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Results in Mathematics

Optimal Lehmer Mean Bounds for the Toader Mean

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Abstract. We find the greatest value p and least value q such that the double inequality $L_p(a,b) < T(a,b) < L_q(a,b)$ holds for all $a,b > 0$ with $a \neq b$, and give a new upper bound for the complete elliptic integral of the second kind. Here $T(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}d\theta$ and $L_p(a,b) = (a^{p+1}+b^{p+1})/(a^p+b^p)$ denote the Toader and p-th Lehmer means of two positive numbers a and b, respectively.

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1. Introduction

For $p \in \mathbb{R}$ and $a, b > 0$ the p-th Lehmer mean $L_p(a,b)$ and the power mean $M_p(a,b)$ are defined by

$$L_p(a,b) = \frac{a^{p+1}+b^{p+1}}{a^p+b^p} \tag{1.1}$$

and

$$M_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases}$$

respectively.

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It is well known that $L_p(a,b)$ and $M_p(a,b)$ are continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b>0$ with $a \neq b$. Many means are special case of these means, for example,

$$L_0(a,b) = M_1(a,b) = (a+b)/2 = A(a,b) \quad \text{is the arithmetic mean,}$$
$$L_{-1/2}(a,b) = M_0(a,b) = \sqrt{ab} = G(a,b) \quad \text{is the geometric mean,}$$
$$L_{-1}(a,b) = M_{-1}(a,b) = 2ab/(a+b) = H(a,b) \quad \text{is the harmonic mean.}$$

Recently, the Lehmer mean has been the subject of intensive research. In particular, many remarkable inequalities for the Lehmer mean can be found in the literature [1,2,5,6,8,10,12].

In [9], Toader introduced the Toader mean $T(a,b)$ of two positive numbers a and b as follows:

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta = \begin{cases} 2a\mathcal{E}\left(\sqrt{1-(b/a)^2}\right)/\pi, & a > b, \\ 2b\mathcal{E}\left(\sqrt{1-(a/b)^2}\right)/\pi, & a < b, \\ a, & a = b. \end{cases} \tag{1.2}$$

Here $\mathcal{E}(r) = \int_0^{\pi/2}(1-r^2\sin^2 t)^{1/2}dt$, $r \in [0,1]$, is the complete elliptic integral of the second kind.

The classical arithmetic-geometric mean $AG(a,b)$ of two positive numbers a and b is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$, which are given by

$$a_0 = a, \qquad b_0 = b,$$
$$a_{n+1} = (a_n+b_n)/2 = A(a_n,b_n), \qquad b_{n+1} = \sqrt{a_n b_n} = G(a_n,b_n).$$

The Gauss identity [4] shows that

$$AG(1,r)\mathcal{K}(\sqrt{1-r^2}) = \frac{\pi}{2} \tag{1.3}$$

for $r \in (0,1)$, where $\mathcal{K}(r) = \int_0^{\pi/2}(1-r^2\sin^2 t)^{-1/2}dt$, $r \in [0,1)$, is the complete elliptic integral of the first kind.

Vuorinen [11] conjectured that

$$M_{3/2}(a,b) < T(a,b) \tag{1.4}$$

for all $a,b>0$ with $a \neq b$. This conjecture was proved by Qiu and Shen [7].

In [3], Alzer and Qiu presented a best possible upper power mean bound for the Toader mean as follows:

$$T(a,b) < M_{\log 2/\log(\pi/2)}(a,b) \tag{1.5}$$

for all $a,b>0$ with $a \neq b$.

The main purpose of this paper is to find the greatest value p and least value q such that the double inequality $L_p(a,b) < T(a,b) < L_q(a,b)$ holds for all $a, b > 0$ with $a \neq b$, and give a new upper bound for the complete elliptic integral of the second kind.

2. Lemmas

In order to establish our main result, we need several Lemmas, which we present in this section.

Lemma 2.1. *Let* $f_1(r) = 2(1+r)^2 - (1+\sqrt{r})^4 + 8(1+r)r^{1/2}$ *and* $f_2(r) = (1+\sqrt{r})^2 + 4[(1+r)/2]^{1/2} r^{1/4}$*, then*

$$\frac{f_1(r)}{f_2(r)} < \frac{1+r^{5/4}}{1+r^{1/4}} \tag{2.1}$$

for all $r \in (0,1)$.

Proof. For $r \in (0,1)$, let $t = r^{1/4}$. Then $t \in (0,1)$ and it is easy to see that inequality (2.1) is equivalent to

$$t(1+t)(t^6+t^5+3t^4-6t^3+3t^2+t+1) < 4t(1+t^5)\left(\frac{1+t^4}{2}\right)^{1/2}. \tag{2.2}$$

Let $f(t) = \{4t(1+t^5)[(1+t^4)/2]^{1/2}\}^2 - [t(1+t)(t^6+t^5+3t^4-6t^3+3t^2+t+1)]^2$, then simple computations lead to

$$\begin{aligned} f(t) &= t^2(1+t)^2(1-t)^4 \\ &\quad \times (7t^8+10t^7+15t^6+2t^5-4t^4+2t^3+15t^2+10t+7) > 0 \end{aligned} \tag{2.3}$$

for $t \in (0,1)$. Therefore, inequality (2.2) follows from (2.3). □

Lemma 2.2. (see [[4], p. 58, 3.22]). *Let* $a_0 = 1, b_0 = r \in (0,1), d_0 = 2, a_{n+1} = (a_n+b_n)/2, b_{n+1} = \sqrt{a_n b_n}$ *and* $d_{n+1} = d_n - 2^n({a_n}^2 - {b_n}^2), n = 0, 1, 2, \ldots,$ *then*

$$\lim_{n\to\infty} d_n = \frac{2\mathcal{E}(\sqrt{1-r^2})}{\mathcal{K}(\sqrt{1-r^2})}.$$

Lemma 2.3. *Let the sequences* $\{a_n\}, \{b_n\}$ *and* $\{d_n\}$ *be defined as in Lemma* 2.2*, then*

(1) $d_n < 2^{n+1}a_n(a_n+b_n)$ *for* $n = 0, 1, 2, \ldots$;

(2) *The sequence* $\{d_n/a_n\}$ *is positive for* $n = 0, 1, 2, \ldots,$ *strictly decreasing and*

$$\lim_{n\to\infty} \frac{d_n}{a_n} = \frac{4}{\pi}\mathcal{E}(\sqrt{1-r^2}). \tag{2.4}$$

Proof. (1) Let $c_n = 2^{n+1}a_n(a_n+b_n)$, we use mathematical induction to prove $d_n < c_n$. We clearly see that $d_0 = 2 < 2(1+r) = c_0$ and $d_1 = 1+r^2 < (1+r)(1+\sqrt{r})^2 = c_1$. If we assume that $d_n < c_n$ for $n = 0, 1, 2, \ldots, k$ $(k \geq 1)$ hold, then

$$\begin{aligned} &c_{k+1} - d_{k+1} \\ &\quad = 2^{k+2}a_{k+1}(a_{k+1}+b_{k+1}) - d_k + 2^k({a_k}^2 - {b_k}^2) \\ &\quad = 2^{k+2}a_{k+1}\left(\frac{a_k+b_k}{2} + \sqrt{a_kb_k}\right) - d_k + 2^{k+2}a_{k+1}\left(\frac{a_k-b_k}{2}\right) \\ &\quad = c_k - d_k + 2^{k+2}a_{k+1}\sqrt{a_kb_k} > 0. \end{aligned}$$

(2) We clearly see that

$$d_n - d_{n+1} = 2^n({a_n}^2 - {b_n}^2) > 0. \tag{2.5}$$

Inequality (2.5) implies that the sequence $\{d_n\}$ is strictly decreasing, then from Lemma 2.2 we know that $d_n > 0$. Therefore, $d_n/a_n > 0$.

From Lemma 2.3(1), we have

$$\begin{aligned} &\frac{d_{n+1}}{a_{n+1}} - \frac{d_n}{a_n} \\ &\quad = \frac{a_nd_{n+1} - a_{n+1}d_n}{a_na_{n+1}} \\ &\quad = \frac{2a_n[d_n - 2^n({a_n}^2 - {b_n}^2)] - (a_n+b_n)d_n}{2a_na_{n+1}} \\ &\quad = \frac{(a_n-b_n)[d_n - 2^{n+1}a_n(a_n+b_n)]}{2a_na_{n+1}} < 0. \end{aligned} \tag{2.6}$$

It follows from (2.6) that $\{d_n/a_n\}$ is strictly decreasing. Equation (2.4) follows from (1.3) and Lemma 2.2. □

Lemma 2.4. *Let $f_1(r)$ and $f_2(r)$ be defined as in Lemma 2.1, then*

$$\frac{f_1(r)}{f_2(r)} > \frac{2}{\pi}\mathcal{E}(\sqrt{1-r^2})$$

for all $r \in (0,1)$.

Proof. Let the sequences $\{a_n\}, \{b_n\}$ and $\{d_n\}$ be defined as in Lemma 2.2, then simple computations lead to

$$d_3 = \frac{1}{4}f_1(r) \tag{2.7}$$

and

$$a_3 = \frac{1}{8}f_2(r). \tag{2.8}$$

Therefore, Lemma 2.4 follows from (2.7) and (2.8) together with (2.4) and the monotonicity of the sequence $\{d_n/a_n\}$. □

3. Main Results

Theorem 3.1. *Inequality $L_0(a,b) < T(a,b) < L_{1/4}(a,b)$ holds for all $a,b>0$ with $a \neq b$, and $L_0(a,b)$ and $L_{1/4}(a,b)$ are the best possible lower and upper Lehmer mean bounds for the Toader mean $T(a,b)$.*

Proof. From (1.4), we clearly see that $T(a,b) > M_{3/2}(a,b) > M_1(a,b) = L_0(a,b)$ for all $a,b>0$ with $a \neq b$.

Next we prove that

$$T(a,b) < L_{1/4}(a,b) \tag{3.1}$$

for all $a,b>0$ with $a \neq b$. Since the Toader mean $T(a,b)$ and Lehmer mean $L_p(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a=1$ and $b=r\in(0,1)$. Then from (1.1) and (1.2) we get

$$\begin{aligned} &T(a,b) - L_{1/4}(a,b) \\ &\quad = \frac{2\mathcal{E}(\sqrt{1-r^2})}{\pi} - \frac{r^{5/4}+1}{r^{1/4}+1} \\ &\quad = \left[\frac{2\mathcal{E}(\sqrt{1-r^2})}{\pi} - \frac{f_1(r)}{f_2(r)}\right] + \left[\frac{f_1(r)}{f_2(r)} - \frac{r^{5/4}+1}{r^{1/4}+1}\right], \end{aligned} \tag{3.2}$$

where $f_1(r)$ and $f_2(r)$ are defined as in Lemma 2.1

Therefore, inequality ((3.1)) follows from Lemma 2.1 and ((2.4)) together with ((3.2)).

At last, we prove that $L_0(a,b)$ and $L_{1/4}(a,b)$ are the best possible lower and upper Lehmer mean bounds for the Toader mean $T(a,b)$.

For any $\varepsilon > 0$ and $0<x<1$, from (1.1) and (1.2) we get

$$\begin{aligned} &T(1,1-x) - L_{1/4-\varepsilon}(1,1-x) \\ &\quad = \frac{2}{\pi}\int_0^{\pi/2} [1-(2x-x^2)\sin^2 t]^{1/2} dt - \frac{1+(1-x)^{5/4-\varepsilon}}{1+(1-x)^{1/4-\varepsilon}} \end{aligned} \tag{3.3}$$

and

$$\lim_{x\to 0}\frac{T(1,x)}{L_\varepsilon(1,x)} = \lim_{x\to 0}\left[\frac{2}{\pi}\mathcal{E}(\sqrt{1-x^2})\frac{1+x^\varepsilon}{1+x^{1+\varepsilon}}\right] = \frac{2}{\pi} < 1. \tag{3.4}$$

Let $x \to 0$, making use of the Taylor expansion one has

$$\begin{aligned} &\frac{2}{\pi}\int_0^{\pi/2} [1-(2x-x^2)\sin^2 t]^{1/2} dt - \frac{1+(1-x)^{5/4-\varepsilon}}{1+(1-x)^{1/4-\varepsilon}} \\ &= 1 - \frac{1}{2}x + \frac{1}{16}x^2 + o(x^2) - \left[1 - \frac{1}{2}x + \left(\frac{1}{16} - \frac{\varepsilon}{4}\right)x^2 + o(x^2)\right] \\ &= \frac{\varepsilon}{4}x^2 + o(x^2). \end{aligned} \tag{3.5}$$

TABLE 1. Comparison of $\mathcal{E}(r)$ with $H(r)$ for some $r \in (0,1)$

r	$\mathcal{E}(r)$	$H(r)$
0.05	1.569814118...	1.569814119...
0.1	1.56686194202...	1.56686194203...
0.2	1.554968546...	1.554968547...
0.3	1.534833465...	1.534833495...
0.4	1.505941612...	1.505941951...
0.5	1.467462209...	1.467464652...
0.6	1.418083394...	1.418096972...
0.7	1.355661136...	1.355727424...

For any $\varepsilon > 0$, Eqs. (3.3) and (3.5) imply that there exists $\delta_1 = \delta_1(\varepsilon) \in (0,1)$ such that $T(1, 1-x) > L_{1/4-\varepsilon}(1, 1-x)$ for $x \in (0, \delta_1)$, and Eq. (3.4) implies that there exists $\delta_2 = \delta_2(\varepsilon) \in (0,1)$ such that $T(1,x) < L_\varepsilon(1,x)$ for $x \in (0, \delta_2)$. □

From Theorem 3.1, we get a upper bound for the complete elliptic integral $\mathcal{E}(r)$ of the second kind as follows.

Corollary 3.2. $\mathcal{E}(r) < \pi[(1-r^2)^{5/8}+1]/\{2[(1-r^2)^{1/8}+1]\}$ *for all* $r \in (0,1)$.

Remark 3.3. Computational and numerical experiments show that the upper bound $\pi[(1-r^2)^{5/8}+1]/\{2[(1-r^2)^{1/8}+1]\}$ for $\mathcal{E}(r)$ is very accurate for some $r \in (0,1)$. In fact, if we let $H(r) = \pi[(1-r^2)^{5/8}+1]/\{2[(1-r^2)^{1/8}+1]\}$, then we have Table 1 via elementary computation.

Remark 3.4. We clearly see that the best possible lower power bound $M_{3/2}(a,b)$ in (1.4) is better than the lower Lehmer mean bound $L_0(a,b) = M_1(a,b)$ in Theorem 3.1. However, we find that the best possible upper power mean bound $M_{\log 2/\log(\pi/2)}(a,b)$ in (1.5) and the best possible upper Lehmer mean bound $L_{1/4}(a,b)$ in Theorem 3.1 are not comparable. In fact, from (1.1) and (1.2) we get

$$\lim_{x\to+\infty} \frac{L_{1/4}(1,x)}{M_{\log 2/\log(\pi/2)}(1,x)} = 2^{\log(\pi/2)/\log 2} = \frac{\pi}{2} > 1$$

and

$$\begin{aligned} &M_{\log 2/\log(\pi/2)}(1,1+x) - L_{1/4}(1,1+x) \\ &\quad = 1 + \frac{1}{2}x + \frac{1}{8}\left[\frac{\log 2}{\log(\pi/2)} - 1\right]x^2 + o(x^2) - \left[1 + \frac{1}{2}x + \frac{1}{16}x^2 + o(x^2)\right] \\ &\quad = \frac{1}{16}\left[\frac{2\log 2}{\log(\pi/2)} - 3\right]x^2 + o(x^2) \\ &\quad = 0.00436\ldots \times x^2 + o(x^2) \quad (x \to 0). \end{aligned}$$

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