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Results in Mathematics

A New Formula for the Bernoulli Polynomials

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Abstract. In this note we show that a seemingly new class of Stirling-type pairs can be applied to produce a new representation of the Bernoulli polynomials at positive rational arguments. A class of generalized harmonic numbers is also investigated, and we point out that these give a new relation for the so-called harmonic polynomials.

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1. Introduction

As Hsu and Shiue noted in their paper [6], a way of generalizing the Stirling numbers is to define connection coefficients of linear transformations between generalized factorials. Doing this, we introduce a new class of numbers generalizing the ordinary Stirling-, r-Stirling- and Whitney numbers. Hence we win a new formula for the Bernoulli numbers and polynomials.

In [6], the notation $(z \mid \alpha)_n = z(z-\alpha)\cdots(z-(n-1)\alpha)$ $(n = 1, 2, \dots)$ was used ($(z \mid \alpha)_0 = 1$). With this, the Stirling-type pair $\{S^1(n,k), S^2(n,k)\} = \{S(n,k;\alpha,\beta,r), S(n,k;\beta,\alpha,-r)\}$ is defined by the relations

$$(t \mid \alpha)_n = \sum_{k=0}^{n} S^1(n,k)(t-r \mid \beta)_k,$$

$$(t \mid \beta)_n = \sum_{k=0}^{n} S^2(n,k)(t-r \mid \beta)_k.$$

The parameters α, β, r are real (or possibly complex) numbers with $(\alpha, \beta, r) \neq (0, 0, 0)$.

There are eleven examples of Stirling-type pairs in [6]. We enlarge this list as follows.

Let $mx + r$ be an arithmetic progression (x varies). Then its nth power can be expressed in terms of the falling factorial $x^{\underline{k}} = x(x-1)\cdots(x-k+1)$ $(k = 0, 1, \ldots, n)$:

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k) x^{\underline{k}} \tag{1}$$

with coefficients to be determined immediately. A reverse relation and coefficients should exist:

$$m^n x^{\underline{n}} = \sum_{k=0}^{n} w_{m,r}(n,k)(mx+r)^k. \tag{2}$$

The coefficients $W_{m,r}(n,k)$ and $w_{m,r}(n,k)$ give back the Stirling-, r-Stirling- [3] and Whitney numbers [2], respectively:

$$\begin{aligned} W_{1,0}(n,k) &= \left\{ {n \atop k} \right\} \equiv S^2(n,k), \\ W_{1,r}(n,k) &= \left\{ {n+r \atop k+r} \right\}_r, \\ W_{m,0}(n,k) &= W_m(n,k). \end{aligned}$$

The first kind is the same. It is natural to call these numbers as r-Whitney numbers of the second kind, while $w_{m,r}(n,k)$ may be called r-Whitney number of the first kind.

To see that these really form a Stirling-type pair, in the final section we determine the exponential generating functions of the r-Whitney numbers:

$$\sum_{n=k}^{\infty} W_{m,r}(n,k) \frac{z^n}{n!} = \frac{e^{rz}}{k!} \left(\frac{e^{mz}-1}{m} \right)^k. \tag{3}$$

$$\sum_{n=k}^{\infty} w_{m,r}(n,k) \frac{z^n}{n!} = (1+mz)^{\frac{-r}{m}} \frac{\ln^k(1+mz)}{m^k k!}. \tag{4}$$

Since the exponential generating function of the general Stirling-type number $S(n, k, \alpha, \beta, r)$ is [6]

$$\frac{(1+\alpha z)^{\frac{r}{\alpha}}}{k!} \left(\frac{(1+\alpha z)^{\frac{\beta}{\alpha}} - 1}{\beta} \right)^k = \sum_{n=0}^{\infty} S(n,k;\alpha,\beta,r) \frac{z^n}{n!},$$

we see, that

$$\begin{aligned} W_{m,r}(n,k) &= S(n,k;0+,m,r), \\ w_{m,r}(n,k) &= S(n,k;m,0+,-r). \end{aligned}$$

Applying Hsu and Shiue's results [6], and following the approach of Benoumhani for Whitney numbers [2], we can readily deduce a number of properties of the r-Whitney numbers:

$$\begin{aligned} w_{m,r}(n,k) &= \sum_{i=0}^{n} \binom{n}{i} m^{i-k} r^{n-i} S^2(i,k), \\ W_{m,r}(n,k) &= \frac{1}{m^k k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (mi+r)^n, \\ w_{m,r}(n,k) &= w_{m,r}(n-1,k-1) + (m-nm-r) w_{m,r}(n-1,k), \\ W_{m,r}(n,k) &= W_{m,r}(n-1,k-1) + (km+r) W_{m,r}(n-1,k), \\ \delta_{n,p} &= \sum_{k=0}^{n} w_{m,r}(n,k) W_{m,r}(k,p). \end{aligned}$$

Closing the introduction, we define the Bernoulli polynomials via their exponential generating function:

$$\frac{ze^{zx}}{e^z-1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}.$$

These polynomials are extremely important in a wide range of mathematics and beyond. We call $B_n = B_n(0)$ as Bernoulli numbers.

2. Applications of r-Whitney Numbers

The newly defined r-Whitney numbers and Bernoulli polynomials are closely connnected. We present the later proven formulae

$$\binom{n+1}{l} B_{n-l+1} = \frac{n+1}{m^{n-l+1}} \sum_{k=0}^{n} W_{m,r}(n,k) \frac{w_{m,r}(k+1,l)}{k+1}, \tag{5}$$

$$\binom{n+1}{l} B_{n-l+1}\left(\frac{r}{m}\right) = \frac{n+1}{m^p} \sum_{k=0}^{n} m^k W_{m,r}(n,k) \frac{S^1(k+1,l)}{k+1}. \tag{6}$$

Note that the first gives back the classical formula [7]

$$\binom{n+1}{l} B_{n-l+1} = (n+1) \sum_{k=0}^{n} S^2(n,k) \frac{S^1(k+1,l)}{k+1},$$

if $m = 1$ and $r = 0$. These formulae connect Whitney numbers, r-Stirling numbers and Bernoulli polynomials, too.

An other application of r-Whitney numbers can be given. To see this, we cite the notion of harmonic polynomials $H_n(z)$ firstly defined by Cheon and

Mikkawy [4]. The numbers

$$H(n,r)=\sum_{1\le n_0+n_1+\cdots+n_r\le n}\frac{1}{n_0n_1\cdots n_r}\quad (n\ge 1,\ r\ge 0)$$

are called generalized harmonic numbers. With these, on can define the harmonic polynomials as

$$H_n(z)=\sum_{k=0}^{n}\frac{(-1)^kH(n+1,k)}{k!}z^k.$$

The generating function for them is

$$\frac{-\ln(1-x)}{x(1-x)^{1-z}}=\sum_{n=0}^{\infty}H_n(z)x^n.$$

The r-Whitney numbers of the first kind enable us to give a new representation of the harmonic polynomials:

$$H_{n-1}\left(1-\frac{r}{m}\right)=\frac{1}{m^{n-1}n!}|w_{m,r}(n,1)|.$$

Since

$$H_{n-1}(0)=H_n=\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}=\frac{1}{n!}S^1(n+1,2)\equiv\frac{1}{n!}\begin{bmatrix}n+1\\2\end{bmatrix},$$

$$H_{n-1}(1-r)=H_n^{(r)}=\binom{n+r-1}{r-1}(H_{n+r-1}-H_{r-1})=\frac{1}{n!}\begin{bmatrix}n+r\\r+1\end{bmatrix}_r.$$

(see [3,8]), our formula seems to be a more general one (put $r=m=1$, but in this case the indices n and k are shifted: $|w_{1,1}(n,k)|=\left[\begin{smallmatrix}n+1\\k+1\end{smallmatrix}\right]$). Here $H_n^{(r)}$ denotes the so-called hyperharmonic numbers cf. [1].

It is worth to introduce the notation

$$H_{n,m}^{(r)}=\frac{1}{m^{n-1}n!}|w_{m,r}(n,1)|. \tag{7}$$

(The r-Whitney numbers of the first kind are signed numbers.)

We prove that

$$H_{n,m}^{(r)}=\sum_{k=1}^{n}\binom{\frac{r}{m}-1+n-k}{n-k}\frac{1}{k}. \tag{8}$$

Since a same formula was proven for harmonic polynomials [4]:

$$H_{n-1}(1-m)=\sum_{k=1}^{n}\binom{m-1+n-k}{n-k}\frac{1}{k},$$

and for hyperharmonic numbers [1,5],

$$H_n^{(r)}=\sum_{k=1}^{n}\binom{r-1+n-k}{n-k}\frac{1}{k},$$

it is clear that our numbers $H_{n,m}^{(r)}$ are harmonic polynomials at rational arguments and extensions of hyperharmonic numbers to any rational upper parameter:

$$H_{n-1}\left(1-\frac{r}{m}\right)=H_{n,m}^{(r)},\quad H_n^{\left(\frac{r}{m}\right)}=H_{n,m}^{(r)}. \tag{9}$$

3. Proofs

First we deduce the exponential generating function of the r-Whitney numbers.

Let us multiply (1) with $\frac{z^n}{n!}$ and take sum on n:

$$e^{(mx+r)z}=\sum_{k=0}^{n}m^k x^{\underline{k}}\sum_{n=k}^{\infty}W_{m,r}(n,k)\frac{z^n}{n!}.$$

On the other hand,

$$e^{(mx+r)z}=e^{rz}(e^{mz}-1+1)^x=e^{rz}\sum_{k=0}^{\infty}x^{\underline{k}}\frac{(e^{mz}-1)^k}{k!}.$$

Comparing the coefficients of $x^{\underline{k}}$, we get (3).

According to (2),

$$m^n\left(\frac{x-r}{m}\right)^{\underline{n}}=\sum_{k=0}^{n}w_{m,r}(n,k)x^k. \tag{10}$$

Again, let us multiply both side with $\frac{z^n}{n!}$ and take sum on n,

$$\sum_{n=0}^{\infty}m^n\left(\frac{x-r}{m}\right)^{\underline{n}}\frac{z^n}{n!}=(1+mz)^{\frac{x-r}{m}}=(1+mz)^{\frac{-r}{m}}\exp\left(\frac{x}{m}\ln(1+mz)\right)$$
$$=(1+mz)^{\frac{-r}{m}}\sum_{k=0}^{\infty}\frac{x^k\ln^k(1+mz)}{m^k k!},$$

and the right hand side of (10) is

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}w_{m,r}(n,k)x^k\right)\frac{z^n}{n!}.$$

Comparing the coefficients of x^k, we get (4).

To prove (5), we use standard techniques from the theory of generating functions. Let us determine the generating function of

$$\sum_{k=0}^{n}W_{m,r}(n,k)\frac{w_{m,r}(k+1,l)}{k+1}.$$

From (4), it is straightforward to see that

$$\sum_{k=0}^{\infty} \frac{w_{m,r}(k+1,l)}{k+1} \frac{z^k}{k!} = \frac{1}{z}(1+mz)^{-\frac{r}{m}} \frac{\ln^l(1+mz)}{m^l l!}.$$

Thus

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{z^n}{n!} \left(\sum_{k=0}^{n} W_{m,r}(n,k) \frac{w_{m,r}(k+1,l)}{k+1} \right) \\
&\quad = \sum_{k=0}^{\infty} \frac{w_{m,r}(k+1,l)}{k+1} \left(\sum_{n=0}^{\infty} W_{m,r}(n,k) \frac{z^n}{n!} \right) \\
&\quad = \sum_{k=0}^{\infty} \frac{w_{m,r}(k+1,l)}{k+1} \frac{e^{rz}}{k!} \left(\frac{e^{mz}-1}{m} \right)^k = \frac{m}{e^{mz}-1} \frac{z^l}{l!} = \frac{z^{l-1}}{l!} \frac{mz}{e^{mz}-1} \\
&\quad = \frac{z^{l-1}}{l!} \sum_{n=0}^{\infty} B_n \frac{(mz)^n}{n!} = \sum_{n=l-1}^{\infty} B_{n-l+1} m^{n-l+1} \frac{z^n}{l!(n-l+1)!} \\
&\quad = \sum_{n=l-1}^{\infty} \binom{n+1}{l} B_{n-l+1} \frac{m^{n-l+1}}{n+1} \frac{z^n}{n!}.
\end{aligned}$$

This proves (5). The proof of identity (6) is totally the same.

Finally, we deal with (8). The generating function of $H_{n,m}^{(r)}$—as one can easily see from (4) to (7)—is

$$\sum_{n=0}^{\infty} H_{n,m}^{(r)} z^n = \frac{-\ln(1-z)}{(1-z)^{\frac{r}{m}}}.$$

This directly shows the identities under (9), and latter immediately implies (8).

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