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Results in Mathematics

Integral Representations for Harmonic Functions of Infinite Order in a Cone

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Abstract. A harmonic function of infinite order defined in an n-dimensional cone and continuous in the closure can be represented in terms of the modified Poisson integral and an infinite sum of harmonic polynomials vanishing on the boundary.

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1. Introduction and Results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n (n \geq 2)$ the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, x_n)$, $X = (x_1, x_2, \dots, x_{n-1})$. The Euclidean distance of two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set $\mathbf{S}$ in $\mathbf{R}^n$ are denoted by $\partial\mathbf{S}$ and $\overline{\mathbf{S}}$, respectively.

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(X, x_n) = (x_1, x_2, \dots, x_{n-1}, x_n)$ by $x_n = r \cos\theta_1$.

For $P \in \mathbf{R}^n$ and $r > 0$, let $B(P, r)$ denote the open ball with center at P and radius r in $\mathbf{R}^n$. $S_r = \partial B(O, r)$. The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. For simplicity,

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a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set Ω, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Lambda \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set $\{(r, \Theta) \in \mathbf{R}^n; r \in \Lambda, (1, \Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Lambda \times \Omega$. In particular, the half space $\mathbf{R}_+ \times \mathbf{S}_+^{n-1} = \{(X, x_n) \in \mathbf{R}^n; x_n > 0\}$ will be denoted by $\mathbf{T}_n$.

By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1}$. We call it a cone. We denote the sets $I \times \Omega$ and $I \times \partial\Omega$ with an interval on $\mathbf{R}$ by $C_n(\Omega; I)$ and $S_n(\Omega; I)$. By $S_n(\Omega; r)$ we denote $C_n(\Omega) \cap S_r$. By $S_n(\Omega)$ we denote $S_n(\Omega; (0, +\infty))$ which is $\partial C_n(\Omega) - \{O\}$.

Furthermore, we denote by $d\sigma_Q$ (resp. dS_r) the $(n-1)$-dimensional volume elements induced by the Euclidean metric on $\partial C_n(\Omega)$ (resp. S_r) and by dw the elements of the Euclidean volume in $\mathbf{R}^n$.

Let $\Omega \subset \mathbf{S}^{n-1}$, Δ be the Laplace operator in $\mathbf{R}^n$ and Δ^* be a Laplace–Beltrami (spherical part of the Laplace) on the unit sphere. It is known (see, e.g. [9, p. 41]) that

$$\begin{aligned} \Delta^*\varphi(\Theta) + \lambda\varphi(\Theta) &= 0 \quad \text{in } \Omega, \\ \varphi(\Theta) &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

has the non-decreasing sequence of positive eigenvalues of (1.1) in the domain Ω, which is denoted by λ_i $(i = 1, 2, 3, \ldots)$. In this expression we write λ_i the same number of times as the dimension of the corresponding eigenspace. When the normalized eigenfunction corresponding λ_i is denoted by $\varphi_i(\Theta)$, the set of sequential eigenfunctions corresponding to the same value of λ_i in the sequence $\varphi_i(\Theta)$ $(i = 1, 2, 3, \ldots)$ makes an orthonormal basis for the eigenspace of the eigenvalue λ_i. Hence for each $\Omega \subset \mathbf{S}^{n-1}$ there is a sequence $\{k_j\}$ of positive integers such that $k_1 = 1$, $\lambda_{k_j} < \lambda_{k_{j+1}}$, $\lambda_{k_j} = \lambda_{k_j+1} = \lambda_{k_j+2} = \cdots = \lambda_{k_{j+1}-1}$ and $\{\varphi_{k_j}, \varphi_{k_j+1}, \ldots, \varphi_{k_{j+1}-1}\}$ is an orthonormal basis for the eigenspace of the eigenvalue λ_{k_j} $(j = 1, 2, 3, \ldots)$.

This paper is essentially based on some results in H. Yoshida and I. Miyamoto (see [11,12]). Hence, in the subsequent consideration, we make the same assumption on Ω as in it: if $n \geq 3$, then Ω is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (see e.g. [5, pp. 88–89] for the definition of $C^{2,\alpha}$-domain), $\varphi_j \in C^2(\overline{\Omega})$ $(j = 1, 2, 3, \ldots)$ and $\frac{\partial\varphi_1}{\partial n} > 0$ on $\partial\Omega$ (here and below, $\frac{\partial}{\partial n}$ denotes differentiation along the interior normal).

For the sequence $\{k_j\}$ mentioned above, by I_{k_l} we denote the set of all positive integers less than k_l $(l = 1, 2, 3, \ldots)$. In spite of the fact $I_{k_1} = \emptyset$, the summation over I_{k_1} of a function $S(k)$ of a variable k will be used by promising $\sum_{k \in I_{k_1}} S(k) = 0$.

We note that each function

$$r^{\aleph_i^{\pm}}\varphi_i(\Theta) \quad (i = 1, 2, 3, \ldots)$$

is harmonic in $C_n(\Omega)$, belongs to the class $C^2(C_n(\Omega)\backslash\{O\})$ and vanishes on $S_n(\Omega)$, where

$$2\aleph_i^{\pm} = -n + 2 \pm \sqrt{(n-2)^2 + 4\lambda_i} \quad (i = 1, 2, 3, \ldots).$$

In the sequel, for the sake of brevity, we shall write φ instead of φ_1, $\aleph^{\pm}$ instead of $\aleph_1^{\pm}$ and χ instead of $\aleph_1^+ - \aleph_1^-$. We use the standard notations $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$ and $[a]$ is the integer part of a, where a is a positive real number.

Let $G_\Omega(P, Q)$ ($P = (r, \Theta), Q = (t, \Phi) \in C_n(\Omega)$) be the Green function of $C_n(\Omega)$. Then the ordinary Poisson kernel relative to $C_n(\Omega)$ is defined by

$$P_\Omega(P, Q) = \frac{1}{c_n} \frac{\partial}{\partial n_Q} G_\Omega(P, Q),$$

where

$$c_n = \begin{cases} 2\pi, & \text{if } n = 2 \\ (n-2)w_n, & \text{if } n \geq 3 \end{cases}$$

$Q \in S_n(\Omega)$, w_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$ and $\frac{\partial}{\partial n_Q}$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$.

Remark 1.1. Let $\Omega = \mathbf{S}_+^{n-1}$. Then

$$G_{\mathbf{S}_+^{n-1}}(P, Q) = \begin{cases} \log|P - Q^*| - \log|P - Q|, & n = 2 \\ |P - Q|^{2-n} - |P - Q^*|^{2-n}, & n \geq 3 \end{cases}$$

where $Q^* = (Y, -y_n)$, that is, Q^* is the mirror image of $Q = (Y, y_n)$ with respect to ∂T_n. Hence, for the two points $P = (X, x_n) \in T_n$ and $Q = (Y, y_n) \in \partial T_n$, we have

$$c_n P_{\mathbf{S}_+^{n-1}}(P, Q) = \frac{\partial}{\partial n_Q} G_{\mathbf{S}_+^{n-1}}(P, Q) = \begin{cases} 2|P - Q|^{-2} x_n, & n = 2 \\ 2(n-2)|P - Q|^{-n} x_n, & n \geq 3. \end{cases}$$

Let $F(\Theta)$ be a function on Ω. The integral

$$\int_\Omega F(\Theta)\varphi_i(\Theta)dS_1,$$

is denoted by $N_i(F)$, when it exists.

For a non-negative integer m and two points $P = (r, \Theta) \in C_n(\Omega)$, $Q = (t, \Phi) \in S_n(\Omega)$, we put

$$\widetilde{K}_{\Omega,m}(P, Q) = \begin{cases} 0, & \text{if } 0 < t < 1 \\ K_{\Omega,m}(P, Q), & \text{if } 1 \leq t < \infty \end{cases}$$

where

$$K_{\Omega,m}(P, Q) = \sum_{i \in I_{k_{m+1}}} 2^{\aleph_i^+ + n - 1} N_i(P_\Omega((1, \Theta), (2, \Phi))) r^{\aleph_i^+} t^{-\aleph_i^+ - n + 1} \varphi_i(\Theta).$$

To obtain the modified Poisson integral representation in a cone, as in [11], we use the following modified kernel function defined by

$$P_{\Omega,m}(P,Q) = P_{\Omega}(P,Q) - \widetilde{K}_{\Omega,m}(P,Q),$$

where $P \in C_n(\Omega)$ and $Q \in S_n(\Omega)$.

Remark 1.2. Suppose $\Omega = \mathbf{S}_+^{n-1}$, $P = (r,\Theta) = (X, x_n) \in T_n$ and $Q = (t,\Phi) = (Y,0) \in \partial T_n$ satisfying $r < t$. Then we have $\aleph_{k_i}^+ = i$ $(i = 1,2,3,\ldots)$ and

$$P_{\mathbf{S}_+^{n-1},m}(P,Q) = \begin{cases} P_{\mathbf{S}_+^{n-1}}(P,Q), & \text{if } 0 < t < 1 \\ P_{\mathbf{S}_+^{n-1}}(P,Q) - \frac{2}{w_n}\sum_{i=0}^{m-1} x_n \frac{r^i}{t^{n+i}} C_i^{\frac{n}{2}}(\cos\eta), & \text{if } 1 \le t < \infty \end{cases} \tag{1.2}$$

where $C_i^{\frac{n}{2}}(\cdot)$ is the Gegenbauer polynomial of degree i and η is the angle between $M = (X,0)$ and $N = (Y,0)$ defined by

$$\cos\eta = \frac{(M,N)}{|M||N|}$$

(see [11, Remarks 1, 2 and 3]).

Write

$$U_{\Omega,u}(P) = \int_{S_n(\Omega)} P_{\Omega}(P,Q)u(Q)d\sigma_Q$$

and

$$U_{\Omega,m,u}(P) = \int_{S_n(\Omega)} P_{\Omega,m}(P,Q)u(Q)d\sigma_Q,$$

where $u(Q)$ is a continuous function on $\partial C_n(\Omega)$.

Firstly let us recall the Dirichlet problem in T_n

$$\begin{cases} \Delta u(P) = 0, & \text{for } P \in T_n \\ u(P) = f(P), & \text{for } a.e. \quad P \in \partial T_n. \end{cases} \tag{1.3}$$

If the integral $\int_{\partial T_n} |f(Q)|(1+|Q|)^{-n} d\sigma_Q$ converges, the solutions of the problem (1.3) can be written as (absolutely convergent) Poisson integral

$$\int_{\partial T_n} P_{\mathbf{S}_+^{n-1}}(P,Q)f(Q)d\sigma_Q. \tag{1.4}$$

If the integral (1.4) diverges, a solution to the problem (1.3) can be given as some regularization of this integral. In particular, Finkelstein and Sheinberg (see [3]) have constructed a solution to the problem (1.3) with an arbitrary continuous function f. This solution is the integral with a modified Poisson kernel derived by subtracting of some special harmonic polynomials from $P_{\mathbf{S}_+^{n-1}}(P,Q)$. This method, ascending to the Weierstrass' theorem about canonical representations of entire functions, has been used by several authors (see e.g. [3,4,10]). Using this modified Poisson kernel, Deng (see [2]) studied the integral representations of harmonic functions of finite order (see e.g. [6, p. 141] for the definition of the order) in a half space.

Next we define three classes of functions as follows.

For real numbers $\alpha \geq 0$, we denote $\mathcal{A}_\alpha$ the class of all measurable functions $f(t,\Phi)$ $(Q=(Y,y_n)\in T_n)$ satisfying the following inequality

$$\int_{T_n} \frac{y_n|f(Y,y_n)|}{1+t^{n+\alpha+2}}dQ<\infty$$

and the class $\mathcal{B}_\alpha$, consists of all measurable functions $g(t,\Phi)$ $(Q=(t,\Phi)=(Y,0)\in\partial T_n)$ satisfying

$$\int_{\partial T_n} \frac{|g(Y,0)|}{1+t^{n+\alpha}}dY<\infty.$$

We use $\mathcal{C}_\alpha$ to denote the class of all continuous functions $u(t,\Phi)$ $((t,\Phi)\in \overline{T_n})$ harmonic in T_n with $u^+(t,\Phi)\in\mathcal{A}_\alpha((t,\Phi)\in T_n)$ and $u^+(t,\Phi)\in\mathcal{B}_\alpha((t,\Phi)\in\partial T_n)$.

For $u\in\mathcal{C}_\alpha$, Deng (see [2]) obtained the following.

Theorem 1.3. *If $u\in\mathcal{C}_\alpha$, m is an integer such that $m<\alpha\leq m+1$ and $P_{\mathbf{S}_+^{n-1},m}$ is defined by (1.2), then the following properties hold*:

(I) *If $\alpha=0$, then the integral*

$$\int_{\partial T_n} P_{\mathbf{S}_+^{n-1}}(P,Q)u(Q)d\sigma_Q$$

is absolutely convergent, it represents a harmonic function $U_{\mathbf{S}_+^{n-1},u}(P)$ in T_n and can be continuously extended to $\overline{T_n}$ such that $u(P)=U_{\mathbf{S}_+^{n-1},u}(P)$ for $P=(r,\Theta)=(X,0)\in\partial T_n$ and there exists a constant b such that $u(P)=bx_n+U_{\mathbf{S}_+^{n-1},u}(P)$ for $P=(r,\Theta)=(X,x_n)\in T_n$.

(II) *If $\alpha>0$, then the integral*

$$\int_{\partial T_n} P_{\mathbf{S}_+^{n-1},m}(P,Q)u(Q)d\sigma_Q$$

is absolutely convergent, it represents a harmonic function $U_{\mathbf{S}_+^{n-1},m,u}(P)$ in T_n and can be continuously extended to $\overline{T_n}$ such that $u(P)=U_{\mathbf{S}_+^{n-1},m,u}(P)$ for $P=(r,\Theta)=(X,0)\in\partial T_n$,

$$\lim_{R\to\infty} R^{-\alpha-1}\sup\{|x_n^{n-1}U_{\mathbf{S}_+^{n-1},m,u}(RP)|:P=(1,\Theta)=(X,x_n)\in T_n\}=0$$

and there exists a harmonic polynomial $Q_{\mathbf{S}_+^{n-1},m}(P)$ of degree not greater than m which vanishes on the boundary ∂T_n such that $u(P)=U_{\mathbf{S}_+^{n-1},m,u}(P)+Q_{\mathbf{S}_+^{n-1},m}(P)$ for $P=(r,\Theta)=(X,x_n)\in T_n$.

Motivated by G. T. Deng's result, a natural question to ask is if we can also obtain the integral representations for harmonic functions of infinite order in $C_n(\Omega)$. In this paper, we give an affirmative answer to this question.

To do this, we define the function $\rho(R)$ under consideration. Hereafter, the function $\rho(R)$ (≥ 1) is always supposed to be nondecreasing and continuously differentiable on the interval $[0, +\infty)$. We assume further that

$$\epsilon_0 = \limsup_{R\to\infty} \frac{(\aleph^+_{k_{[\rho(R)]+1}})' R \ln R}{\aleph^+_{k_{[\rho(R)]+1}}} < 1. \tag{1.5}$$

For positive real numbers β, we denote $\mathcal{A}_{\Omega,\beta,\rho}$ the class of all measurable functions $f(t,\Phi)$ $(Q = (t,\Phi) \in C_n(\Omega))$ satisfying the following inequality

$$\int\limits_{C_n(\Omega)} \frac{|f(t,\Phi)|\varphi}{1 + t^{n+\aleph^+_{k_{[\rho(t)]+1}}+\beta-1}} dw < \infty \tag{1.6}$$

and the class $\mathcal{B}_{\Omega,\beta,\rho}$, consists of all measurable functions $g(t,\Phi)$ $(Q = (t,\Phi) \in S_n(\Omega))$ satisfying

$$\int\limits_{S_n(\Omega)} \frac{|g(t,\Phi)|}{1 + t^{n+\aleph^+_{k_{[\rho(t)]+1}}+\beta-3}} \frac{\partial\varphi}{\partial n} d\sigma_Q < \infty. \tag{1.7}$$

Similarly, we will also consider the class of all continuous functions $u(t,\Phi)$ $((t,\Phi) \in \overline{C_n(\Omega)})$ harmonic in $C_n(\Omega)$ with $u^+(t,\Phi) \in \mathcal{A}_{\Omega,\beta,\rho}$ $((t,\Phi) \in C_n(\Omega))$ and $u^+(t,\Phi) \in \mathcal{B}_{\Omega,\beta,\rho}$ $((t,\Phi) \in S_n(\Omega))$ is denoted by $\mathcal{C}_{\Omega,\beta,\rho}$.

Now we state our results.

Theorem 1.4. *If $u \in \mathcal{C}_{\Omega,\beta,\rho}$, then $u \in \mathcal{B}_{\Omega,\beta,\rho}$.*

Theorem 1.5. *If $u \in \mathcal{C}_{\Omega,\beta,\rho}$, then the following properties hold*:

(I) *$U_{\Omega,[\rho(t)],u}(P)$ is a harmonic function on $C_n(\Omega)$ and can be continuously extended to $\overline{C_n(\Omega)}$ such that*

$$U_{\Omega,[\rho(t)],u}(P) = u(P)$$

for $P = (r,\Theta) \in S_n(\Omega)$.

(II) *There exists a harmonic polynomial $h(P) = \sum_{i=1}^{\infty} A_i r^{\aleph_i^+} \varphi_i(\Theta)$ vanishing continuously on $\partial C_n(\Omega)$ such that*

$$u(P) = U_{\Omega,[\rho(t)],u}(P) + h(P)$$

for $P = (r,\Theta) \in C_n(\Omega)$, where A_i $(i = 1, 2, 3, \ldots)$ is a constant.

If we put $\rho(t) \equiv m$ and $\beta = 1$, as an application of Theorems 1.4 and 1.5, we easily get the following Corollary 1.6.

Corollary 1.6. *If $u \in \mathcal{C}_{\Omega,1,m}$, then the following properties hold*:

(I) $u \in \mathcal{B}_{\Omega,1,m}$

(II) *The integral*

$$\int_{S_n(\Omega)} P_{\Omega,m,u}(P,Q)u(Q)d\sigma_Q,$$

is absolutely convergent, it represents a harmonic function $U_{\Omega,m,u}(P)$ *in* $C_n(\Omega)$ *and can be continuously extended to* $\overline{C_n(\Omega)}$ *such that* $U_{\Omega,m,u}(P) = u(P)$ *for* $P = (r,\Theta) \in S_n(\Omega)$.

(III) *There exists a harmonic polynomial* $h(P) = \sum_{i=1}^{k_{m+1}-1} B_i r^{\aleph_i^+}\varphi_i(\Theta)$ *vanishing continuously on* $\partial C_n(\Omega)$ *such that* $u(P) = U_{\Omega,m,u}(P) + h(P)$ *for* $P = (r,\Theta) \in C_n(\Omega)$, *where* B_i $(i = 1,2,3,\ldots,k_{m+1}-1)$ *is a constant.*

Remark 1.7. (I) and (II) in Corollary 1.6 follow readily from Theorems 1.4 and 1.5(I) respectively. By following the method of Yoshida and Miyamoto (see [11, Theorem 3]), we can show that Corollary 1.6(III) holds. So we omit the details of the proof here.

2. Lemmas

The following Lemma generalizes the Carleman's formula (referring to the holomorphic functions in the half space) (see [1]) to the harmonic functions in a cone, which is due to A. Yu. Rashkovskii and L. I. Ronkin (see [7], [8, p. 224]).

Lemma 2.1. *If* $u(t,\Phi)$ *is a harmonic function on a domain containing* $C_n(\Omega;(1,R))$, *then*

$$m_+(R) + \int_{S_n(\Omega;(1,R))} u^+\left(\frac{1}{t^{-\aleph^-}} - \frac{t^{\aleph^+}}{R^{\chi}}\right)\frac{\partial\varphi}{\partial n}d\sigma_Q + d_1 + \frac{d_2}{R^{\chi}}$$
$$= m_-(R) + \int_{S_n(\Omega;(1,R))} u^-\left(\frac{1}{t^{-\aleph^-}} - \frac{t^{\aleph^+}}{R^{\chi}}\right)\frac{\partial\varphi}{\partial n}d\sigma_Q,$$

where

$$m_\pm(R) = \chi\int_{S_n(\Omega;R)} \frac{u^\pm\varphi}{R^{1-\aleph^-}}dS_R,$$

$$d_1 = \int_{S_n(\Omega;1)} \aleph^- u\varphi - \varphi\frac{\partial u}{\partial n}dS_1 \quad \text{and} \quad d_2 = \int_{S_n(\Omega;1)} \varphi\frac{\partial u}{\partial n} - \aleph^+ u\varphi dS_1.$$

Lemma 2.2 ([11, Lemma 3]). *For a non-negative integer* m, *we have*

$$|P_\Omega(P,Q) - K_{\Omega,m}(P,Q)| \le M_1(2r)^{\aleph^+_{k_{m+1}}} t^{-\aleph^+_{k_{m+1}}-n+1}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ *and any* $Q = (t,\Phi) \in S_n(\Omega)$ *satisfying* $0 < \frac{r}{t} < \frac{1}{2}$, *where* M_1 *is a constant independent of* P, Q *and* m.

Lemma 2.3 ([11, Lemma 5]). *If u is a locally integrable and upper semi-continuous function on $\partial C_n(\Omega)$. For any fixed $P \in C_n(\Omega)$, $V(P,Q)$ $(Q \in \partial C_n(\Omega))$ is a locally integrable function on $\partial C_n(\Omega)$. Put*

$$W(P,Q) = P_\Omega(P,Q) - V(P,Q) \quad (P \in C_n(\Omega), Q \in \partial C_n(\Omega)).$$

Suppose that the following conditions (I) *and* (II) *are satisfied:*

(I) *For any $Q' \in \partial C_n(\Omega)$ and any $\epsilon_1 > 0$, there exist a neighborhood $B(Q')$ of Q' in $\mathbf{R}^n$ and a number R $(0 < R < \infty)$ such that*

$$\int_{S_n(\Omega;[R,\infty))} |W(P,Q)||u(Q)|d\sigma_Q < \epsilon_1 \tag{2.1}$$

for any $P = (r,\Theta) \in C_n(\Omega) \cap B(Q')$.

(II) *For any $Q' \in \partial C_n(\Omega)$ and any number R $(0 < R < \infty)$,*

$$\limsup_{P \to Q', P \in C_n(\Omega)} \int_{S_n(\Omega;(0,R))} |V(P,Q)||u(Q)|d\sigma_Q = 0. \tag{2.2}$$

Then

$$\limsup_{P \to Q', P \in C_n(\Omega)} \int_{S_n(\Omega)} W(P,Q)u(Q)d\sigma_Q \leq u(Q')$$

for any $Q' \in \partial C_n(\Omega)$.

Lemma 2.4 ([12, Theorem 3.1]). *If $h(r,\Theta)$ is a harmonic function in $C_n(\Omega)$ vanishing continuously on $\partial C_n(\Omega)$, then*

$$h(r,\Theta) = \sum_{i=1}^{\infty} D_i r^{\aleph_i^+} \varphi_i(\Theta),$$

where D_i $(i = 1,2,3,\ldots)$ is a constant satisfying $D_i r^{\aleph_i^+} = N_i(h(r,\Theta))$ for every r $(0 < r < \infty)$.

3. Proof of Theorem 1.4

For any ϵ $(0 < \epsilon < 1 - \epsilon_0)$, there exists a sufficiently large positive number R_ϵ such that $R > R_\epsilon$, by (1.5) we have

$$\aleph^+_{k_{[\rho(R)]+1}} < \aleph^+_{k_{[\rho(e)]+1}} (\ln R)^{\epsilon_0+\epsilon}. \tag{3.1}$$

Since $u \in \mathcal{C}_{\Omega,\beta,\rho}$, we obtain by (1.6)

$$\begin{aligned}\frac{1}{\chi}\int_1^\infty \frac{m_+(R)}{R^{\aleph^+_{k_{[\rho(R)]+1}}-\aleph^++\beta}}dR &= \int_{C_n(\Omega;(1,\infty))} \frac{u^+\varphi}{t^{n+\aleph^+_{k_{[\rho(t)]+1}}+\beta-1}}dw \\ &\le 2\int_{C_n(\Omega)} \frac{u^+\varphi}{1+t^{n+\aleph^+_{k_{[\rho(t)]+1}}+\beta-1}}dw \\ &< \infty. \end{aligned} \tag{3.2}$$

From (1.7), we conclude that

$$\begin{aligned}&\int_1^\infty \frac{1}{R^{\aleph^+_{k_{[\rho(R)]+1}}-\aleph^++\beta}} \int_{S_n(\Omega;(1,R))} u^+\left(\frac{1}{t^{-\aleph^-}}-\frac{t^{\aleph^+}}{R^\chi}\right)\frac{\partial\varphi}{\partial n}d\sigma_Q dR \\ &= \int_{S_n(\Omega;(1,\infty))} u^+t^{\aleph^+}\int_t^\infty \frac{1}{R^{\aleph^+_{k_{[\rho(R)]+1}}-\aleph^++\beta}}\left(\frac{1}{t^\chi}-\frac{1}{R^\chi}\right)dR\frac{\partial\varphi}{\partial n}d\sigma_Q \\ &\le \frac{\chi}{(\chi+\beta)\beta}\int_{S_n(\Omega;(1,\infty))} \frac{u^+t^{\aleph^+}}{t^{\aleph^+_{k_{[\rho(t)]+1}}-\aleph^-+\beta-1}}\frac{\partial\varphi}{\partial n}d\sigma_Q \\ &\le \frac{2\chi}{(\chi+\beta)\beta}\int_{S_n(\Omega)} \frac{u^+}{1+t^{n+\aleph^+_{k_{[\rho(t)]+1}}+\beta-3}}\frac{\partial\varphi}{\partial n}d\sigma_Q \\ &< \infty. \end{aligned} \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we obtain by Lemma 2.1

$$\begin{aligned}&\int_1^\infty \frac{1}{R^{\aleph^+_{k_{[\rho(R)]+1}}-\aleph^++\frac{\beta}{2}}} \int_{S_n(\Omega;(1,R))} u^-\left(\frac{1}{t^{-\aleph^-}}-\frac{t^{\aleph^+}}{R^\chi}\right)\frac{\partial\varphi}{\partial n}d\sigma_Q dR \\ &\le \int_1^\infty \frac{1}{R^{\aleph^+_{k_{[\rho(R)]+1}}-\aleph^++\frac{\beta}{2}}} \int_{S_n(\Omega;(1,R))} u^+\left(\frac{1}{t^{-\aleph^-}}-\frac{t^{\aleph^+}}{R^\chi}\right)\frac{\partial\varphi}{\partial n}d\sigma_Q dR \\ &\quad+\int_1^\infty \frac{m_+(R)}{R^{\aleph^+_{k_{[\rho(R)]+1}}-\aleph^++\frac{\beta}{2}}}dR+\int_1^\infty \frac{1}{R^{\aleph^+_{k_{[\rho(R)]+1}}-\aleph^++\frac{\beta}{2}}}\left(d_1+\frac{d_2}{R^\chi}\right)dR \\ &< \infty. \end{aligned}$$

Set

$$\mathcal{H}(\beta) = \lim_{t\to\infty}\frac{1}{t^{-\aleph^+_{k_{[\rho(t)]+1}}+\aleph^--\beta+1}}\int_t^\infty \frac{1}{R^{\aleph^+_{k_{[\rho(R)]+1}}-\aleph^++\frac{\beta}{2}}}\left(\frac{1}{t^\chi}-\frac{1}{R^\chi}\right)dR.$$

By the L'Hospital's rule and (3.1), we have

$$\begin{aligned}
\mathcal{H}(\beta) &= \chi \lim_{t\to\infty} \frac{t^{\aleph^+_{k_{[\rho(t)]+1}}-\aleph^+ +\beta-1}}{\aleph^+_{k_{[\rho(t)]+1}}} \frac{\int_t^\infty \frac{1}{R^{\aleph^+_{k_{[\rho(R)]+1}}-\aleph^+ +\frac{\beta}{2}}}dR}{\frac{(\aleph^+_{k_{[\rho(t)]+1}})' t\ln t}{\aleph^+_{k_{[\rho(t)]+1}}} + \frac{\beta-\aleph^- -1}{\aleph^+_{k_{[\rho(t)]+1}}} + 1}\\
&> \frac{\chi}{\epsilon_0+1} \lim_{t\to\infty} \frac{t^{\aleph^+_{k_{[\rho(t)]+1}}-\aleph^+ +\beta-1}}{\aleph^+_{k_{[\rho(t)]+1}}} \int_t^\infty \frac{1}{R^{\aleph^+_{k_{[\rho(R)]+1}}-\aleph^+ +\frac{\beta}{2}}}dR\\
&= \frac{\chi}{\epsilon_0+1} \lim_{t\to\infty} \frac{\frac{t^{\frac{\beta}{2}}}{(\aleph^+_{k_{[\rho(t)]+1}})^2}}{\frac{(\aleph^+_{k_{[\rho(t)]+1}})' t\ln t}{\aleph^+_{k_{[\rho(t)]+1}}}\left(1-\frac{1}{\aleph^+_{k_{[\rho(t)]+1}}\ln t}\right) + \frac{\beta-\aleph^+ -1}{\aleph^+_{k_{[\rho(t)]+1}}} + 1}\\
&> \frac{\chi}{(\epsilon_0+1)^2} \lim_{t\to\infty} \frac{t^{\frac{\beta}{2}}}{(\aleph^+_{k_{[\rho(t)]+1}})^2}\\
&> \frac{\chi\beta}{4(\epsilon_0+1)^2(\epsilon_0+\epsilon)(\aleph^+_{k_{[\rho(e)]+1}})^2} \lim_{t\to\infty} \frac{t^{\frac{\beta}{2}}}{(\ln t)^{2(\epsilon_0+\epsilon)-1}}\\
&= \begin{cases} +\infty, & \text{if } \ 2(\epsilon_0+\epsilon)\le 1\\ \frac{\chi\beta^2 \lim\limits_{t\to\infty} t^{\frac{\beta}{2}}(\ln t)^{2-2(\epsilon_0+\epsilon)}}{8(\epsilon_0+1)^2(\epsilon_0+\epsilon)(\aleph^+_{k_{[\rho(e)]+1}})^2(2(\epsilon_0+\epsilon)-1)}, & \text{if } \ 2(\epsilon_0+\epsilon)> 1 \end{cases}\\
&= +\infty,
\end{aligned}$$

which yields that there exists a positive constant M_2 such that for any $t\ge 1$

$$\int_t^\infty \frac{t^{\aleph^+}}{R^{\aleph^+_{k_{[\rho(R)]+1}}-\aleph^+ +\frac{\beta}{2}}}\left(\frac{1}{t^\chi}-\frac{1}{R^\chi}\right)dR \ge \frac{M_2}{t^{n+\aleph^+_{k_{[\rho(t)]+1}}+\beta-3}}.$$

i.e.,

$$\begin{aligned}
&M_2 \int\limits_{S_n(\Omega;(1,\infty))} \frac{u^-}{t^{n+\aleph^+_{k_{[\rho(t)]+1}}+\beta-3}} \frac{\partial\varphi}{\partial n} d\sigma_Q\\
&\le \int\limits_{S_n(\Omega;(1,\infty))} u^- t^{\aleph^+} \int_t^\infty \frac{1}{R^{\aleph^+_{k_{[\rho(R)]+1}}-\aleph^+ +\frac{\beta}{2}}}\left(\frac{1}{t^\chi}-\frac{1}{R^\chi}\right)dR \frac{\partial\varphi}{\partial n} d\sigma_Q\\
&< \infty.
\end{aligned}$$

Then Theorem 1.4 is proved from $|u| = u^+ + u^-$.

4. Proof of Theorem 1.5

Let l_1 be any positive number such that $l_1 \geq 2\beta$. For any fixed $P = (r, \Theta) \in C_n(\Omega)$, take a number σ satisfying $\sigma > \sigma_r = \max\{[2r] + 1, \vartheta_r\}$, where $\vartheta_r = \exp\left(\frac{l_1}{\beta}\aleph^+_{k_{[\rho(e)]+1}} 2^{1+\epsilon_0+\epsilon} \ln 2r\right)^{\frac{1}{1-\epsilon_0-\epsilon}}$.

By (3.1), we remark that there exists a constant $M(r)$ dependent only on r such that $M(r) \geq (2r)^{\aleph^+_{k_{[\rho(i+1)]+1}}} i^{-\frac{\beta}{l_1}}$ from $\sigma \geq \vartheta_r$.

By Lemma 2.2 and Theorem 1.4, we have

$$\begin{aligned}
&\int\limits_{S_n(\Omega;(\sigma,\infty))} |P_{\Omega,[\rho(t)]}(P,Q)||u(Q)|d\sigma_Q \\
&\leq M_1 \sum_{i=\sigma_r}^{\infty} \int\limits_{S_n(\Omega;[i,i+1))} \frac{(2r)^{\aleph^+_{k_{[\rho(t)]+1}}}}{t^{\aleph^+_{k_{[\rho(t)]+1}}+n-1}} |u(t,\Phi)|d\sigma_Q \\
&\leq M_1 \sum_{i=\sigma_r}^{\infty} \frac{(2r)^{\aleph^+_{k_{[\rho(i+1)]+1}}}}{i^{\frac{\beta}{l_1}}} \int\limits_{S_n(\Omega;[i,i+1))} \frac{|u(t,\Phi)|}{t^{n+\aleph^+_{k_{[\rho(t)]+1}}+\frac{\beta}{l_1}-2}} d\sigma_Q \\
&\leq M_1 M(r) \int\limits_{S_n(\Omega;[\sigma_r,\infty))} \frac{|u(t,\Phi)|}{1+t^{n+\aleph^+_{k_{[\rho(t)]+1}}+\frac{\beta}{l_1}-2}} d\sigma_Q \\
&< \infty.
\end{aligned}$$

Hence $U_{\Omega,[\rho(t)],u}(P)$ is absolutely convergent and finite for any $P \in C_n(\Omega)$. Thus $U_{\Omega,[\rho(t)],u}(P)$ is harmonic on $C_n(\Omega)$.

Next we prove that $\lim_{P\in C_n(\Omega),P\to Q'} U_{\Omega,[\rho(t)],u}(P) = u(Q')$ for any $Q' = (t', \Phi') \in \partial C_n(\Omega)$. Setting $V(P,Q) = \widetilde{K}_{\Omega,[\rho(t)]}(P,Q)$, which is locally integrable on $\partial C_n(\Omega)$ for any fixed $P \in C_n(\Omega)$. Then we apply Lemma 2.3 to $u(Q)$ and $-u(Q)$.

For any $\epsilon > 0$ and a positive number δ, by the above inequality we can choose a number σ, $\sigma > \max\{[2(t'+\delta)]+1, \vartheta_{t'+\delta}\}$ such that (2.1) holds, where $P \in C_n(\Omega) \cap B(Q', \delta)$.

Since $\lim_{\Theta\to\Phi'} \varphi_i(\Theta) = 0$ $(i = 1, 2, 3\ldots)$ as $P = (r,\Theta) \to Q' = (t', \Phi') \in S_n(\Omega)$, $\lim_{P\in C_n(\Omega),P\to Q'} \widetilde{K}_{\Omega,[\rho(t)]}(P,Q) = 0$, where $Q \in S_n(\Omega)$ and $Q' \in \partial C_n(\Omega)$. Then (2.2) holds.

So (I) is proved. Finally (I) and Lemma 2.4 give the conclusion of (II). Then we complete the proof of Theorem 1.5.

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