(s, x(s))ds$$
$$+ \int_a^t T(t-s)G(s, x(s))dw(s)$$

for all $t \geq a$ and for each $a \in \mathbb{R}$.

Let us list the following assumptions:

(H1) The operator $A : D(A) \subset L^2(\Omega; \mathbb{H}) \to L^2(\Omega; \mathbb{H})$ is the infinitesimal generator of an analytic semigroup of linear operators $\{T(t)\}_{t\geq 0}$ on $L^2(\Omega; \mathbb{H})$ and M, δ are positive numbers such that $\|T(t)\| \leq Me^{-\delta t}$ for $t \geq 0$.

(H2) There exists a positive number $\alpha \in (0, 1)$ such that $g : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \to L^2(\Omega; \mathbb{H}_\alpha)$ is quadratic mean almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \mathbb{B}_1$ where $\mathbb{B}_1 \subset L^2(\Omega; \mathbb{H})$ being a compact subspace. Moreover, g is Lipschitz in the sense that: there exists $L_g > 0$ such that

$$E\|(-A)^\alpha g(t, x) - (-A)^\alpha g(t, y)\|^2 \leq L_g E\|x - y\|^2,$$

for all $t \in \mathbb{R}$ and for each stochastic processes $x, y \in L^2(\Omega; \mathbb{H})$.

(H3) The function $G : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \to L^2(\Omega; L_2^0)$ is quadratic mean almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \mathbb{B}_2$ where $\mathbb{B}_2 \subset L^2(\Omega; \mathbb{H})$ being a compact subspace. Moreover, G is Lipschitz in the sense that: there exists $L_G > 0$ such that

$$E\|G(t,x) - G(t,y)\|^2_{L_2^0} \leq L_G E\|x-y\|^2,$$

for all $t \in \mathbb{R}$ and for each stochastic processes $x, y \in L^2(\Omega; \mathbb{H})$.

3. Main Results

In this section, we present and prove our main theorem.

Theorem 3.1. *Assume the conditions (H1)–(H3) are satisfied, then the problem (1.1) admits a unique quadratic mean almost periodic mild solution on* $\mathbb{R}$ *provide that*

$$L_0 = \left[3L_g\|(-A)^{-\alpha}\|^2 + 3M_{1-\alpha}^2 L_g \delta^{-2\alpha}[\Gamma(\alpha)]^2 + \frac{3TrQM^2L_G}{2\delta}\right] < 1, \quad (3.1)$$

where $\Gamma(\cdot)$ *is the gamma function.*

Proof. Let $\Lambda : AP(\mathbb{R}; L^2(\Omega; \mathbb{H})) \to C(\mathbb{R}, L^2(\Omega; \mathbb{H}))$ be the operator defined by

$$\Lambda x(t) = g(t, x(t)) + \int_{-\infty}^{t} AT(t-s)g(s, x(s))ds + \int_{-\infty}^{t} T(t-s)G(s, x(s))dw(s), \ t \in \mathbb{R}.$$

First we prove that Λx is well defined. From Lemma 2.5, we infer that $s \to g(s, x(s))$ is in $AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$. Thus using Lemma 2.2 (ii) it follows that there exists a constant $N_g > 0$ such that $E\|(-A)^\alpha g(t, x(t))\|^2 \leq N_g$, for all $t \in \mathbb{R}$. Moreover, from the continuity of $s \to AT(t-s)$ and $s \to T(t-s)$ in the uniform operator topology on $(-\infty, t)$ for each $t \in \mathbb{R}$ and the estimate

$$\begin{aligned}
&E\left\|\int_{-\infty}^{t} AT(t-s)g(s, x(s))ds\right\|^2 \\
&= E\left\|\int_{-\infty}^{t} (-A)^{1-\alpha}T(t-s)(-A)^\alpha g(s, x(s))ds\right\|^2 \\
&\leq M_{1-\alpha}^2 E\left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1}\|(-A)^\alpha g(s, x(s))\|ds\right)^2 \\
&\leq M_{1-\alpha}^2 \left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1}ds\right)
\end{aligned}$$

$$\times \left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1} E\|(-A)^{\alpha} g(s,x(s))\|^2 ds \right)$$

$$\leq N_g M_{1-\alpha}^2 \left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1} ds \right)^2$$

$$= N_g M_{1-\alpha}^2 \delta^{-2\alpha} [\Gamma(\alpha)]^2,$$

it follows that $s \to AT(t-s)g(s,x(s))$ and $s \to T(t-s)G(s,x(s))$ are integrable on $(-\infty,t)$ for every $t \in \mathbb{R}$, therefore Λx is well defined and continuous.

Next, we show that $\Lambda x(t) \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}))$. We define

$$\Lambda_1 x(t) = \int_{-\infty}^{t} AT(t-s)g(s,x(s))ds$$

and

$$\Lambda_2 x(t) = \int_{-\infty}^{t} T(t-s)G(s,x(s))dw(s).$$

Let us show that $\Lambda_1 x(t)$ is quadratic mean almost periodic. Now since $g(\cdot, x(\cdot)) \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$, by Definition 2.2, it follows that for any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains at least a number τ with the property that

$$E\|(-A)^{\alpha} g(t+\tau, x(t+\tau)) - (-A)^{\alpha} g(t,x(t))\|^2 < \frac{\varepsilon}{M_{1-\alpha}^2 \delta^{-2\alpha} [\Gamma(\alpha)]^2},$$

for each $t \in \mathbb{R}$.

Now, using Cauchy–Schwarz inequality, we obtain that

$$E\|\Lambda_1 x(t+\tau) - \Lambda_1 x(t)\|^2$$

$$= E \left\| \int_{-\infty}^{t} AT(t-s)[g(s+\tau, x(s+\tau)) - g(s,x(s))]ds \right\|^2$$

$$= E \left\| \int_{-\infty}^{t} (-A)^{1-\alpha} T(t-s)[(-A)^{\alpha} g(s+\tau, x(s+\tau)) - (-A)^{\alpha} g(s,x(s))]ds \right\|^2$$

$$\leq M_{1-\alpha}^2 E \left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1} \right.$$

$$\times\|(-A)^\alpha g(s+\tau,x(s+\tau))-(-A)^\alpha g(s,x(s))\|ds\Bigg)^2$$

$$\le M_{1-\alpha}^2 E\left[\left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1}ds\right)\right.$$

$$\times\left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1}\|(-A)^\alpha g(s+\tau,x(s+\tau))\right.$$

$$\left.\left.-(-A)^\alpha g(s,x(s))\|^2 ds\right)\right] \le M_{1-\alpha}^2\left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1}ds\right)$$

$$\times\left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1}E\|(-A)^\alpha g(s+\tau,x(s+\tau))\right.$$

$$\left.-(-A)^\alpha g(s,x(s))\|^2 ds\right) \le M_{1-\alpha}^2\left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1}ds\right)^2$$

$$\times\sup_{t\in\mathbb{R}} E\|(-A)^\alpha g(t+\tau,x(t+\tau))-(-A)^\alpha g(t,x(t))\|^2$$

$$\le \frac{\varepsilon}{\delta^{-2\alpha}[\Gamma(\alpha)]^2}\left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1}ds\right)^2 = \varepsilon.$$

Hence, $\Lambda_1 x(\cdot)$ is quadratic mean almost periodic.

Similarly, by using Lemma 2.5, one can easily see that $s\to G(s,x(s))$ is quadratic mean almost periodic. Therefore, it follows from Definition 2.2 that for any $\varepsilon>0$, there exists $l(\varepsilon)>0$ such that every interval of length $l(\varepsilon)$ contains at least a number τ with the property that

$$E\|G(t+\tau,x(t+\tau))-G(t,x(t))\|_{L_2^0}^2 < \frac{2\delta\varepsilon}{TrQM^2},$$

for each $t\in\mathbb{R}$. Now, let us prove that $\Lambda_2 x(t)$ is quadratic mean almost periodic. We adopt the techniques developed in [4]. Let $\tilde{w}(t) := w(t+\tau)-w(\tau)$ for each $t\in\mathbb{R}$, note that $\tilde{w}$ is also a Brownian motion and has the same distribution as w.

Now, we consider

$$
\begin{aligned}
& E\|\Lambda_2 x(t+\tau)-\Lambda_2 x(t)\|^2 \\
= & E\left\|\int_{-\infty}^{t+\tau} T(t+\tau-s)G(s,x(s))dw(s)-\int_{-\infty}^{t} T(t-s)G(s,x(s))dw(s)\right\|^2 \\
= & E\left\|\int_{-\infty}^{t} T(t-s)[G(s+\tau,x(s+\tau))-G(s,x(s))]d\tilde{w}(s)\right\|^2.
\end{aligned}
$$

Thus using an estimate on Ito integral established in Ichikawa [20], we obtain that

$$
\begin{aligned}
E\|\Lambda_2 & x(t+\tau)-\Lambda_2 x(t)\|^2 \\
& = E\left\|\int_{-\infty}^{t} T(t-s)[G(s+\tau,x(s+\tau))-G(s,x(s))]d\tilde{w}(s)\right\|^2 \\
& \le TrQE\left[\int_{-\infty}^{t} \|T(t-s)[G(s+\tau,x(s+\tau))-G(s,x(s))]\|^2 ds\right] \\
& \le TrQE\left[\int_{-\infty}^{t} \|T(t-s)\|^2\|G(s+\tau,x(s+\tau))-G(s,x(s))]\|^2_{L_2^0} ds\right] \\
& \le TrQM^2\int_{-\infty}^{t} e^{-2\delta(t-s)}E\|G(s+\tau,x(s+\tau))-G(s,x(s))]\|^2_{L_2^0}\, ds \\
& \le TrQM^2\left(\int_{-\infty}^{t} e^{-2\delta(t-s)}ds\right)\sup_{t\in\mathbb{R}} E\|G(t+\tau,x(t+\tau))-G(t,x(t))]\|^2_{L_2^0} \\
& < 2\delta\varepsilon\int_{-\infty}^{t} e^{-2\delta(t-s)}ds=\varepsilon.
\end{aligned}
$$

Thus, $\Lambda_2 x(\cdot)$ is quadratic mean almost periodic. And in view of the above, it is clear that Λ maps $AP(\mathbb{R};L^2(\Omega;\mathbb{H}))$ into itself.

Now the remaining task is to prove that Λ is a strict contraction on $AP(\mathbb{R};L^2(\Omega;\mathbb{H}))$. Indeed, for each $t\in\mathbb{R}, x,y\in AP(\mathbb{R};L^2(\Omega;\mathbb{H}))$, we have

$$
\begin{aligned}
E\|\Lambda x(t)-\Lambda y(t)\|^2 & \le 3E\|g(t,x(t))-g(t,y(t))\|^2_\alpha \\
& +3E\left(\left\|\int_{-\infty}^{t} AT(t-s)[g(s,x(s))-g(s,y(s))]ds\right\|\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&+3E\left(\left\|\int_{-\infty}^{t} T(t-s)[G(s,x(s))-G(s,y(s))]dw(s)\right\|\right)^2\\
&\le 3\|(-A)^{-\alpha}\|^2 E\|(-A)^{\alpha}g(t,x(t))-(-A)^{\alpha}g\left(t,y(t)\right)\|^2\\
&+3E\left(\left\|\int_{-\infty}^{t} (-A)^{1-\alpha}T(t-s)[(-A)^{\alpha}g(s,x(s))-(-A)^{\alpha}g(s,y(s))]ds\right\|\right)^2\\
&+3TrQE\left(\int_{-\infty}^{t} \|T(t-s)[G(s,x(s))-G(s,y(s))]\|^2 ds\right)\\
&\le 3\|(-A)^{-\alpha}\|^2 \sup_{t\in\mathbb{R}} E\|(-A)^{\alpha}g(t,x(t))-(-A)^{\alpha}g(t,y(t))\|^2\\
&+3M_{1-\alpha}^2 E\left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1}\|(-A)^{\alpha}g(s,x(s))-(-A)^{\alpha}g(s,y(s))\|ds\right)^2\\
&+3TrQE\left(\int_{-\infty}^{t} \|T(t-s)\|^2\|G\left(s,x(s)\right)-G(s,y(s))\|_{L_2^0}^2 ds\right)\\
&\le 3L_g\|(-A)^{-\alpha}\|^2 \sup_{t\in\mathbb{R}} E\|x(t)-y(t)\|^2+3M_{1-\alpha}^2\\
&\quad E\left[\left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1}ds\right)\right.\\
&\quad \left.\times\left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1}\|(-A)^{\alpha}g(s,x(s))-(-A)^{\alpha}g(s,y(s))\|^2 ds\right)\right]\\
&+3TrQM^2\int_{-\infty}^{t} e^{-2\delta(t-s)}E\|G\left(s,x(s)\right)-G(s,y(s))\|_{L_2^0}^2 ds\\
&\le 3L_g\|(-A)^{-\alpha}\|^2 \sup_{t\in\mathbb{R}} E\|x(t)-y(t)\|^2+3M_{1-\alpha}^2\left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1}ds\right)\\
&\quad\times\left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1}E\|(-A)^{\alpha}g(s,x(s))-(-A)^{\alpha}g(s,y(s))\|^2 ds\right)\\
&+3TrQM^2L_G\left(\int_{-\infty}^{t} e^{-2\delta(t-s)}ds\right)\sup_{t\in\mathbb{R}} E\|x(t)-y(t)\|^2
\end{aligned}
$$

$$\leq 3L_g\|(-A)^{-\alpha}\|^2 \sup_{t\in\mathbb{R}} E\|x(t)-y(t)\|^2$$
$$+3M_{1-\alpha}^2 L_g \left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{\alpha-1}ds\right)^2 \sup_{t\in\mathbb{R}} E\|x(t)-y(t)\|^2$$
$$+3TrQM^2L_G\frac{1}{2\delta}\sup_{t\in\mathbb{R}} E\|x(t)-y(t)\|^2$$
$$\leq 3L_g\|(-A)^{-\alpha}\|^2 \sup_{t\in\mathbb{R}} E\|x(t)-y(t)\|^2 + 3M_{1-\alpha}^2 L_g\delta^{-2\alpha}[\Gamma(\alpha)]^2$$
$$\times \sup_{t\in\mathbb{R}} E\|x(t)-y(t)\|^2 + \frac{3TrQM^2L_G}{2\delta}\sup_{t\in\mathbb{R}} E\|x(t)-y(t)\|^2$$
$$= \left[3L_g\|(-A)^{-\alpha}\|^2 + 3M_{1-\alpha}^2 L_g\delta^{-2\alpha}[\Gamma(\alpha)]^2 + \frac{3TrQM^2L_G}{2\delta}\right]$$
$$\times \sup_{t\in\mathbb{R}} E\|x(t)-y(t)\|^2,$$

by using the arithmetic geometic inequality, Cauchy–Schwarz inequality and Ito isometry identity.

Note that

$$\sup_{t\in\mathbb{R}} E\|x(t)-y(t)\|^2 \leq \left[\sup_{t\in\mathbb{R}}(E\|x(t)-y(t)\|^2)^{\frac{1}{2}}\right]^2.$$

Thus, it follows that, for each $t\in\mathbb{R}$,

$$(E\|\Lambda x(t)-\Lambda y(t)\|^2)^{\frac{1}{2}} \leq \sqrt{L_0}\|x-y\|_{AP(\mathbb{R};L^2(\Omega;\mathbb{H}))}.$$

Hence

$$\|\Lambda x-\Lambda y\|_{AP(\mathbb{R};L^2(\Omega;\mathbb{H}))} = \sup_{t\in\mathbb{R}}\left(E\|\Lambda x(t)-\Lambda y(t)\|^2\right)^{\frac{1}{2}}$$
$$\leq \sqrt{L_0}\|x-y\|_{AP(\mathbb{R};L^2(\Omega;\mathbb{H}))},$$

which implies that Λ is a contraction by (3.1). So by the contraction principle, we conclude that there exists a unique fixed point $x(\cdot)$ for Λ in $AP(\mathbb{R};L^2(\Omega;\mathbb{H}))$, such that $\Lambda x = x$, that is

$$x(t) = g(t,x(t)) + \int_{-\infty}^{t} AT(t-s)g(s,x(s))ds + \int_{-\infty}^{t} T(t-s)G(s,x(s))dw(s)$$

for all $t\in\mathbb{R}$. If we let $x(a) = g(a,x(a)) + \int_{-\infty}^{a} AT(a-s)g(s,x(s))ds + \int_{-\infty}^{a} T(a-s)G(s,x(s))dw(s)$, then

$$T(t-a)x(a) = T(t-a)g(a,x(a)) + \int_{-\infty}^{a} AT(t-s)g(s,x(s))ds + \int_{-\infty}^{a} T(t-s)G(s,x(s))dw(s).$$

But for $t \geq a$,

$$\begin{aligned}\int_{a}^{t} T(t-s)G(s,x(s))dw(s) &= \int_{-\infty}^{t} T(t-s)G(s,x(s))dw(s) \\ &\quad - \int_{-\infty}^{a} T(t-s)G(s,x(s))dw(s) = x(t) - g(t,x(t)) \\ &\quad - \int_{-\infty}^{t} AT(t-s)g(s,x(s))ds - T(t-a)[x(a) - g(a,x(a))] \\ &\quad + \int_{-\infty}^{a} AT(t-s)g(s,x(s))ds = x(t) - g\,(t,x(t)) \\ &\quad - \int_{a}^{t} AT(t-s)g(s,x(s))ds - T(t-a)[x(a) - g(a,x(a))].\end{aligned}$$

In conclusion, $x(t) = T(t-a)[x(a)-g(a,x(a))]+g(t,x(t))+\int_{a}^{t} AT(t-s)g(s,x(s))ds + \int_{a}^{t} T(t-s)G(s,x(s))dw(s)$ is a mild solution of the problem (1.1) and $x(\cdot) \in AP(\mathbb{R};L^2(\Omega;\mathbb{H}))$. The proof is finished. □

4. An Example

In this section we consider a simple example to illustrate our main theorem. We examine the existence and uniqueness of quadratic mean almost periodic solutions for the following stochastic partial differential equation

$$\frac{\partial}{\partial t}[x(t,\xi) - g(t,x(t,\xi))] = \frac{\partial^2}{\partial \xi^2} x(t,\xi) + G(t,x(t,\xi))dw(t),\ t \in \mathbb{R},\ \xi \in \mathcal{D}, \tag{4.1}$$

where $\mathcal{D} \subset \mathbb{R}^n (n \geq 1)$ is a bounded subset with C^2 boundary $\partial\mathcal{D}$.

Let $H := L^2(\mathcal{D})$ be equipped with its natural topology and define an operator A on $L^2(\mathbb{R}; H)$ by

$$Ax(t, \cdot) = \frac{\partial^2}{\partial \xi^2} x(t, \cdot), \; x \in H^2(\mathcal{D} \cap H_0^1(\mathcal{D}).$$

It is well known that (for example, see [17,24,26]) A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ on $L^2(\mathbb{R}; H)$ satisfying (H1). Therefore, under assumptions (H2)–(H3), if we assume that (3.1) holds, an application of Theorem 3.1 yields that (4.1) has a unique quadratic mean almost periodic mild solution.

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