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Results in Mathematics

On Nonlocal Boundary Value Problems for Nonlinear Integro-differential Equations of Arbitrary Fractional Order

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Abstract. In this paper, we prove the existence of solutions of a nonlocal boundary value problem for nonlinear integro-differential equations of fractional order given by

$$
{}^cD^q x(t) = f(t, x(t), (\phi x)(t), (\psi x)(t)), \quad 0 < t < 1,
$$
$$
x(0) = \beta x(\eta), x'(0) = 0, x''(0) = 0, \ldots, x^{(m-2)}(0) = 0, x(1) = \alpha x(\eta),
$$

where $q \in (m-1, m], m \in \mathbb{N}, m \geq 2$, $0 < \eta < 1$, and ϕx and ψx are integral operators. The existence results are established by means of the contraction mapping principle and Krasnoselskii's fixed point theorem. An illustrative example is also presented.

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Keywords. Nonlinear fractional differential equations, nonlocal boundary conditions, existence, fixed point theorem.

1. Introduction

Multi-point nonlinear boundary value problems, which refer to a different family of boundary conditions in the study of disconjugacy theory [1] and take into account the boundary data at intermediate points of the interval under consideration, have been addressed by many authors, for example, see [2–9] and the references therein. Multi-point boundary conditions are important in various physical problems of applied science when the controllers at the end points of the interval (under consideration) dissipate or add energy according to the censors located at intermediate points.

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Fractional differential equations appear naturally in various fields of science and engineering such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electrodynamics of complex medium, viscoelasticity, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, etc. In consequence, differential equations of fractional order have been addressed by several researchers with the sphere of study ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. For some recent work on this branch of differential equations, see [10–30] and the references therein. Recently, Ahmad and Nieto [10] studied a nonlocal boundary value problem for higher order nonlinear differential equations of fractional order.

In this paper, motivated by [10], we discuss the existence of solutions of a nonlocal boundary value problem for nonlinear integro-differential equations of fractional order q:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t), (\phi x)(t), (\psi x)(t)), \quad q \in (m-1, m], m \in \mathbb{N}, m \geq 2, 0 < t < 1, \\ x(0) = \beta x(\eta), x'(0) = 0, x''(0) = 0, \ldots, x^{(m-2)}(0) = 0, x(1) = \alpha x(\eta), \\ 0 < \eta < 1, \quad (\alpha - \beta)\eta^{m-1} \neq 1 - \beta, \quad \beta, \alpha \in \mathbb{R}, \end{cases} \tag{1.1}$$

where ${}^cD$ is the Caputo fractional derivative, $f : [0,1] \times X \times X \times X \to X$ is continuous, and for $\gamma, \delta : [0,1] \times [0,1] \to [0, \infty)$,

$$(\phi x)(t) = \int_0^t \gamma(t,s)x(s)\, ds, \quad (\psi x)(t) = \int_0^t \delta(t,s)x(s)\, ds.$$

Here, $(X, \|.\|)$ is a Banach space and $\mathcal{C} = \mathrm{C}([0,1], X)$ denotes the Banach space of all continuous functions from $[0,1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|.\|$.

By a solution of (1.1), we mean a function $x \in \mathcal{C}$ of class $C^m[0,1]$ which satisfies the nonlocal fractional boundary value problem (1.1).

2. Preliminaries

Let us recall some basic definitions [11,12,14] on fractional calculus.

Definition 2.1. For a continuous function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)\, ds, \quad n-1 < q < n, \quad n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann–Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}}\, ds, \quad q > 0,$$

provided the integral exists.

Definition 2.3. The Riemann–Liouville fractional derivative of order q for a continuous function $g(t)$ is defined by

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \Big(\frac{d}{dt}\Big)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}}\, ds, \quad n = [q] + 1,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

To study the nonlinear problem (1.1), we first consider the associated linear problem and obtain its solution.

Lemma 2.4. *For a given $\sigma \in C[0,1]$, the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = \sigma(t), \quad 0 < t < 1, \quad q \in (m-1, m], \quad m \in \mathbb{N}, \quad m \geq 2, \\ x(0) = \beta x(\eta), \quad x'(0) = 0, \quad x''(0) = 0, \ldots, x^{(m-2)}(0) = 0, \quad x(1) = \alpha x(\eta), \\ 0 < \eta < 1, \quad (\alpha - \beta)\eta^{m-1} \neq 1 - \beta, \quad \beta, \alpha \in \mathbb{R}, \end{cases} \tag{2.1}$$

is given by

$$\begin{aligned} x(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)\, ds + \Big(\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \sigma(s)\, ds \\ &- \Big(\frac{\beta + (\alpha-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \sigma(s)\, ds. \end{aligned} \tag{2.2}$$

Proof. As argued in [10], the general solution of (2.1) can be written as

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)\, ds - c_0 - c_1 t - c_2 t^2 - \cdots - c_{m-1} t^{m-1}, \tag{2.3}$$

where $c_0, c_1, c_2, \ldots, c_{m-1} \in \mathbb{R}$ are arbitrary constants. In view of the relations ${}^cD^q I^q x(t) = x(t)$ and $I^q I^p x(t) = I^{q+p} x(t)$ for $q, p > 0, x \in L(0,1)$, we obtain

$$x'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} \sigma(s)\, ds - c_1 - 2c_2 t - \cdots - (m-1)c_{m-1} t^{m-2},$$

$$x''(t) = \int_0^t \frac{(t-s)^{q-3}}{\Gamma(q-2)} \sigma(s)\, ds - 2c_2 - \cdots - (m-1)(m-2)c_{m-1} t^{m-3}, \ldots$$

Applying the boundary conditions for (2.1), we find that

$$c_0 = \Big(\frac{\beta}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \sigma(s)\, ds$$
$$- \Big(\frac{\beta\eta^{m-1}}{\beta - 1 + (\alpha-\beta)} \eta^{m-1}\Big) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \sigma(s)\, ds,$$

$c_1 = 0, \ldots, c_{m-2} = 0$, and

$$c_{m-1} = \Big(\frac{\alpha-\beta}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \sigma(s)\, ds$$
$$+ \Big(\frac{\beta-1}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \sigma(s)\, ds,$$

Substituting the values of $c_0, c_1, \ldots, c_{m-1}$ in (2.3), we obtain (2.2). This completes the proof. $\square$

3. Main Results

To prove the main results, we need the following assumptions:

($\mathbf{A_1}$) There exist positive functions $L_1(t), L_2(t), L_3(t)$ such that

$$\begin{aligned} &\|f(t, x(t), (\phi x)(t), (\psi x)(t)) - f(t, y(t), (\phi y)(t), (\psi y)(t))\| \\ &\quad \le L_1(t)\|x-y\| + L_2(t)\|\phi x - \phi y\| + L_3(t)\|\psi x - \psi y\|, \\ &\quad\ \forall t \in [0,1], x, y \in X. \end{aligned}$$

We set

$$\gamma_0 = \sup_{t\in[0,1]} \Big| \int_0^t \gamma(t,s)\, ds \Big|, \delta_0 = \sup_{t\in[0,1]} \Big| \int_0^t \delta(t,s)\, ds \Big|,$$
$$I_L^q = \sup_{t\in[0,1]} \{|I^q L_1(t)|, |I^q L_2(t)|, |I^q L_3(t)|\},$$
$$I^q L(1) = \max\{|I^q L_1(1)|, |I^q L_2(1)|, |I^q L_3(1)|\},$$

and

$$I^q L(\eta) = \max\{|I^q L_1(\eta)|, |I^q L_2(\eta)|, |I^q L_3(\eta)|\}.$$

($\mathbf{A_2}$) There exists a number κ such that $\Lambda \le \kappa < 1$, where

$$\Lambda = (1+\gamma_0+\delta_0)\Big\{I_L^q + \frac{(|\beta\eta^{m-1}| + |1-\beta|)I^q L(1) + (|\beta| + |\alpha-\beta|)I^q L(\eta)}{|\beta - 1 + (\alpha-\beta)\eta^{m-1}|}\Big\}.$$

($\mathbf{A_3}$) $\|f(t,x(t),(\phi x)(t),(\psi x)(t))\| \le \mu(t), \forall (t,x,\phi x,\psi x) \in [0,1]\times X \times X \times X, \mu \in L^1([0,1],R^+)$.

Theorem 3.1. *Assume that* (A_1) *and* (A_2) *hold. Then the boundary value problem* (1.1) *has a unique solution.*

Proof. Define $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(\mathcal{G}x)(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s),(\phi x)(s),(\psi x)(s))\,ds \\ &+\Big(\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s),(\phi x)(s),(\psi x)(s))\,ds \\ &-\Big(\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s,x(s),(\phi x)(s),(\psi x)(s))\,ds,\end{aligned}$$

$t \in [0,1]$. Let us set $\sup_{t\in[0,1]} |f(t,0,0,0)| = M$, and choose

$$r \ge \frac{M}{(1-\lambda)\Gamma(q+1)}\Big\{1 + \frac{(|\beta\eta^{m-1}| + |1-\beta|) + (|\beta| + |\alpha-\beta|\eta^q)}{|\beta-1+(\alpha-\beta)\eta^{m-1}|}\Big\},$$

where λ is such that $\Lambda \le \lambda < 1$. Now we show that $\mathcal{G}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$. For $x \in B_r$, we have

$$\begin{aligned}\|(\mathcal{G}x)(t)\| \le &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f(s,x(s),(\phi x)(s),(\psi x)(s))\,ds\| \\ &+\Big|\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \|f(s,x(s),(\phi x)(s),(\psi x)(s))\|ds \\ &+\Big|\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \|f(s,x(s),(\phi x)(s),(\psi x)(s))\|ds \\ \le &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (\|f(s,x(s),(\phi x)(s),(\psi x)(s)) - f(s,0,0,0)\| + \|f(s,0,0,0)\|)\,ds\end{aligned}$$

$$
\begin{aligned}
&+\Big|\frac{\beta\eta^{m-1}+(1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}(\|f(s,x(s),(\phi x)(s),(\psi x)(s))\\
&-f(s,0,0,0)\|+\|f(s,0,0,0)\|)\,ds\\
&+\Big|\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}(\|f(s,x(s),(\phi x)(s),(\psi x)(s))\\
&-f(s,0,0,0)\|+\|f(s,0,0,0)\|)\,ds\\
\le&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}(L_1(s)\|x(s)\|+L_2(s)\|(\phi x)(s)\|+L_3(s)\|(\psi x)(s)\|+M)\,ds\\
&+\Big|\frac{\beta\eta^{m-1}+(1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}(L_1(s)\|x(s)\|\\
&+L_2(s)\|(\phi x)(s)\|+L_3(s)\|(\psi x)(s)\|+M)\,ds\\
&+\Big|\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}(L_1(s)\|x(s)\|+L_2(s)\|(\phi x)(s)\|\\
&+L_3(s)\|(\psi x)(s)\|+M)\,ds\\
\le&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}(L_1(s)\|x(s)\|+\gamma_0L_2(s)\|x(s)\|+\delta_0L_3(s)\|x(s)\|+M)\,ds\\
&+\Big|\frac{\beta\eta^{m-1}+(1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}(L_1(s)\|x(s)\|+\gamma_0L_2(s)\|x(s)\|\\
&+\delta_0L_3(s)\|x(s)\|+M)\,ds\\
&+\Big|\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}(L_1(s)\|x(s)\|+\gamma_0L_2(s)\|x(s)\|\\
&+\delta_0L_3(s)\|x(s)\|+M)\,ds\\
\le&(I^qL_1(t)+\gamma_0I^qL_2(t)+\delta_0I^qL_3(t))r+\frac{Mt^q}{\Gamma(q+1)}\\
&+\Big|\frac{\beta\eta^{m-1}+(1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\Big((I^qL_1(1)+\gamma_0I^qL_2(1)+\delta_0I^qL_3(1))r+\frac{M}{\Gamma(q+1)}\Big)\\
&+\Big|\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\Big((I^qL_1(\eta)+\gamma_0I^qL_2(\eta)+\delta_0I^qL_3(\eta))r+\frac{M\eta^q}{\Gamma(q+1)}\Big)\\
\le&I_L^q\{1+\gamma_0+\delta_0\}r+\frac{M}{\Gamma(q+1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(|\beta\eta^{m-1}|+|1-\beta|)}{|\beta-1+(\alpha-\beta)\eta^{m-1}|}\Big((I^qL(1)\{1+\gamma_0+\delta_0\}r+\frac{M}{\Gamma(q+1)}\Big)\\
&+\frac{(|\beta|+|\alpha-\beta|)}{|\beta-1+(\alpha-\beta)\eta^{m-1}|}\Big((I^qL(\eta)\{1+\gamma_0+\delta_0\}r+\frac{M\eta^q}{\Gamma(q+1)}\Big)\\
&=(1+\gamma_0+\delta_0)\Big\{I_L^q+\frac{(|\beta\eta^{m-1}|+|1-\beta|)}{|\beta-1+(\alpha-\beta)\eta^{m-1}|}I^qL(1)\\
&+\frac{(|\beta|+|\alpha-\beta|)}{|\beta-1+(\alpha-\beta)\eta^{m-1}|}I^qL(\eta)\Big\}r\\
&+\frac{M}{\Gamma(q+1)}\Big\{1+\frac{(|\beta\eta^{m-1}|+|1-\beta|)}{|\beta-1+(\alpha-\beta)\eta^{m-1}|}+\frac{(|\beta|+|\alpha-\beta|)\eta^q}{|\beta-1+(\alpha-\beta)\eta^{m-1}|}\Big\}\\
&\le(\Lambda+1-\lambda)r\le r.
\end{aligned}
$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
&\|(\mathcal{G}x)(t)-(\mathcal{G}y)(t)\|\\
&\le\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\|f(s,x(s),(\phi x)(s),(\psi x)(s)-f(s,y(s),(\phi y)(s),(\psi y)(s))\|\,ds\\
&+\Big|\frac{\beta\eta^{m-1}+(1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\|f(s,x(s),(\phi x)(s),(\psi x)(s)\\
&-f(s,y(s),(\phi y)(s),(\psi y)(s))\|\,ds\\
&+\Big|\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}\|f(s,x(s),(\phi x)(s),(\psi x)(s)\\
&-f(s,y(s),(\phi y)(s),(\psi y)(s))\|\,ds\\
&\le\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\Big(L_1(s)\|x-y\|+L_2(s)\|\phi x-\phi y\|+L_3(s)\|\psi x-\psi y\|\Big)\,ds\\
&+\Big|\frac{\beta\eta^{m-1}+(1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\Big(L_1(s)\|x-y\|+L_2(s)\|\phi x-\phi y\|\\
&+L_3(s)\|\psi x-\psi y\|\Big)\,ds\\
&+\Big|\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}\Big(L_1(s)\|x-y\|+L_2(s)\|\phi x-\phi y\||\\
&+L_3(s)\|\psi x-\psi y\|\Big)\,ds\\
&\le\Big(I^qL_1(t)+\gamma_0I^qL_2(t)+\delta_0I^qL_3(t)\Big)\|x-y\|
\end{aligned}
$$

$$
\begin{aligned}
&+\Big|\frac{\beta\eta^{m-1}+(1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\Big(I^qL_1(1)+\gamma_0I^qL_2(1)+\delta_0I^qL_3(1)\Big)\|x-y\|\\
&+\Big|\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\Big(I^qL_1(\eta)+\gamma_0I^qL_2(\eta)+\delta_0I^qL_3(\eta)\Big)\|x-y\|\\
\le{}& (1+\gamma_0+\delta_0)\Big\{I_L^q+\frac{(|\beta\eta^{m-1}|+|1-\beta|)I^qL(1)+(|\beta|+|\alpha-\beta|)I^qL(\eta)}{|\beta-1+(\alpha-\beta)\eta^{m-1}|}\Big\}\\
&\times\|x-y\|\\
\le{}& \|x-y\|,
\end{aligned}
$$

where we have used the assumption (A_2) in the last inequality. Clearly $\mathcal{G}$ is a contraction. Thus, by the contraction mapping principle, we obtain the conclusion of the theorem. This completes the proof. □

Now, we state Krasnoselskii's fixed point theorem [31] which is needed to prove our next existence result.

Theorem 3.2. *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be operators such that (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.3. *Suppose that $f : [0,1] \times X \times X \times X \to X$ is jointly continuous and maps bounded subsets of $[0,1] \times X \times X \times X$ into relatively compact subsets of X and the assumptions (A_1) and (A_3) hold with*

$$
\Lambda_1=(1+\gamma_0+\delta_0)\Big\{\frac{(|\beta\eta^{m-1}|+|1-\beta|)I^qL(1)+(|\beta|+|\alpha-\beta|)I^qL(\eta)}{|\beta-1+(\alpha-\beta)\eta^{m-1}|}\Big\}<1.
$$

Then there exists at least one solution of the boundary value problem (1.1) on $[0,1]$.

Proof. Let us fix

$$
r\ge\frac{\|\mu\|_{L^1}}{\Gamma(q)}\Big\{1+\frac{(|\beta\eta^{m-1}|+|1-\beta|)+(|\beta|+|\alpha-\beta|)\eta^{q-1})}{|\beta-1+(\alpha-\beta)\eta^{m-1}|}\Big\},
$$

and consider $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$. We define the operators Θ_1 and Θ_2 on B_r by

$$
(\Theta_1x)(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s),(\phi x)(s),(\psi x)(s))\,ds,
$$

$$
\begin{aligned}
(\Theta_2x)(t)={}&\Big(\frac{\beta\eta^{m-1}+(1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s,x(s),(\phi x)(s),(\psi x)(s))\,ds\\
&-\Big(\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}f(s,x(s),(\phi x)(s),(\psi x)(s))\,ds.
\end{aligned}
$$

For $x, y \in B_r$, we find that

$$\|\Theta_1 x + \Theta_2 y\| \;\leq \frac{\|\mu\|_{L^1}}{\Gamma(q)}\Big\{1 + \frac{(|\beta\eta^{m-1}| + |1-\beta|) + (|\beta| + |\alpha - \beta|\eta^{q-1})}{|\beta - 1 + (\alpha-\beta)\eta^{m-1}|}\Big\} \leq r.$$

Thus, $\Theta_1 x + \Theta_2 y \in B_r$. It follows from the assumption (A_1) that Θ_2 is a contraction mapping for $\Lambda_1 < 1$. Continuity of f implies that the operator Θ_1 is continuous. Also, Θ_1 is uniformly bounded on B_r as

$$\|\Theta_1 x\| \leq \frac{\|\mu\|_{L^1}}{\Gamma(q)}.$$

To show that the operator Θ_1 is compact, we use the classical Arzela–Ascoli's Theorem. Let $\mathcal{A}$ be a bounded subset of $\mathcal{C}$. We have to show that $\Phi(\mathcal{A})$ is equicontinuous and for each t, the set $\Phi(\mathcal{A})(t)$ is relatively compact in X. In view of $(A_1), (A_3)$, we define $\sup_{(t,x,\phi x,\psi x)\in[0,1]\times B_r\times B_r\times B_r}\|f(t,x,\phi x,\psi x)\| = f_{\max} < \infty$, and consequently we have

$$\begin{aligned}
&\|(\Theta_1 x)(t_1) - (\Theta_1 x)(t_2)\| \\
&\quad = \Big\|\frac{1}{\Gamma(q)}\int_0^{t_1}\Big((t_2-s)^{q-1} - (t_1-s)^{q-1}\Big)f(s,x(s),\phi x(s),\psi x(s))\,ds \\
&\qquad + \int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s),\phi x(s),\psi x(s))\,ds\Big\| \\
&\quad \leq \frac{f_{\max}}{\Gamma(q+1)}|2(t_2-t_1)^q + t_1^q - t_2^q|,
\end{aligned}$$

which is independent of x. Thus, Θ_1 is equicontinuous. Using the fact that f maps bounded subsets into relatively compact subsets, we have that $\Theta_1(\mathcal{A})(t)$ is relatively compact in X for every t. Therefore, Θ_1 is relatively compact on B_r. Hence, by the Arzela–Ascoli's Theorem, Θ_1 is compact on B_r. Thus all the assumptions of Theorem 3.2 are satisfied and so the boundary value problem (1.1) has at least one solution on $[0, 1]$. □

Example. Consider the following boundary value problem

$$\begin{cases} {}^cD^{\frac{5}{2}}x(t) = \frac{t}{2}\frac{|x|}{1+|x|} + \frac{1}{5}\int_0^t \frac{e^{-(s-t)}}{5}x(s)\,ds + \frac{1}{5}\int_0^t \frac{e^{-(s-t)/2}}{5}x(s)\,ds, \quad t\in[0,1], \\ \qquad x(0) = \frac{1}{2}x(\frac{1}{3}), \quad x'(0) = 0, \quad x(1) = x(\frac{1}{3}). \end{cases} \tag{3.1}$$

Here, $m = 3, q = \frac{5}{2}, \gamma(t,s) = \frac{e^{-(s-t)}}{5}, \delta = \frac{e^{-(s-t)/2}}{5}\beta = \frac{1}{2}, \alpha = 1, \eta = \frac{1}{3}$. With $\gamma_0 = \frac{e-1}{5}, \delta_0 = \frac{2(\sqrt{e}-1)}{5}$, we find that

$$\Lambda = \frac{12(e + 2(\sqrt{e}+1))}{125\sqrt{\pi}} < 1.$$

Thus, by Theorem 3.1, the boundary value problem (3.1) has a unique solution on $[0,1]$.

4. Conclusions

It is worth-mentioning that the nonlocal problem (1.1) is a generalized form of the problem considered in [10] in the sense that it contains a nonlinearity of the form $f(t,x(t),(\phi x)(t),(\psi x)(t))$ in contrast to $f(t,x(t))$ and involves the boundary condition $x(0)=\beta x(\eta)$ instead of $x(0)=0$.

Our main results are based on a generalized variant of a Lipschitz condition, that is, there exist positive functions $L_1(t), L_2(t), L_3(t)$ such that

$$\begin{aligned}&\|f(t,x(t),(\phi x)(t),(\psi x)(t))-f(t,y(t),(\phi y)(t),(\psi y)(t))\|\\&\quad\leq L_1(t)\|x-y\|+L_2(t)\|\phi x-\phi y\|+L_3(t)\|\psi x-\psi y\|,\quad \forall t\in[0,1], x,y\in X.\end{aligned}$$

On the other hand, the results of [10] are proved by requiring a Lipschitz condition. In case $L_1(t), L_2(t)$, and $L_3(t)$ are constant functions, that is $L_1(t)=L_1, L_2(t)=L_2$, and $L_3(t)=L_3$ (L_1, L_2, L_3 are positive real numbers), then the assumption (A_1) reduces to a Lipschitz condition and Λ given by (A_2) takes the form

$$\Lambda=\frac{(L_1+\gamma_0L_2+\delta_0L_3)}{\Gamma(q+1)}\Big\{1+\frac{(|\beta\eta^{m-1}|+|1-\beta|)+(|\beta|+|\alpha-\beta|\eta^q)}{|\beta-1+(\alpha-\beta)\eta^{m-1}|}\Big\}.$$

Furthermore, the solution for an $m-th$ order linear nonlocal boundary value problem [32] can be obtained by fixing $q=m$ in (2.2). Thus, our results are new and generalize some earlier ones.

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