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**Results in Mathematics**

# On the Cauchy Problems of Fractional Evolution Equations with Nonlocal Initial Conditions

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**Abstract.** In this paper, new criterions which allow us to relax the compactness and Lipschitz continuity on nonlocal item, ensuring the existence and uniqueness of mild solutions for the Cauchy problems of fractional evolution equations with nonlocal initial conditions, are established. The results obtained in this paper essentially extend some existing results in this area. Finally, we present two applications to the abstract results.

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**Keywords.** Cauchy problem of fractional evolution equation, mild solution, nonlocal initial condition, compact semigroup.

## 1. Introduction

In this paper we study the existence of mild solutions to the Cauchy problem of fractional evolution equation with nonlocal initial condition

$$\begin{cases} {}^cD_t^\alpha u(t) = Au(t) + F(t, u(t)), & t \in [0, T], \\ u(0) = H(u). \end{cases} \tag{1.1}$$

Here, ${}^cD_t^\alpha, 0 < \alpha < 1$, is the Caputo fractional derivative of order $\alpha$, the operator $(A, \mathcal{D}(A))$ is the infinitesimal generator of a compact semigroup of strongly continuous operators $\{T(t)\}_{t\geq 0}$ on a Banach space $(\mathbb{E}, \|\cdot\|)$, and $H : C([0, T]; \mathbb{E}) \to \mathbb{E}, F : [0, T] \times \mathbb{E} \to \mathbb{E}$ are given functions to be specified later. As can be seen, $H$ constitutes a nonlocal condition.

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The motivation for this study is that evolution equations involve fractional derivatives in time have, in some cases, better effects in applications than traditional evolution equations of integer order in time (see [14,16,23] and the references therein). Recently, there have been many papers concerning this topic (cf., e.g. [1,2,5–7,9,10,13,22,25,26]). It is worth mentioning that this class of equations can provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. What it need to emphasize is that this is the main advantage of fractional models in comparison with integer-order models, in which such effects are in fact neglected.

In particular, stimulated by the observation that nonlocal initial conditions are more realistic than usual ones in treating physical problems (see [3,4,8] for more detailed information about the importance of nonlocal initial conditions in applications), the study of fractional evolution equations with nonlocal initial conditions has been investigated to a large extent. One direction of the study is the existence of mild solutions to this class of equation, among others, we refer to, e.g., [2,9,25,26] and the references therein. Other contributions about the nonlocal problems, please see [3,11,17–20,24,25] and the references therein.

However, much of the previous research on mild solutions was done under the restriction that the nonlocal item is compact or Lipschitz continuous. This condition turns out to be quite restrictive and is not satisfied usually in practical applications. Thus, there naturally arises a question: “whether there exists a mild solution when the nonlocal item loses the compactness and Lipschitz continuity”.

In this paper, among others, we will give an affirmative answer to this question. New criterions, ensuring the existence and uniqueness of mild solutions to (1.1), are established. More precisely, with the help of the compactness of the semigroup generated by $A$, we will first prove an existence result of mild solution to (1.1), which allows us to relax the compactness and Lipschitz continuity on the nonlocal item $H$. In fact, in the proof of the result we only need to suppose the continuity and the growth condition on the nonlocal item and do not impose any other conditions. Then, under a hypothesis on the nonlocal item $H$ which is more general than those in many previous publications, an existence and uniqueness result of mild solutions to (1.1) is also established.

The rest of this paper is organized as follows. In Sect. 2, we present some preliminaries. Section 3 is devoted to main results and their proofs. Finally, in Sect. 4, two examples are given to illustrate the feasibility of our abstract results.

## 2. Preliminaries

Throughout this paper, we denote by $\mathcal{L}(\mathbb{E})$ the Banach space of all linear and bounded operators on $\mathbb{E}$, by $C([0,T];\mathbb{E})$ the Banach space of all continuous functions from $[0,T]$ into $\mathbb{E}$ with the uniform norm topology $\|u\|_\infty = \sup\{\|u(t)\|, t \in [0,T]\}$. The linear operator $(A, \mathcal{D}(A))$ is the infinitesimal generator of a compact and uniformly bounded semigroup of strongly continuous operators $\{T(t)\}_{t\geq 0}$ on $\mathbb{E}$. Let $M$ be a constant such that

$$M = \sup\{\|T(t)\|_{\mathcal{L}(\mathbb{E})},\ t \in [0,\infty)\}.$$

For the sake of convenience, we write

$$\Omega_r = \{u \in C([0,T];\mathbb{E}); \|u(t)\| \leq r,\ \forall t \in [0,T]\},$$

where $r$ is any positive constant.

In the following we recall some definitions of fractional calculus (see, e.g., [15,22,23] for more details).

**Definition 2.1.** The Riemann–Liouville fractional integral operator of order $\alpha > 0$ of function $f$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

**Definition 2.2.** The Caputo fractional derivative of order $\alpha > 0, m-1 < \alpha < m, m \in \mathbb{N}$, is defined as

$${}^cD^\alpha f(t) = I^{m-\alpha} D_t^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} D_s^m f(s)ds,$$

where $D_t^m := \frac{d^m}{dt^m}$ and $f$ is an abstract function with value in $\mathbb{E}$. If $0 < \alpha < 1$, then

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds.$$

Throughout this paper, we let $0 < \alpha < 1$. Define two families $\{\mathcal{S}_\alpha(t)\}_{t\geq 0}$ and $\{\mathcal{P}_\alpha(t)\}_{t\geq 0}$ of linear operators by

$$\mathcal{S}_\alpha(t)x = \int_0^\infty \Psi_\alpha(s) T(st^\alpha)x ds, \quad \mathcal{P}_\alpha(t)x = \int_0^\infty \alpha s \Psi_\alpha(s) T(st^\alpha)x ds, \quad x \in \mathbb{E},$$

where

$$\Psi_\alpha(s) = \frac{1}{\pi\alpha} \sum_{n=1}^\infty (-s)^{n-1} \frac{\Gamma(1+\alpha n)}{n!} \sin(n\pi\alpha), \quad s \in (0,\infty)$$

is the function of Wright type defined on $(0,\infty)$ which satisfies

$$\Psi_\alpha(s)\ge 0,\quad s\in(0,\infty),\qquad \int_0^\infty \Psi_\alpha(s)ds=1,\quad \text{and}$$

$$\int_0^\infty s^\gamma\Psi_\alpha(s)ds=\frac{\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma)},\quad \gamma\in[0,1]. \tag{2.1}$$

Then, from [25, Lemma 3.2, Lemma 3.3] it follows that for all $t\ge 0,\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are linear and bounded operators on $\mathbb{E}$, for every $x\in\mathbb{E},t\to\mathcal{S}_\alpha(t)x,t\to\mathcal{P}_\alpha(t)x$ are continuous functions from $[0,\infty)$ into $\mathbb{E}$, and $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are compact operators for $t>0$.

In fact, we also have the following result.

**Lemma 2.1.** *For $t>0,\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are continuous in the uniform operator topology.*

*Proof.* First note that

$$\|\mathcal{S}_\alpha(t)\|_{\mathcal{L}(\mathbb{E})}\le M,\quad \|\mathcal{P}_\alpha(t)\|_{\mathcal{L}(\mathbb{E})}\le\frac{\alpha M}{\Gamma(1+\alpha)}$$

for $0\le t<\infty$. Given $\epsilon>0$, it follows from (2.1) that there exist $\delta_1,\ \delta_2>0$ with $\delta_1<\delta_2$ such that for $x\in\mathbb{E}$,

$$\frac{1}{2M}\int_0^{\delta_1}\Psi_\alpha(s)\|x\|ds\le\frac{\epsilon}{3}\|x\|,\quad \frac{1}{2M}\int_{\delta_2}^\infty\Psi_\alpha(s)\|x\|ds\le\frac{\epsilon}{3}\|x\|. \tag{2.2}$$

On the other hand, for $t_1,\ t_2>0$, since the compactness of $T(t)$ for $t>0$ implies the continuity in the uniform operator topology, there exists a $\delta_0>0$ such that

$$\|T(st_1^\alpha)-T(st_2^\alpha)\|_{\mathcal{L}(\mathbb{E})}\le\frac{\epsilon}{3},\quad s\in[\delta_1,\delta_2]$$

when $|t_1-t_2|\le\delta_0$. This together with (2.2) yields that if $|t_1-t_2|\le\delta_0$, then for $x\in\mathbb{E}$,

$$\begin{aligned}\|\mathcal{S}_\alpha(t_1)x-\mathcal{S}_\alpha(t_2)x\|&\le\frac{1}{2M}\int_0^{\delta_1}\Psi_\alpha(s)ds+\frac{1}{2M}\int_{\delta_2}^\infty\Psi_\alpha(s)ds\\&\quad+\int_{\delta_1}^{\delta_2}\Psi_\alpha(s)\|(T(st_1^\alpha)x-T(st_2^\alpha)x)\|ds\\&\le\epsilon\|x\|,\end{aligned}$$

which implies that for $t>0,\mathcal{S}_\alpha(t)$ is continuous in the uniform operator topology. A similar argument enables us to give the characterization of continuity on $\mathcal{P}_\alpha(t)$. This completes the proof. □

Based on the work in [25, Lemma 3.1 and Definition 3.1], in this paper we adopt the following definition of mild solution to (1.1).

**Definition 2.3.** By a mild solution of (1.1), we mean a function $u \in C([0,T];\mathbb{E})$ satisfying

$$u(t) = \mathcal{S}_\alpha(t)H(u) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)F(s,u(s))ds, \quad t \in [0,T].$$

## 3. Main Results

Let $m \geq 1$ be fixed. First consider the nonlocal Cauchy problem in the form

$$\begin{cases} {}^cD_t^\alpha u(t) = Au(t) + F(t,u(t)), & t \in [0,T], \\ u(0) = T\left(\frac{1}{m}\right)H(u). \end{cases} \tag{3.1}$$

We can prove the following result.

**Lemma 3.1.** *Let the following hypotheses hold.*

($H_1$) *$F : [0,T]\times\mathbb{E} \to \mathbb{E}$ is a Carathéodory function, and there exists a constant $\beta \in [0,\alpha)$ and a function $f_r(\cdot) \in L^{1/\beta}(0,T;\mathbb{R}^+)$ such that for a.e. $t \in [0,T]$ and all $u \in \mathbb{E}$ satisfying $\|u\| \leq r$,*

$$\|F(t,u)\| \leq f_r(t), \quad \text{and} \quad \liminf_{r\to+\infty}\frac{\|f_r\|_{L^{1/\beta}(0,T)}}{r} = \sigma < \infty.$$

($H_2$) *$H : C([0,T];\mathbb{E}) \to \mathbb{E}$ is continuous, there exists a nondecreasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $u \in \Omega_r$,*

$$\|H(u)\| \leq \Phi(r), \quad \text{and} \quad \liminf_{r\to+\infty}\frac{\Phi(r)}{r} = \mu < \infty.$$

*Then for every $m \geq 1$, the Cauchy problem (3.1) has at least a mild solution $u_m$ provided that*

$$M\mu + \frac{\alpha\sigma M T^{\alpha-\beta}}{\Gamma(1+\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta} < 1. \tag{3.2}$$

*Proof.* Let $m \geq 1$ be fixed. It is clear that we will obtain the result if we show that the mapping $J^\alpha : C([0,T];\mathbb{E}) \to C([0,T];\mathbb{E})$ defined by

$$(J^\alpha u)(t) = \mathcal{S}_\alpha(t)T\left(\frac{1}{m}\right)H(u) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)F(s,u(s))ds$$

has a fixed point.

To accomplish this goal, we first see that $J^\alpha$ is well defined. Note also, from ($H_1$) and Hölder's inequality, that

$$\int_0^t (t-s)^{\alpha-1}f_\rho(s)ds \leq \left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta} T^{\alpha-\beta}\|f_\rho\|_{L^{1/\beta}(0,T)}, \quad t \in [0,T], \tag{3.3}$$

which together with $(H_2)$ yields that for any $u \in \Omega_r$,

$$\begin{aligned}\|(J^\alpha u)(t)\| &\leq \left\|\mathcal{S}_\alpha(t)T\left(\frac{1}{m}\right)\right\|_{\mathcal{L}(\mathbb{E})} \|H(u)\| \\ &\quad + \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|_{\mathcal{L}(\mathbb{E})}\|F(t,x)\|ds \\ &\leq M\Phi(r) + \frac{\alpha M}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1} f_r(s)ds \\ &\leq M\Phi(r) + \frac{\alpha M T^{\alpha-\beta}}{\Gamma(1+\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}\|f_r\|_{L^{1/\beta}(0,T)}.\end{aligned}$$

This implies that there exists a integer $r_0 > 0$ such that $J^\alpha$ maps $\Omega_{r_0}$ into itself. In fact, if this is not the case, then for each $n > 0$, there would exist $u_n \in \Omega_n$ and $t_n \in [0,T]$ such that $\|(J^\alpha u_n)(t_n)\| > n$. Thus, we obtain

$$n < \|(J^\alpha u_n)(t_n)\| \leq M\Phi(n) + \frac{\alpha M T^{\alpha-\beta}}{\Gamma(1+\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}\|f_n\|_{L^{1/\beta}(0,T)}.$$

Dividing on both sides by $n$ and taking the lower limit as $n \to +\infty$, we get

$$1 \leq M\mu + \frac{\alpha\sigma M T^{\alpha-\beta}}{\Gamma(1+\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta},$$

which contradicts (3.2).

Next, we shall prove that $J^\alpha$ is continuous on $\Omega_{r_0}$. Let $\{u_n\}_{n=1}^\infty \subset \Omega_{r_0}$ be a sequence such that $u_n \to u$ as $n \to \infty$ in $C([0,T];\mathbb{E})$. Therefore, it follows from the continuity of $F$ with respect to the second variable that for a.e. $s \in [0,T], F(s,u_n(s)) \to F(s,u(s))$ as $n \to \infty$. Hence, by (3.3) and the continuity of operator $H$, the Lebesgue dominated convergence theorem gives that for each $t \in [0,T]$,

$$\|(J^\alpha u_n)(t) - (J^\alpha u)(t)\| \to 0 \quad \text{as } n \to \infty,$$

which implies that

$$\|J^\alpha u_n - J^\alpha u\|_\infty \to 0 \quad \text{as } n \to \infty.$$

That is to say that $J^\alpha$ is continuous on $\Omega_{r_0}$.

Let us decompose the operator $J^\alpha$ as follows:

$$J^\alpha = J_H^\alpha + J_F^\alpha,$$

where

$$(J_{H}^{\alpha}u)(t) = \mathcal{S}_\alpha(t)T\left(\frac{1}{m}\right)H(u), \quad t \in [0,T],$$

$$(J_{F}^{\alpha}u)(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)F(s,u(s))ds, \quad t \in [0,T].$$

Observe that

$$(J_{H}^{\alpha}u)(t) = \begin{cases} T\left(\frac{1}{m}\right)H(u) & \text{if } t = 0, \\ \mathcal{S}_\alpha(t)T\left(\frac{1}{m}\right)H(u) & \text{if } 0 < t \le T. \end{cases}$$

Therefore, from $(H_2)$ and the compactness of $T(\frac{1}{m})$ $(m \ge 1)$ we deduce that $J_{H}^{\alpha}$ maps $\Omega_{r_0}$ into $C([0,T];\mathbb{E})$ is compact. Also, in view of $(H_1)$, the presentation of operator $\mathcal{P}_\alpha(t)$, and the compactness of operator $T(t)$ for $t > 0$, the same idea with that in [25, Theorem 3.1] (see also [2, Theorem 3.1]) can be used to prove that for each $t \in (0,T]$, the set $\{(J_{F}^{\alpha}u)(t); u \in \Omega_{r_0}\}$ is relatively compact in $\mathbb{E}$ and the set $\{(J_{F}^{\alpha}u)(\cdot); u \in \Omega_{r_0}\}$ is equicontinuous on $[0,T]$. Hence, applying the Arzela–Ascoli theorem we can conclude that $J_{F}^{\alpha}$ is compact on $\Omega_{r_0}$.

Thus, we can make use of Schauder's fixed point theorem to deduce that for each $m \ge 1, \Gamma^\alpha$ has at least a fixed point $u_m \in \Omega_{r_0}$, which means that $u_m$ is a mild solution to (3.1). The proof is completed. □

We now return to the Cauchy problem (1.1). One of main results in this paper is the following.

**Theorem 3.1.** *Assume that the hypotheses in Lemma 3.1 are satisfied. Suppose in addition that*

$(H_3)$ *There is a $\varsigma \in (0,T)$ such that for any $u, w \in C([0,T];\mathbb{E})$ satisfying $u(t) = w(t)$ $(t \in [\varsigma, T]), H(u) = H(w)$.*

*Then the Cauchy problem (1.1) has at least one mild solution.*

*Remark* 3.1. Note that Assumption $(H_3)$ is the case when the values of the solution $u(t)$ for $t$ near zero do not affect $H(u)$. A case in point was presented in [8], where the operator $H$ is given as follows:

$$H(u) = \sum_{i=1}^{p} C_i u(t_i),$$

where $C_i$ $(i = 1, \cdots, p)$ are given constants and $0 < t_1 < \cdots < t_{p-1} < t_p < +\infty$ $(p \in \mathbb{N})$, which is used to describe the diffusion phenomenon of a small amount of gas in a transparent tube.

*Proof of Theorem 3.1.* In view of $(H_1)$ and $(H_2)$, Lemma 3.1 implies that there exists a $r_0 > 0$ such that for every $m \ge 1$, the Cauchy problem (3.1) has at least a mild solution $u_m \in \Omega_{r_0}$, i.e., $u_m$ satisfies the integral equation

$$u_m(t) = \mathcal{S}_\alpha(t)H(u_m) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)F(s,u_m(s))ds, \quad t \in [0,T]. \tag{3.4}$$

We first prove that the set $\{u_m\}_{m=1}^{\infty}$ is precompact in $C([0,T];\mathbb{E})$. Let $\zeta \in (0,\varsigma)$ be fixed, where $\varsigma$ is the constant in $(H_3)$.

From the compactness of $\mathcal{S}_\alpha(t)$ for $t > 0$ and $(H_2)$ it follows that for each $t \in (0,T]$, the set

$$\left\{\mathcal{S}_\alpha(t)T\left(\frac{1}{m}\right)H(u_m); m \geq 1\right\}$$

is relatively compact on $\mathbb{E}$. Also, Lemma 2.1 together with $(H_2)$ gives that for $t_1, t_2 \in [\zeta, T]$ with $t_1 \leq t_2$,

$$\begin{aligned}
&\left\|\mathcal{S}_\alpha(t_2)T\left(\frac{1}{m}\right)H(u_m) - \mathcal{S}_\alpha(t_1)T\left(\frac{1}{m}\right)H(u_m)\right\| \\
&\quad = \left\|(\mathcal{S}_\alpha(t_2) - \mathcal{S}_\alpha(t_1))T\left(\frac{1}{m}\right)H(u_m)\right\| \\
&\quad \to 0, \quad \text{as } t_2 \to t_1,
\end{aligned}$$

uniformly for $m \geq 1$. Therefore, applying Arzela-Ascoli theorem one has that the set

$$\left.\left\{\mathcal{S}_\alpha(t)T\left(\frac{1}{m}\right)H(u_m); m \geq 1\right\}\right|_{[\zeta,T]}$$

is precompact in $C([\zeta,T];\mathbb{E})$.

The same idea with the proof of [25, Theorem 3.1] can be used to prove that the set

$$\left.\left\{\int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)F(s,u_m(s))ds; m \geq 1\right\}\right|_{[0,T]}$$

is precompact in $C([0,T];\mathbb{E})$. Hence, we prove that the set $\{u_m; m \geq 1\}\,|_{[\zeta,T]}$ is precompact in $C([\zeta,T];\mathbb{E})$, which implies that, without loss of generality, we may assume that

$$u_m \to u \quad \text{as } m \to \infty \tag{3.5}$$

in $C([\zeta,T];\mathbb{E})$.

To prove that the set $\{u_m\}_{m=1}^{\infty}$ is precompact in $C([0,T];\mathbb{E})$, it will suffice to show that the set

$$\left.\left\{\mathcal{S}_\alpha(t)T\left(\frac{1}{m}\right)H(u_m); m \geq 1\right\}\right|_{[0,\zeta]}$$

is precompact in $C([0,\zeta];\mathbb{E})$. Write

$$\widetilde{u}_m(t)=\begin{cases}u_m(t) & \text{if } t\in[\varsigma,T],\\ u_m(\varsigma) & \text{if } t\in[0,\varsigma].\end{cases}$$

Then, by (3.5) we may assume that

$$\widetilde{u}_m\to u \quad \text{as } m\to\infty$$

in $C([0,T];\mathbb{E})$. Thus, from the continuity of operator $H$ and the strong continuity of $T(t)$ we get

$$\begin{aligned}&\left\|H(u)-T\left(\frac{1}{m}\right)H(u_m)\right\|\\ &\quad=\left\|H(u)-T\left(\frac{1}{m}\right)H(\widetilde{u}_m)\right\|\\ &\quad\le\left\|H(u)-T\left(\frac{1}{m}\right)H(u)\right\|+\left\|T\left(\frac{1}{m}\right)(H(u)-H(\widetilde{u}_m))\right\|\\ &\quad\le\left\|H(u)-T\left(\frac{1}{m}\right)H(u)\right\|+M\|H(u)-H(\widetilde{u}_m)\|\\ &\quad\to 0 \quad \text{as } m\to\infty,\end{aligned}$$

which implies that the set $\{T\left(\frac{1}{m}\right)H(u_m);m\ge 1\}$ is relatively compact in $\mathbb{E}$. This together with the strong continuity of $\mathcal{S}_\alpha(t)$ concludes that for $t_1,t_2\in[0,\zeta],t_1\le t_2$,

$$\begin{aligned}&\left\|\mathcal{S}_\alpha(t_2)T\left(\frac{1}{m}\right)H(u_m)-\mathcal{S}_\alpha(t_1)T\left(\frac{1}{m}\right)H(u_m)\right\|\\ &\quad\le\|\left(\mathcal{S}_\alpha(t_2)-\mathcal{S}_\alpha(t_1)\right)T\left(\frac{1}{m}\right)H(u_m)\|\\ &\quad\to 0 \quad \text{as } t_2\to t_1,\end{aligned}$$

uniformly for $m\ge 1$. Since for each $t\in[0,T]$, the set

$$\left\{\mathcal{S}_\alpha(t)T\left(\frac{1}{m}\right)H(u_m);m\ge 1\right\}$$

is relatively compact in $\mathbb{E}$, again by Arzela-Ascoli theorem one has that the set

$$\left.\left\{\mathcal{S}_\alpha(t)T\left(\frac{1}{m}\right)H(u_m);m\ge 1\right\}\right|_{[0,\zeta]}$$

is precompact in $C([0,\zeta];\mathbb{E})$. Consequently, the result that the set $\{u_m\}_{m=1}^{\infty}$ is precompact in $C([0,T];\mathbb{E})$ follows.

Now, without loss of generality, we let

$$u_m\to u, \quad \text{as } m\to\infty$$

in $C([0,T];\mathbb{E})$. Letting $m\to\infty$ in (3.4), we find

$$u(t)=\mathcal{S}_\alpha(t)H(u)+\int_0^t(t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)F(s,u(s))ds,\quad t\in[0,T],$$

which implies that $u$ is a mild solution to the Cauchy problem (1.1). This completes the proof. □

We start with the following theorem which assures the existence and uniqueness of mild solution to (1.1). For the sake of convenience, we write $M_0:=\frac{2^{3-2\alpha}\alpha^2M^2\Gamma(2\alpha-1)}{\Gamma^2(1+\alpha)}$.

**Theorem 3.2.** *Let $\frac{1}{2}<\alpha<1$. Assume that*

($H_4$) *$F:[0,T]\times\mathbb{E}\to\mathbb{E}$ is continuous in $t$ on $[0,T]$ and there exists a function $L_F$ such that*

$$\|F(t,u)-F(t,v)\|\le L_F\|u-v\|$$

*for all $u,v\in\mathbb{E}$.*

($H_5$) *$H:C([0,T];\mathbb{E})\to\mathbb{E}$ and there exists a nonnegative function $\Psi$ satisfying*

$$\begin{aligned}&\Psi(\tau_1)\le\Psi(\tau_2),\quad \forall\tau_1,\tau_2\in C([0,T];[0,\infty))\ \text{with}\ \tau_1(t)\le\tau_2(t),\ \text{and}\\ &\Psi(\lambda\tau)\le\lambda\Psi(\tau),\quad \forall\lambda>0,\ \tau\in C([0,T];[0,\infty)),\end{aligned}\tag{3.6}$$

*such that*

$$\|H(u)-H(v)\|\le\Psi(\|u-v\|),$$

*for all $u,v\in C([0,T];\mathbb{E})$.*

($H_6$) $M\Psi\left(\sqrt{2e^{(M_0L_F^2+2)T}}\right)<1$.

*Then the Cauchy problem (1.1) has a unique mild solution.*

*Remark* 3.2. Note that the conditions ($H_5$) on nonlocal item $H$ here are more general than Lipschitz continuity. In fact, if $H:C([0,T];\mathbb{E})\to\mathbb{E}$ is Lipschitz continuous with Lipschitz constant $L$, then we may take

$$\Psi(w)=L\max_{\tau\in[0,T]}w(\tau).$$

*Proof of Theorem* 3.2. We assume that $u_1\in C([0,T];\mathbb{E})$ is fixed and introduce an equivalent norm $\|u\|'_\infty=\sup\{\|e^{-\theta t}u(t)\|,t\in[0,T]\}$ in $C([0,T];\mathbb{E})$, where $\theta$ is a positive constant yet to be determined. Define a operator on $C([0,T];\mathbb{E})$ by

$$(J_0^\alpha u)(t)=\mathcal{S}_\alpha(t)H(u_1)+\int_0^t(t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)F(s,u(s))ds,\quad t\in[0,T].$$

It is clear that $J_0^\alpha u \in (C([0,T];\mathbb{E}), \|\cdot\|'_\infty)$ for all $u \in (C([0,T];\mathbb{E}), \|\cdot\|'_\infty)$. Furthermore, if $u, v \in (C([0,T];\mathbb{E}), \|\cdot\|'_\infty)$, then it follows from $(H_4)$ that

$$\begin{aligned}
&e^{-\theta t}\|(J_0^\alpha u)(t) - (J_0^\alpha v)(t)\| \\
&\quad \le e^{-\theta t}\int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|_{\mathcal{L}(\mathbb{E})}\|F(s,u(s)) - F(s,v(s))\|ds \\
&\quad \le \frac{M\alpha L_F\|u-v\|'_\infty}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-\theta(t-s)}ds.
\end{aligned}$$

Letting $\theta > 0$ be an appropriate constant such that

$$\nu := \sup_{t\in[0,T]}\left\{\frac{M\alpha L_F}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-\theta(t-s)}ds\right\} < 1,$$

it follows that

$$\|J_0^\alpha u - J_0^\alpha v\|'_\infty \le \nu\|u-v\|'_\infty,$$

which proves that $J_0^\alpha$ is a contractive operator on $(C([0,T];\mathbb{E}), \|\cdot\|'_\infty)$. Thus, we infer that $J_0^\alpha$ has a unique fixed point $u_2 \in C([0,T];\mathbb{E})$.

Now, by mathematical induction we can deduce that there exists a sequence $\{u_n\}_{n=1}^\infty \subset C([0,T];\mathbb{E})$ such that

$$u_n(t) = \mathcal{S}_\alpha(t)H(u_{n-1}) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)F(s,u_n(s))ds, \quad t \in [0,T],\ n \ge 2. \tag{3.7}$$

By $(H_4)$, $(H_5)$ and Hölder's inequality we have

$$\begin{aligned}
&\|u_3(t) - u_2(t)\| \\
&\quad \le M\Psi(\|u_2 - u_1\|) + \frac{\alpha M L_F}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\|u_3(s) - u_2(s)\|ds \\
&\quad \le M\Psi(\|u_2 - u_1\|) \\
&\qquad + \frac{\alpha M L_F}{\Gamma(1+\alpha)}\left(\int_0^t (t-s)^{2\alpha-2}e^{2s}ds\right)^{1/2}\left(\int_0^t \|u_3(s) - u_2(s)\|^2 e^{-2s}ds\right)^{1/2}
\end{aligned}$$

for $t \in [0, T]$, which together with $\frac{1}{2} < \alpha < 1$ gives that

$$\begin{aligned} & e^{-2t}\|u_3(t) - u_2(t)\|^2 \\ & \quad \le 2M^2 e^{-2t}\Psi^2(\|u_2(\cdot) - u_1(\cdot)\|) \\ & \qquad + \frac{2\alpha^2 M^2 L_F^2 e^{-2t}}{\Gamma^2(1+\alpha)} \int_0^t (t-s)^{2\alpha-2} e^{2s} ds \cdot \int_0^t \|u_3(s) - u_2(s)\|^2 e^{-2s} ds \\ & \quad = 2M^2 e^{-2t}\Psi^2(\|u_2(\cdot) - u_1(\cdot)\|) + M_0 L_F^2 \int_0^t \|u_3(s) - u_2(s)\|^2 e^{-2s} ds \end{aligned}$$

for $t \in [0, T]$. By Bellman–Gronwall's inequality,

$$\|u_3(t) - u_2(t)\|^2 \le 2M^2 e^{(M_0 L_F^2 + 2)t}\Psi^2(\|u_2(\cdot) - u_1(\cdot)\|), \quad t \in [0, T].$$

That is to say

$$\|u_3(t) - u_2(t)\| \le M\sqrt{2e^{(M_0 L_F^2 + 2)t}}\Psi(\|u_2(\cdot) - u_1(\cdot)\|), \quad t \in [0, T].$$

Applying (3.6) and mathematical induction, we have

$$\begin{aligned} & \|u_n(t) - u_{n-1}(t)\| \\ & \quad \le M\sqrt{2e^{(M_0 L_F^2 + 2)t}} \left( M\Psi\left(\sqrt{2e^{(M_0 L_F^2 + 2)\cdot}}\right)\right)^{n-3} \Psi(\|u_2(\cdot) - u_1(\cdot)\|) \end{aligned}$$

for $t \in [0, T]$ and $n \ge 3$. This together with $(H_6)$ yields that for any $n > m \ge 3$,

$$\begin{aligned} \|u_n(t) - u_m(t)\| & \le \sum_{j=m}^{n-1} \|u_{j+1}(t) - u_j(t)\| \\ & \le M\sqrt{2e^{(M_0 L_F^2 + 2)T}}\Psi(\|u_2(\cdot) - u_1(\cdot)\|) \\ & \quad \times \sum_{j=m}^{n-1} \left( M\Psi\left(\sqrt{2e^{(M_0 L_F^2 + 2)\cdot}}\right)\right)^{j-2} \\ & \to 0 \quad (t \in [0, T]) \end{aligned}$$

as $m \to \infty$, which implies that $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $C([0, T]; \mathbb{E})$. Therefore, there exists a function $u \in C([0, T]; \mathbb{E})$ such that

$$\|u_n - u\|_\infty \to 0,$$

Moreover, from (3.7) we deduce that $u$ is a mild solution to (1.1).

In the sequel, we prove the uniqueness. Let $v$ be also a mild solution to (1.1). Then, for $t \in [0, T]$,

$$\|u(t) - v(t)\| \le M\Psi(\|u(\cdot) - v(\cdot)\|) + \frac{\alpha M L_F}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1}\|u(s) - v(s)\| ds.$$

Again applying Bellman-Gronwall's inequality we obtain that for $t \in [0, T]$, $n \geq 1$,

$$\|u(t) - v(t)\| \leq M\sqrt{2e^{(M_0 L_F^2 + 2)T}} \left( M\Psi \left( \sqrt{2e^{(M_0 L_F^2 + 2)T}} \right) \right)^{n-1} \Psi(\|u(\cdot) - v(\cdot)\|).$$

Letting $n \to \infty$, one has $u(t) \equiv v(t)$ on $[0, T]$, which implies that $u$ is a unique mild solution to (1.1). This completes the proof. □

## 4. Examples

In this section, we present two examples, which do not aim at generality but indicate how our theorems can be applied to concrete problems.

Throughout this section we let $\mathbb{E} = L^2[0, \pi]$ and let the operators $A = \frac{\partial^2}{\partial x^2} : \mathcal{D}(A) \subset \mathbb{E} \mapsto \mathbb{E}$ be defined by

$$\mathcal{D}(A) = \{u \in \mathbb{E}; u, u' \text{ are absolutely continuous}, u'' \in \mathbb{E}, \text{ and } u(0) = u(\pi) = 0\}.$$

Then, $A$ has a discrete spectrum and the eigenvalues are $-n^2, n \in \mathbb{N}$, with the corresponding normalized eigenvectors $y_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$. Moreover, $A$ generates a compact, analytic semigroup $\{T(t)\}_{t \geq 0}$ on $\mathbb{E}$ :

$$T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}(u, y_n)y_n, \quad \|T(t)\|_{\mathcal{L}(\mathbb{E})} \leq e^{-t} \quad \text{for all } t \geq 0.$$

(see [12]). Denote by $E_{\alpha,\beta}$ the generalized Mittag-Leffler special function defined by

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)} \quad \alpha, \beta > 0, t \in \mathbb{R}$$

(cf., e.g., [21]). Therefore, we have that for $u \in \mathbb{E}$,

$$\mathcal{S}_\alpha(t)u = \sum_{n=1}^{\infty} E_\alpha(-n^2 t^\alpha)(u, y_n)y_n, \quad \|\mathcal{S}_\alpha(t)\|_{\mathcal{L}(\mathbb{E})} \leq 1 \quad \text{for all } t \geq 0,$$

$$\mathcal{P}_\alpha(t)u = \sum_{n=1}^{\infty} e_\alpha(-n^2 t^\alpha)(u, y_n)y_n, \quad \|\mathcal{P}_\alpha(t)\|_{\mathcal{L}(\mathbb{E})} \leq \frac{\alpha}{\Gamma(1+\alpha)} \quad \text{for all } t \geq 0,$$

where $E_\alpha(t) := E_{\alpha,1}(t)$ and $e_\alpha(t) := E_{\alpha,\alpha}(t)$.

*Example* 4.1. Consider the fractional partial differential equation with nonlocal initial condition of form

$$\begin{cases} {}^c\partial_t^{\frac{1}{2}} u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = \frac{\sin u(t,x)}{t^{\frac{1}{3}}}, \quad 0 \leq t \leq T,\ 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi) = 0, \quad 0 \leq t \leq T, \\ u(0,x) = u_0(x) + \sum_{i=1}^{p} C_i u^{\frac{1}{3}}(t_i, x), \quad 0 \leq x \leq \pi, \end{cases} \tag{4.1}$$

in $\mathbb{E}$, where $0 < t_1 < \cdots < t_{p-1} < t_p < T$ and $C_i$ $(i = 1, \ldots, p)$ are given constants. Define

$$u(t)x = u(t,x),$$
$$F(t,u(t))(x) = \frac{\sin u(t,x)}{t^{\frac{1}{3}}},$$
$$H(u)(x) = u_0(x) + \sum_{i=1}^{p} C_i u^{\frac{1}{3}}(t_i, x).$$

Then (4.1) can be reformulated as the abstract problem (1.1) and hypotheses $(H_1), (H_2)$ and $(H_3)$ hold, where

$$\frac{1}{3} < \beta < \frac{1}{2}, \quad f_r(t) = \pi^{\frac{1}{2}} t^{-\frac{1}{3}}, \quad \Phi(r) = \|u_0\| + \pi r^{\frac{1}{3}} \sum_{i=1}^{p} |C_i|, \quad \sigma = 0, \quad \mu = 0.$$

Hence, (4.1) has at least one mild solution due to Theorem 3.1.

*Example* 4.2. In $\mathbb{E}$ consider the fractional partial differential equation with nonlocal initial condition in the form

$$\begin{cases} {}^c\partial_t^\alpha u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = e^{-t} \frac{|u(t,x)|}{1+|u(t,x)|}, \quad 0 \le t \le T, \ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, \quad 0 \le t \le T, \\ u(0,x) = \int_{s_1}^{s_2} K(s) u(s,x) ds, \ 0 \le x \le \pi, \end{cases} \tag{4.2}$$

where $\frac{1}{2} < \alpha < 1, s_1, s_2 \in [0,T]$ with $s_1 < s_2$, and $K : [s_1, s_2] \to \mathbb{R}^+$ is a continuous function.

Define

$$u(t)x = u(t,x),$$
$$F(t,u(t))(x) = e^{-t} \frac{|u(t,x)|}{1+|u(t,x)|},$$
$$H(u)(x) = \int_{s_1}^{s_2} K(s) u(s,x) ds,$$

and take

$$\Psi(w) = k_0 (s_2 - s_1)^{\frac{1}{2}} \left( \int_{s_1}^{s_2} w^2(s) ds \right)^{\frac{1}{2}},$$

where $k_0 = \max_{\tau \in [s_1,s_2]} K(\tau)$. Then note that the assumptions $(H_4)$ and $(H_5)$ hold with $L_F = 1$. Furthermore, we obtain that when $k_0$ is small enough such that the assumption $(H_6)$ is satisfied, (4.2) has a unique mild solution due to Theorem 3.2.

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