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Results in Mathematics

A New Analytical Approach to Solve Magnetohydrodynamics Flow Over a Nonlinear Porous Stretching Sheet by Laplace Padé Decomposition Method

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Abstract. The aspire of this article is to bring in a new approximate method, that is to say the Laplace Padé decomposition method which is a mixture of Laplace decomposition and Padé approximation to offer an analytical approximate way out to magnetohydrodynamics flow over a nonlinear porous stretching sheet. This new iteration approach provides us with a convenient way to approximate solution. A closed agreement between the obtained solution and some well-known results has been established. The proposed procedure can be applied to handle other nonlinear problems.

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Keywords. Laplace decomposition method, Padé approximation, MHD flow, porous stretching sheet, approximate solution.

1. Introduction

Nonlinear phenomenon come into view in a various areas of scientific field such as plasma physics, solid state physics, fluid dynamics and chemical kinetics. For the reason that of the increased interest in nonlinear physical model, an extensive variety of analytical and numerical methods have been used in the investigation of these scientific models. Mathematical modeling of numerous physical systems leads to nonlinear ordinary and partial differential equations in a variety of fields of physics and engineering. An efficient scheme is essential to analyze the mathematical model that provides solutions conforming to physical reality. Normal analytic procedures linearize the system or assume that nonlinearity are relatively not important. Such assumptions occasionally strongly influence the solution with respect to the actual physics of the phenomenon such as perturbation. Thus looking for exact solutions of nonlinear ordinary or partial differential equation is of great significance. A variety of powerful mathematical techniques such as Adomian decomposition method [1,6–9], homotopy perturbation method [3,11–13,25], variational methods [26–30] and homotopy analysis method [14–16,22,23] have been projected for obtaining exact and approximate analytical solutions. Some of these methods use precise transformations in order to trim down the equations into simpler ones or system of equations and others give the solution in a series form that converges to the exact solution.

The Laplace decomposition method (LDM) was first proposed, by Khuri [17] and is used to furnish approximate solutions of the nonlinear initial value problems. The LDM is valuable to attain exact and approximate solutions of linear and nonlinear differential equations. Recently Majid et al. [10,18–21] introduced various modifications in Laplace decomposition technique to deal with nonlinear behavior of the physical models. It is worth mentioning that the proposed scheme is an elegant recipe of LDM and Padé approximation [2]. The advantage of proposed idea is its capability of combining two powerful techniques for obtaining fast convergent series for nonlinear equations.

2. Description of Laplace Decomposition Method

Consider equation $F(u(x)) = g(x)$, where F represents a general nonlinear ordinary or partial differential operator together with both linear and nonlinear terms. The nonlinear term can be split as $L + R$, where L is the highest order linear operator and R is the remaining of the linear operator. Consequently, the equation can be written as

$$Lu + Ru + Nu = g(x), \tag{2.1}$$

where Nu, indicates the nonlinear terms. By applying Laplace transform on both sides of Eq. (2.1), we get

$$\pounds[Lu + Ru + Nu = g(x)]. \tag{2.2}$$

Using the differential property of Laplace transform, we have

$$s^n \pounds[u] - \sum_{k=1}^{n} s^{k-1} u^{(n-k)}(0) + \pounds[Ru] + \pounds[Nu] = \pounds[g(x)]. \tag{2.3}$$

Operating inverse Laplace transform on both sides of Eq. (2.3), we get

$$u = G(x) - \pounds^{-1}\left[\frac{1}{s^n}[\pounds[Nu] + \pounds[Ru]]\right]. \tag{2.4}$$

The LDM assumes the solution u can be expanded into infinite series as

$$u = \sum_{m=0}^{\infty} u_m. \tag{2.5}$$

Also the nonlinear term Nu can be written as

$$Nu = \sum_{m=0}^{\infty} A_m, \tag{2.6}$$

where A_m are the Adomian polynomials [9]. By substituting Eqs. (2.5) and (2.6) in Eq. (2.4), the solution can be written as

$$\sum_{m=0}^{\infty} u_m(x) = G(x) - \pounds^{-1}\left[\frac{1}{s^n}\left[\pounds\left[\sum_{m=0}^{\infty} A_m\right] + \pounds\left[R\sum_{m=0}^{\infty} u_m\right]\right]\right]. \tag{2.7}$$

In Eq. (2.7), the Adomian polynomials can be generated by several means. Here we used the following recursive formulation:

$$A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \quad m = 0, 1, 2, \ldots \tag{2.8}$$

In general, the recursive relation is given by

$$u_0(x) = G(x), \tag{2.9}$$

$$u_{m+1}(x) = -\pounds^{-1}\left[\frac{1}{s^n}\left[\pounds\left[\sum_{m=0}^{\infty} A_m\right] + \pounds\left[R\sum_{m=0}^{\infty} u_m\right]\right]\right], \quad m \geq 0, \tag{2.10}$$

where $G(x)$ represents the term arising from source term and prescribe initial conditions. The proposed method does not resort to linearization, assumptions of weak nonlinearity and it is more realistic compared to the method of simplifying the physical problems.

3. Padé Approximates

A Padé approximate is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $u(x)$. The $[L/M]$ Padé approximates to a function $u(x)$ are given by [2]

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)}, \tag{3.1}$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a degree of at most M. The power series in terms of x is given below

$$u(x) = \sum_{i=0}^{\infty} a_i x^i, \tag{3.2}$$

$$u(x) = \frac{P_L(x)}{Q_M(x)} + O(x^{L+M+1}). \tag{3.3}$$

Determine the coefficients of $P_L(x)$ and $Q_M(x)$ by Eq. (3.3). We can multiply the numerator and denominator by a constant and leave $[L/M]$ unchanged, we imposed the normalization condition

$$Q_0(x) = q_0 = 1. \tag{3.4}$$

Expanding polynomials $P_L(x)$ and $Q_M(x)$ in power series in terms of x of order L and M which is given below :

$$\begin{aligned} P_L(x) &= p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L, \\ Q_M(x) &= 1 + q_1 x + q_2 x^2 + \cdots + q_M x^M. \end{aligned} \tag{3.5}$$

Using Eq. (3.5) in Eq. (3.3), we can write Eq. (3.3) in the notation of formal power series

$$\sum_{i=0}^{\infty} a_i x^i = \frac{p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L}{1 + q_1 x + q_2 x^2 + \cdots + q_M x^M} + O(x^{L+M+1}). \tag{3.6}$$

By cross-multiplication of Eq. (3.6), we get

$$\begin{aligned} &(p_0 + p_1 x + \cdots + p_L x^L)(a_0 + a_1 x + a_1 x^2 \cdots) \\ &\quad = 1 + q_1 x + \cdots + q_M x^M + O(x^{L+M+1}). \end{aligned} \tag{3.7}$$

From Eq. (3.7) we obtain the set of linear equations

$$\begin{cases} a_0 = p_0, \\ a_1 + a_0 q_1 = p_1, \\ a_2 + a_1 q_1 + a_0 q_2 = p_2, \\ \vdots \\ a_L + a_{L-1} q_1 + a_0 q_L = p_L, \end{cases} \tag{3.8}$$

and

$$\begin{cases} a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0, \\ a_{L+2} + a_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0, \\ \vdots \\ a_{L+M} + a_{L+M-1} q_1 + a_L q_M = 0. \end{cases} \tag{3.9}$$

From Eq. (3.9), we can obtain $q_i, 1 \leq i \leq M$. Once the values of $q_1, q_2, \ldots, q_M$ are all known Eq. (3.8) gives an explicit formula for the unknown quantities $p_1, p_2, \ldots, p_L$. We calculate diagonal approximates like [2/2], [3/3],[4/4],or [5/5] which are more accurate than nondiagonal approximates and can be calculated easily by built-in utilities of Mathematica 7 and Maple 14.

4. Mathematical Formulation of the Problem

Let us consider the magnetohydrodynamics (MHD) flow of an incompressible viscous fluid over a nonlinear porous stretching sheet at $y = 0$. The fluid is electrically conducting under the influence of an applied magnetic field $B(x)$

normal to the stretching sheet. The induced magnetic field is neglected. The resulting boundary-layer equations are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{4.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\sigma\frac{B_0^2(x)}{\rho}u, \tag{4.2}$$

where u and v are the velocity components in the x-and y-directions, respectively, ν is the kinematic viscosity, ρ is the density and σ is the electrical conductivity of the fluid. In Eq. (4.2), the external electric field and polarization effects are negligible, therefore our required magnetic field is given as

$$B(x)=B_0x^{\frac{n-1}{2}}. \tag{4.3}$$

The boundary conditions corresponding to the nonlinear porous stretching sheet are given below

$$\begin{aligned}&u(x,0)=cx^n, \quad v(x,0)=-V_0,\\ &u(x,y)\to 0 \quad \text{as} \quad y\to\infty,\end{aligned} \tag{4.4}$$

where c is the stretching parameter and V_0 is the porosity of the plate (where $V_0>0$ corresponds to suction velocity and $V_0<0$ holds for injection velocity). Upon making use of the following similarity transformation

$$\begin{aligned}\eta&=\sqrt{\frac{c(n+1)}{2\nu}}x^{\frac{n-1}{2}}y, \quad u=cx^n f^{'}(\eta),\\ v&=-\sqrt{\frac{c\nu(n+1)}{2}}x^{\frac{n-1}{2}}\left[f(\eta)+\frac{n-1}{n+1}\eta f'(\eta)\right].\end{aligned} \tag{4.5}$$

The resulting nonlinear differential equation and boundary conditions are of the following form

$$f'''+ff''-\beta f'^2-Mf'=0, \tag{4.6}$$

$$f(0)=K, \quad f'(0)=1, \quad f'(\infty)=0, \tag{4.7}$$

where

$$\beta=\frac{2n}{n+1}, \quad M=\frac{2\sigma B_0^2}{\rho c(1+n)}, \quad K=\frac{V_0}{\sqrt{\frac{c(n+1)}{2\nu}}x^{\frac{n-1}{2}}}. \tag{4.8}$$

Applying Laplace transform algorithm we get

$$s^3\pounds[f]-s^2f(0)-sf'(0)-f''(0)=\pounds\left[\beta f'^2-ff''+Mf'\right]. \tag{4.9}$$

Using given boundary condition Eqs. (4.9) becomes

$$s^3\pounds[f]-s^2K-s-\zeta=\pounds\left[\beta f'^2-ff''+Mf'\right], \tag{4.10}$$

$$\pounds[f]=\frac{s^2K+s+\zeta}{s^3}+\frac{1}{s^3}\pounds\left[\beta f'^2-ff''+Mf'\right]. \tag{4.11}$$

Applying inverse Laplace transform to Eq. (4.11), we get

$$f(\eta) = K + \eta + \frac{\zeta\eta^2}{2} + \pounds^{-1}\left[\frac{1}{s^3}\pounds[\beta {f'}^2 - ff'' + Mf']\right]. \tag{4.12}$$

The LDM assumes a series solution of the function $f(\eta)$ is given by

$$f(\eta) = \sum_{m=0}^{\infty} f_m(\eta), \tag{4.13}$$

Using Eq. (4.13) into Eq. (4.12) we get

$$\sum_{m=0}^{\infty} f_m(\eta) = K + \eta + \frac{\zeta\eta^2}{2} + \pounds^{-1}\left[\frac{1}{s^3}\pounds\left[\beta\sum_{m=0}^{\infty} A_m(\eta) - \sum_{m=0}^{\infty} B_m(\eta) + Mf'_m\right]\right]. \tag{4.14}$$

In Eq. (4.14), $A_m(\eta)$ and $B_m(\eta)$ are Adomian polynomials [6] that represents nonlinear terms. Therefore, Adomian polynomials are given below

$$\sum_{m=0}^{\infty} A_m(\eta) = f'^2(\eta), \tag{4.15}$$

$$\sum_{m=0}^{\infty} B_m(\eta) = f(\eta)f''(\eta). \tag{4.16}$$

The few components of the Adomian polynomials are given as follow:

$$\begin{aligned}
A_0(\eta) &= f_0'^2(\eta),\\
A_1(\eta) &= 2f_0'(\eta)f_1'(\eta),\\
A_2(\eta) &= f_1'^2(\eta) + 2f_0'(\eta)f_2'(\eta),\\
&\vdots\\
A_m(\eta) &= \sum_{i=0}^{m} f_i'(\eta)f_{m-i}'(\eta).
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
B_0(\eta) &= f_0(\eta)f_0''(\eta),\\
B_1(\eta) &= f_0(\eta)f_1''(\eta) + f_1(\eta)f_0''(\eta),\\
B_2(\eta) &= f_0(\eta)f_2''(\eta) + f_1(\eta)f_1''(\eta) + f_2(\eta)f_0''(\eta)\\
&\vdots\\
B_m(\eta) &= \sum_{i=0}^{m} f_i(\eta)f_{m-i}''(\eta).
\end{aligned} \tag{4.18}$$

From Eqs. (4.17–4.18), our required recursive relation is given as follows:

$$f_0(\eta) = K + \eta + \frac{\zeta\eta^2}{2}, \tag{4.19}$$

$$f_{m+1}(\eta) = \pounds^{-1}\left[\frac{1}{s^3}\pounds\left[\beta\sum_{m=0}^{\infty}A_m(\eta) - \sum_{m=0}^{\infty}B_m(\eta) + Mf'_m\right]\right], \quad m \geq 0. \tag{4.20}$$

The first few components of $f_m(\eta)$ given as follows

$$f_0(\eta) = K + \eta + \frac{\zeta\eta^2}{2}, \tag{4.21}$$

$$\begin{aligned} f_1(\eta) = {} & \frac{M\eta^3}{6} - \frac{K\eta^3\zeta}{6} - \frac{\eta^4\zeta}{24} + \frac{M\eta^4\zeta}{24} - \frac{\eta^5\zeta^2}{120} \\ & + \frac{\eta^3\beta}{6} + \frac{\eta^4\zeta\beta}{12} + \frac{\eta^5\zeta^2\beta}{60}, \end{aligned} \tag{4.22}$$

$$\begin{aligned} f_2(\eta) = {} & -\frac{KM\eta^4}{24} - \frac{M\eta^5}{60} + \frac{M^2\eta^5}{120} + \frac{K^2\eta^4\zeta}{24} + \frac{K\eta^5\zeta}{40} - \frac{KM\eta^5\zeta}{60} \\ & + \frac{\eta^6\zeta}{240} - \frac{M\eta^6\zeta}{90} + \frac{M^2\eta^6\zeta}{720} + \frac{K\eta^6\zeta^2}{144} + \frac{11\eta^7\zeta}{5040} - \frac{M\eta^7\zeta^2}{630} \\ & + \frac{11\eta^8\zeta^3}{40320} - \frac{K\eta^4\beta}{24} - \frac{\eta^5\beta}{60} + \frac{M\eta^5\beta}{40} - \frac{K\eta^5\beta\zeta}{30} - \frac{\eta^6\beta\zeta}{60} \\ & + \frac{M\eta^6\beta\zeta}{72} - \frac{K\eta^6\beta\zeta^2}{90} - \frac{2\eta^7\beta\zeta^2}{315} + \frac{M\eta^7\beta\zeta^2}{504} - \frac{\eta^8\beta\zeta^3}{1260} \\ & + \frac{\eta^5\beta^2}{60} + \frac{\eta^6\beta^2\zeta}{72} + \frac{\eta^7\beta^2\zeta^2}{252} + \frac{\eta^8\beta^2\zeta^3}{504}, \end{aligned} \tag{4.23}$$

Accordingly, the solution of Eq. (4.6) in a series form is given by

$$\begin{aligned} f(\eta) = {} & K + \eta + \frac{\zeta\eta^2}{2} + \frac{M\eta^3}{6} - \frac{K\eta^3\zeta}{6} - \frac{\eta^4\zeta}{24} + \frac{M\eta^4\zeta}{24} - \frac{\eta^5\zeta^2}{120} \\ & + \frac{\eta^3\beta}{6} + \frac{\eta^4\zeta\beta}{12} + \frac{\eta^5\zeta^2\beta}{60} + -\frac{KM\eta^4}{24} - \frac{M\eta^5}{60} + \frac{M^2\eta^5}{120} \\ & + \frac{K^2\eta^4\zeta}{24} + \frac{K\eta^5\zeta}{40} + -\frac{KM\eta^5\zeta}{60} + \frac{\eta^6\zeta}{240} - \frac{M\eta^6\zeta}{90} + \frac{M^2\eta^6\zeta}{720} \\ & + \frac{K\eta^6\zeta^2}{144} + + \frac{11\eta^7\zeta}{5040} - \frac{M\eta^7\zeta^2}{630} + \frac{11\eta^8\zeta^3}{40320} - \frac{K\eta^4\beta}{24} - \frac{\eta^5\beta}{60} \\ & + \frac{M\eta^5\beta}{40} - \frac{K\eta^5\beta\zeta}{30} - \frac{\eta^6\beta\zeta}{60} + \frac{M\eta^6\beta\zeta}{72} - \frac{K\eta^6\beta\zeta^2}{90} - \frac{2\eta^7\beta\zeta^2}{315} \\ & + \frac{M\eta^7\beta\zeta^2}{504} - \frac{\eta^8\beta\zeta^3}{1260} + \frac{\eta^5\beta^2}{60} + \frac{\eta^6\beta^2\zeta}{72} + \frac{\eta^7\beta^2\zeta^2}{252} \\ & + \frac{\eta^8\beta^2\zeta^3}{504} + \cdots \end{aligned} \tag{4.24}$$

Our plan in this section is principally concerned with the mathematical behavior of the solution $f(\eta)$ in order to establish the value of free parameter $\zeta = f''(0)$. It was properly shown by Baker [2] that this objective can effortlessly achieved by forming the Padé approximates which have the advantage of manipulating the polynomial approximation into a rational function to obtain the more information about $f(\eta)$. In reality, Padé approximates will converges on the entire real axis if $f(\eta)$ is free of singularities on the entire real axis. Additionally, the diagonal approximates are most correct approximates, for that reason we will construct only diagonal approximates. By means of the boundary condition $f'(\infty) = 0$, the diagonal approximates $[M/M]$ vanish if the coefficients of η with the highest power in the numerator vanishes. Choosing the coefficients of the maximum power of η equal to zero, we get a polynomial equations in ζ which can be solved very straightforwardly by means of the built in utilities in the most manipulation languages such as Maple and Mathematica.

Tables 1, 2, 3 and 4, obviously elucidates that present solution technique that is Laplace Padé decomposition method (LPDM) shows an excellent agreement with the solutions previously available in literature. This study shows that LPDM suits for MHD flow problems (Fig. 1).

TABLE 1. Comparison of the values of $\zeta = f''(0)$ obtained by Laplace Padé decomposition method and exact solution

β	K	M	[11/11]	[12/12]	[13/13]	[14/14]	[15/15]	Exact solution [24]
1	0	0.0	−0.99999	−1.000000	−1.00000	−1.00000	−1.00000	−1.00000
		1.0	−1.41425	−1.41423	−1.41422	−1.41421	−1.41421	−1.41421
		5.0	−2.44949	−2.44949	−2.44949	−2.44949	−2.44949	−2.44948
		10.0	−3.31662	−3.31662	−3.31662	−3.31662	−3.31662	−3.31662
		50.0	−7.14144	−7.14142	−7.14142	−7.14142	−7.14142	−7.14142
		100.0	−10.04989	−10.04987	−10.04987	−10.04987	−10.04987	−10.04987

TABLE 2. Comparison of the values of $\zeta = f''(0)$ obtained by shooting method and Laplace Padé decomposition method

β	K	M	Shooting method [4]	Present method (LPDM)
5.0	0.0	0.0	−1.9025	−1.9031
		1.0	−2.1529	−2.1529
		5.0	−2.9414	−2.9414
		10.0	−3.6956	−3.6956
		50.0	−7.3256	−7.3256
		100.0	−10.1816	−10.1816

Table 3. Comparison of the values of $\zeta = f''(0)$ obtained by Crocco transform and Laplace Padé decomposition method

β	K	M	Crocco transform [4]	Present method (LPDM)
5.0	0.0	0.0	−1.9025	−1.9031
		1.0	−2.1529	−2.1529
		5.0	−2.9414	−2.9414
		10.0	−3.6956	−3.6956
		50.0	−7.3256	−7.3256
		100.0	−10.1816	−10.1816

Table 4. Comparison of the values of $\zeta = f''(0)$ obtained by Homotopy analysis and Laplace Padé decomposition methods

β	K	M	HAM [5]	Present method (LPDM)
1.0	0.0	1.0	−1.40992	−1.41421
		5.0	−2.44892	−2.44949
		10.0	−3.31285	−3.31662

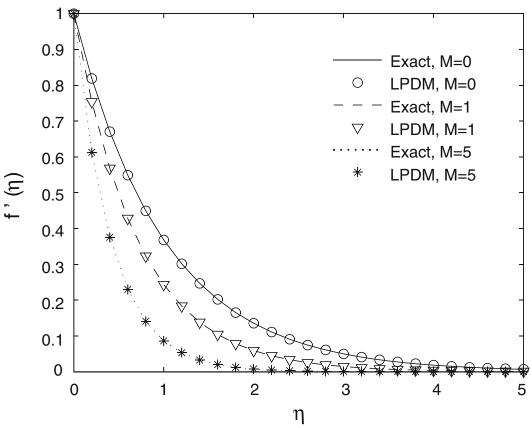

Figure 1. Comparison of solution obtained by LPDM and exact solution [24] for $\beta = 1$ and $K = 0$

In order to observe the variations of M, β and K on f', we plot Figs. 2, 3 and 4. It is noted from Fig. 2 that f' decreases when M is increases. Further Fig. 3 and 4 show that f' decreases by increasing β and K, respectively.

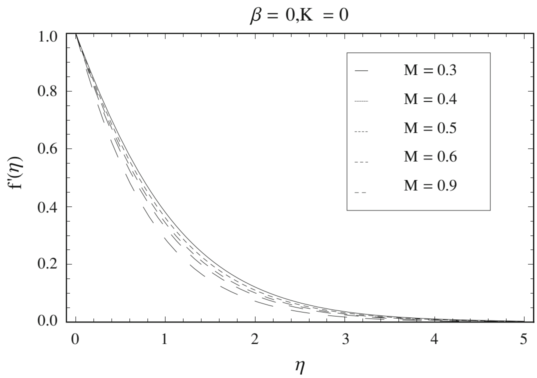

FIGURE 2. Magnetic effects on velocity profile

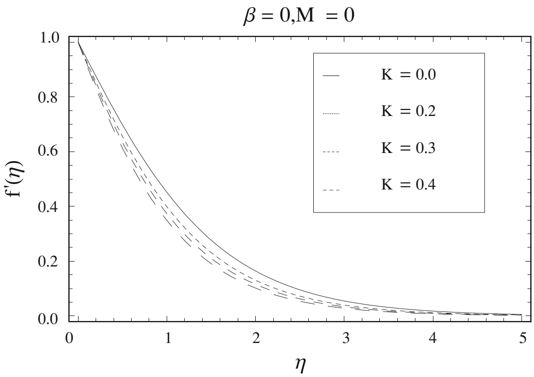

FIGURE 3. Porous effects on velocity profile

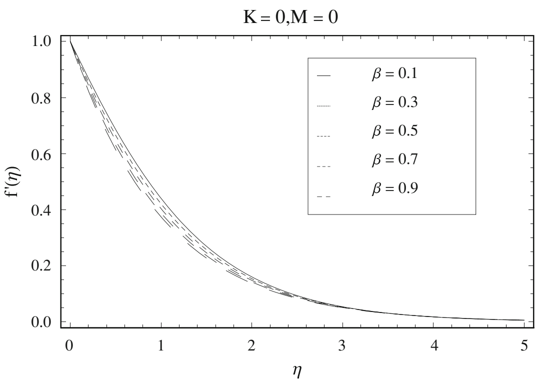

FIGURE 4. Stretching effects on velocity profile

5. Concluding Remarks

The objective now is to offer series solution of a MHD flow over a nonlinear porous stretching sheet equation via LPDM. Such a investigation is even not presented so far in literature. Series solution is obtained by means of LPDM. The convergence of LPDM is also shown in Tables 1, 2, 3 and 4. The results of LPDM are compared with shooting method, Crocco transform and homotopy analysis method. The results of these methods have closed agreement with each other. The analysis given here shows more confidence on LPDM. For that reason, this method can be applied to other difficult nonlinear equations in boundary layer theory and does not involve linearization, discretization, perturbation and occupy less memory space in executions of a recursive relation.

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